Instability of standing wave, global existence and blowup for the Klein–Gordon–Zakharov system with different-degree nonlinearities

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\begin{abstract}
This paper discusses the Klein–Gordon–Zakharov system with different-degree nonlinearities in two and three space dimensions. Firstly, we prove the existence of standing wave with ground state by applying an intricate variational argument. Next, by introducing an auxiliary functional and an equivalent minimization problem, we obtain two invariant manifolds under the solution flow generated by the Cauchy problem to the aforementioned Klein–Gordon–Zakharov system. Furthermore, by constructing a type of constrained variational problem, utilizing the above two invariant manifolds as well as applying potential well argument and concavity method, we derive a sharp threshold for global existence and blowup. Then, combining the above results, we obtain two conclusions of how small the initial data are for the solution to exist globally by using dilation transformation. Finally, we prove a modified instability of standing wave to the system under study.
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1. Introduction

In the present paper, we consider the rescaled Klein–Gordon–Zakharov system in two and three space dimensions:

\[ c^{-2}u_{tt} - \Delta u + c^2 u = -nu - |u|^2 u, \quad t > 0, \quad x \in \mathbb{R}^N, \]

\[ n_t + \nabla \cdot V = 0, \quad t > 0, \quad x \in \mathbb{R}^N, \]

\[ \alpha^{-2}V_t + \nabla n + \nabla |u|^2 = 0, \quad t > 0, \quad x \in \mathbb{R}^N, \]

\[ \begin{cases} u(0, x) = u_0(x), & \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^N, \\ n(0, x) = n_0(x), & \quad V(0, x) = v_0(x), \quad x \in \mathbb{R}^N, \end{cases} \]

where \( N = 2, 3 \), \( (c, \alpha) \) are two parameters (see [10] for the rescaling with physical constants), \( u = u(t, x) \) is an unknown complex vector-valued function, \( V = V(t, x) \) is an unknown real vector-valued function and \( n = n(t, x) \) is an unknown real scalar-valued function. When \( N = 3 \) and with the absence of \( -|u|^2u \), the system (1.1)-(1.3) describes the interaction between Langmuir waves and ion sound waves in a plasma (see Bergé, Bidégaray and Colin [2], Dendy [3], Masmoudi and Nakanishi [10], Thornhill and ter Haar [17] as well as Zakharov [20]). Ohta and Todorova [12] proved the strong instability of standing wave solutions to the system (1.5)-(1.6) when \( c = 1, \alpha > 0 \) with arbitrary space dimension. Ohta and Todorova [12] proved the strong instability of standing wave solutions to the system (1.5)-(1.6) when \( c = 1, \alpha > 0 \).
and $N = 2, 3$. When $0 < \alpha < c$, Masmoudi and Nakanishi [11] proved existence and uniqueness of solutions to (1.5)–(1.6) in the energy space $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ on some time interval which is uniform with respect to two large parameters $c$ and $\alpha$.

When $N = 2$ and $c = \alpha = 1$, Guo and Yuan [7] proved without assuming that the Cauchy data are small, the existence and uniqueness of the global smooth solution for the Cauchy problem of the Klein–Gordon–Zakharov system (1.1)–(1.4) via the so-called continuous method and delicate a priori estimates and studied the asymptotic behavior of the solutions to the Klein–Gordon–Zakharov system (1.1)–(1.3) with a small parameter approaching zero.

To our knowledge, for the Klein–Gordon–Zakharov system (1.1)–(1.3) (system with two different-degree nonlinearities, two parameters $(c, \alpha)$ and $N = 2, 3$), no results are known so far on the existence and instability of standing wave as well as sharp threshold of global existence and blowup. In the present paper, we are interested in the sharp threshold of global existence, existence and instability of standing wave for the system (1.1)–(1.3) with $\alpha < c$. More specifically, by applying and generalizing the methods studying nonlinear wave equations (see for example [8,15,16]) and the Klein–Gordon–Zakharov system with same-degree nonlinearity (see for example [4,13,14,18,21]), we shall derive a sharp threshold of global existence for the Cauchy problem (1.1)–(1.4). On the other hand, using the sharp threshold, we answer the question of how small the Cauchy data $(u_0, u_1, n_0, v_0)$ are for the global solutions of the Cauchy problem (1.1)–(1.4) to exist in the light of the relation between $\int_{\mathbb{R}^N} |u_0|^2 dx$ and 0. Finally, combining the above results, we shall obtain the modified instability of standing wave for the Klein–Gordon–Zakharov system (1.1)–(1.3).

Now we recall that the local well-posedness for the Klein–Gordon–Zakharov system (1.5)–(1.6) with initial data (1.7) was performed by Ohta and Todorova [12], Ozawa, Tsutaya and Tsutsumi [13] when $\alpha \neq c$, $N = 2, 3$ and Masmoudi and Nakanishi [11] when $\alpha < c$, $N = 3$. Using the idea of the papers of Ozawa, Tsutaya and Tsutsumi [13], Guo and Yuan [7] as well as Masmoudi and Nakanishi [11], we can prove the local well-posedness of the Klein–Gordon–Zakharov system (1.1)–(1.3) in the energy space $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ with $\alpha < c$, that is, for any $(u_0, u_1, n_0, v_0) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ there exists a unique solution

$$ (u, u_t, n, V) \in C([0, T_{\text{max}}); H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)) $$

of (1.1)–(1.3) with the following conserved energy

$$ \mathcal{E}(u, u_t, n, V) = \mathcal{E}(u_0, u_1, n_0, v_0) $$

for all $t \in [0, T_{\text{max}})$, where $\mathcal{E}(u, u_t, n, V)$ is defined by

$$ \mathcal{E}(u, u_t, n, V) = c^2 \int_{\mathbb{R}^N} |u_1|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + c^2 \int_{\mathbb{R}^N} |u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} |u|^4 dx $$

$$ + \frac{1}{2} \int_{\mathbb{R}^N} |n|^2 dx + \frac{1}{2} \alpha^{-2} \int_{\mathbb{R}^N} |V|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} |n|^2 dx. $$

Since the system (1.1)–(1.3) includes different-degree nonlinearities and derivative nonlinearity, in order to get the results in this paper aforementioned, we must improve some techniques different from those proposed in the papers studying the Klein–Gordon–Zakharov system with quadratic nonlinearity and the nonlinear coupled system of the Klein–Gordon and wave equations without derivative and different-degree nonlinearities (see [4,5,12,21,22]). On the one hand, for the derivative nonlinearity, we must introduce a homogeneous Sobolev space $\dot{H}^{-1}(\mathbb{R}^N)$ which is defined by

$$ \dot{H}^{-1}(\mathbb{R}^N) = \{ n \mid \exists v : \mathbb{R}^N \to \mathbb{R}^N \text{ such that } n = -\nabla \cdot v, $$

$$ v \in L^2(\mathbb{R}^N) \text{ and } \| n \|_{\dot{H}^{-1}(\mathbb{R}^N)} = \| v \|_{L^2(\mathbb{R}^N)}. \} $$

(1.10)
Unlike the nonlinear Klein–Gordon system without derivative nonlinearity, we cannot obtain the blowup in finite time in $L^2(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Instead by defining

$$F_{(\alpha,\alpha)}(t) := 2c^{-2} \int_{\mathbb{R}^N} |u|^2 \, dx + \alpha^{-2} \int_{\mathbb{R}^N} |g|^2 \, dx,$$  \hspace{1cm} (1.11)

where $g$ will be defined by assumption (A1) at the end of this section, we can only obtain that the solution of the Cauchy problem (1.1)–(1.4) blows up in finite time in $H^1(\mathbb{R}^N) \times H^{-1}(\mathbb{R}^N)$. On the other hand, the different-degree nonlinearities in the system (1.1)–(1.3) yield the following:

In Lemma 3.2, the zero point $\lambda$ of $K(\phi, \psi) = 0$ is not unique, which brings about much more complexities to show the existence and instability of the standing wave for the system (1.1)–(1.3).

For simplicity, throughout this paper, we denote various positive constants by $C$, $\int_{\mathbb{R}^N} \cdot \, dx$ by $\int \cdot \, dx$ and make the following assumption:

(A1) There exists a real vector-valued function $g(t, x) \in L^2(\mathbb{R}^N)$ such that

$$g_1(t, x) = V(t, x).$$

In addition, for simplicity, we introduce a modified $H^1(\mathbb{R}^N)$-norm as

$$\|u\|_{H^1(\mathbb{R}^N)}^2 = c^2 \int |u|^2 \, dx + \int |\nabla u|^2 \, dx. \hspace{1cm} (1.12)$$

**Remark 1.1.** In fact, from the mathematical point of view, when $\alpha \geq c > 0$, the results in the present paper are still true. But it has been pointed out in Masmoudi and Nakanishi [11] that the system (1.5)–(1.6) does not have the null form structure as in the sixteenth reference of [11] and this suggests that when $\alpha = c$ the system (1.5)–(1.6) may be locally ill-posed in $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. Thus, as the system (1.5)–(1.6), in view of physical relativity and the local well-posedness, we only consider the case $0 < \alpha < c$ for the system (1.1)–(1.3) in the present paper.

2. Preliminaries

In this section, we first define several functionals and manifolds, then consider two constrained minimization problems and finally give some elementary results.

For $(\phi, \psi) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, we define two functionals and a manifold as

$$J(\phi, \psi) := \int |\nabla \phi|^2 \, dx + c^2 \int |\phi|^2 \, dx + \frac{1}{2} \int |\phi|^4 \, dx + \frac{1}{2} \int |\psi|^2 \, dx + \int \psi |\phi|^2 \, dx,$$ \hspace{1cm} (2.1)

$$K(\phi, \psi) := 2 \int |\nabla \phi|^2 \, dx + 2c^2 \int |\phi|^2 \, dx + 2 \int |\phi|^4 \, dx + \int |\psi|^2 \, dx + 3 \int \psi |\phi|^2 \, dx,$$ \hspace{1cm} (2.2)

$$\mathcal{M} := \{ (\phi, \psi) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \mid K(\phi, \psi) = 0, \hspace{0.2cm} (\phi, \psi) \neq (0, 0) \}. \hspace{1cm} (2.3)$$

In addition, we define a constrained variational problem

$$d_{\mathcal{M}} := \inf_{(\phi, \psi) \in \mathcal{M}} J(\phi, \psi). \hspace{1cm} (2.4)$$

**Remark 2.1.** Since functionals $J(u)$ and $K(u)$ include two different high-degree nonlinearities (higher than two-degree), in order to prove $d_{\mathcal{M}} > 0$, it is more difficult than functionals $J(u)$ and $K(u)$ without different-degree nonlinearities. We will prove $d_{\mathcal{M}} > 0$ in Section 3.
Besides functionals \(J(\phi, \psi)\) and \(K(\phi, \psi)\) defined by (2.1) and (2.2), we also introduce another functional

\[
I(\phi, \psi) := J(\phi, \psi) - \frac{1}{\nu + 1} K(\phi, \psi)
\]

\[
= \frac{\nu - 1}{\nu + 1} \int |\nabla \phi|^2 \, dx + \frac{\nu - 1}{\nu + 1} c^2 \int |\phi|^2 \, dx + \frac{\nu - 3}{2(\nu + 1)} \int |\phi|^4 \, dx
\]

\[
+ \frac{v - 1}{2(v + 1)} \int |\psi|^2 \, dx + \frac{v - 2}{v + 1} \int \psi |\phi|^2 \, dx
\]

(2.5)

and set

\[
\mathcal{M}^- := \{(\phi, \psi) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \mid K(\phi, \psi) \leq 0, (\phi, \psi) \neq (0, 0)\},
\]

(2.6)

where \(\nu > 1\) is a constant. In addition, we define a constrained variational problem

\[
d_{\mathcal{M}^-} := \inf_{(\phi, \psi) \in \mathcal{M}^-} I(\phi, \psi).
\]

(2.7)

In the following, we give some elementary results according to the above definitions.

**Lemma 2.1.** If \(K(\phi, \psi) \leq 0\) and \((\phi, \psi) \neq (0, 0)\), then \(I(s^{1/2} \phi, s \psi)\) is an increasing function of \(s \in (0, \infty)\), where functionals \(K(\phi, \psi)\) and \(I(\phi, \psi)\) are defined by (2.2) and (2.5).

**Proof.** From (2.5) it follows that

\[
I(s^{1/2} \phi, s \psi) = \frac{\nu - 1}{\nu + 1} s \int |\nabla \phi|^2 \, dx + \frac{\nu - 1}{\nu + 1} s c^2 \int |\phi|^2 \, dx + \frac{\nu - 3}{2(\nu + 1)} s^2 \int |\phi|^4 \, dx
\]

\[
+ \frac{v - 1}{2(v + 1)} s^2 \int |\psi|^2 \, dx + \frac{v - 2}{v + 1} s^2 \int \psi |\phi|^2 \, dx.
\]

(2.8)

By \(K(\phi, \psi) \leq 0\) and (2.2), we have

\[
K(\phi, \psi) = 2 \int |\nabla \phi|^2 \, dx + 2c^2 \int |\phi|^2 \, dx + 2 \int |\phi|^4 \, dx + \int |\psi|^2 \, dx + 3 \int \psi |\phi|^2 \, dx \leq 0
\]

which yields that

\[
2 \int |\nabla \phi|^2 \, dx + 2c^2 \int |\phi|^2 \, dx \leq -2 \int |\phi|^4 \, dx - \int |\psi|^2 \, dx - 3 \int \psi |\phi|^2 \, dx.
\]

(2.9)

By the inequality

\[
\left| \int \psi |\phi|^2 \, dx \right| \leq \frac{1}{2} \int |\psi|^2 \, dx + \frac{1}{2} \int |\phi|^4 \, dx,
\]

(2.10)

\((\phi, \psi) \neq (0, 0)\) and (2.9) imply that

\[
0 < 2 \int |\nabla \phi|^2 \, dx + 2c^2 \int |\phi|^2 \, dx \leq \frac{1}{2} \left( \int |\psi|^2 \, dx - \int |\phi|^4 \, dx \right).
\]

(2.11)

From \(K(\phi, \psi) \leq 0\), \(\nu > 1\), \((\phi, \psi) \neq (0, 0)\), (2.10) and (2.11), for \(s \in (0, \infty)\) we get
\[
\frac{dl(s^\frac{1}{2} \phi, s\psi)}{ds} = \frac{v - 1}{v + 1} \int |\nabla \phi|^2 \, dx + \frac{v - 1}{v + 1} c^2 \int |\phi|^2 \, dx + \frac{v - 3}{v + 1} s \int |\phi|^4 \, dx \\
+ \frac{v - 1}{v + 1} s \int |\psi|^2 \, dx + \frac{2(v - 2)}{v + 1} s \int |\phi|^2 \, dx \\
\geq \frac{v - 1}{v + 1} \int |\nabla \phi|^2 \, dx + \frac{v - 1}{v + 1} c^2 \int |\phi|^2 \, dx \\
+ \frac{1}{v + 1} s \left( \int |\psi|^2 \, dx - \int |\phi|^4 \, dx \right) > 0,
\]

which concludes our proof. \(\square\)

Using Lemma 2.1, we can get the following conclusion.

**Proposition 2.1.** \(\mathcal{M}\) and \(\mathcal{M}^-\) are non-empty, and

\[
d_{\mathcal{M}} = \inf_{(\phi, \psi) \in \mathcal{M}} I(\phi, \psi) = d_{\mathcal{M}^-} = \inf_{(\phi, \psi) \in \mathcal{M}^-} I(\phi, \psi).
\]

(2.12)

Furthermore, \(I(\phi, \psi) > d_{\mathcal{M}}\) if \(K(\phi, \psi) < 0\).

**Proof.**

**Step 1.** First we prove that \(\mathcal{M}^-\) is non-empty. Choose any \((\phi, \psi) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)\) with \((\phi, \psi) \neq (0, 0)\) and consider

\[
P(s) = K(s^\frac{1}{2} \phi, s\psi) \\
= 2s \left( \int |\nabla \phi|^2 \, dx + c^2 \int |\phi|^2 \, dx \right) + 2s^2 \int |\phi|^4 \, dx \\
+ s^2 \int |\psi|^2 \, dx + 3s^2 \int |\phi|^2 \, dx,
\]

thus if we take \(2 \int |\phi|^4 \, dx + \int |\psi|^2 \, dx + 3 \int |\phi|^2 \, dx < 0\), then we see that \(P(s) < 0\) for sufficiently large \(s > 1\) and hence \((s^\frac{1}{2} \phi, s\psi) \in \mathcal{M}^-\).

**Step 2.** Next, to prove that \(\mathcal{M}\) is non-empty, we choose \((\phi, \psi) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)\) such that \(K(\phi, \psi) < 0\) and consider \(K(s^\frac{1}{2} \phi, s\psi)\). Now for \(s = 1\), \(K(\phi, \psi) < 0\); for \(s > 0\) close to zero, \(K(s^\frac{1}{2} \phi, s\psi) > 0\) from the expression for \(K(s^\frac{1}{2} \phi, s\psi)\). Therefore by continuity, there exists an \(s_0 \in (0, 1)\) such that \(K(s_0^\frac{1}{2} \phi, s_0\psi) = 0\), i.e., \((s_0^\frac{1}{2} \phi, s_0\psi) \in \mathcal{M}\).

**Step 3.** Furthermore, we show that (2.12) holds. By Lemma 2.1, \(I(s^\frac{1}{2} \phi, s\psi)\) is an increasing function of \(s \in (0, \infty)\) provided that \(K(\phi, \psi) \leq 0\) and \((\phi, \psi) \neq (0, 0)\). Since \(K(s_0^\frac{1}{2} \phi, s_0\psi) = 0\), by (2.5) we have

\[
d_{\mathcal{M}} \leq I(s_0^\frac{1}{2} \phi, s_0\psi) \\
= I(s_0^\frac{1}{2} \phi, s_0\psi) + \frac{1}{v + 1} K(s_0^\frac{1}{2} \phi, s_0\psi) \\
= I(s_0^\frac{1}{2} \phi, s_0\psi),
\]

which together with Lemma 2.1 implies that for \(s_0 \in (0, 1)\),
\[ d_{\mathcal{M}} \leq \inf_{(s_0 \phi, s_0 \psi) \in \mathcal{M}^-} J(s_0 \phi, s_0 \psi) \]
\[ = \inf_{(s_0 \phi, s_0 \psi) \in \mathcal{M}^-} I(s_0 \phi, s_0 \psi) \]
\[ \leq \inf_{(\phi, \psi) \in \mathcal{M}^-} I(\phi, \psi) = d_{\mathcal{M}^-}. \]  
\hfill (2.13)

But by definition (2.4)

\[ d_{\mathcal{M}} = \inf_{(\phi, \psi) \in \mathcal{M}} J(\phi, \psi) = \inf_{(\phi, \psi) \in \mathcal{M}} I(\phi, \psi) \geq \inf_{(\phi, \psi) \in \mathcal{M}^-} I(\phi, \psi) = d_{\mathcal{M}^-}. \]  
\hfill (2.14)

Thus (2.13) and (2.14) imply (2.12).

Finally, we show: If \( K(\phi, \psi) < 0 \), then \( I(\phi, \psi) > d_{\mathcal{M}} \). Since \( K(\phi, \psi) < 0 \), from Step 2 it follows that there exists an \( s \in (0, 1) \) such that \( K(s \phi, s \psi) = 0 \) and \( (\phi, \psi) \neq (0, 0) \). From (2.5) we have

\[ J(s \phi, s \psi) = I(s \phi, s \psi) \geq d_{\mathcal{M}} \]

which yields from Lemma 2.1 that

\[ d_{\mathcal{M}} \leq I(s \phi, s \psi) < I(\phi, \psi). \]

Therefore, if \( K(\phi, \psi) < 0 \), then \( I(\phi, \psi) > d_{\mathcal{M}} \).

So far, the proof of Proposition 2.1 is completed. \( \square \)

3. Existence of standing wave

In this section, we discuss the existence of standing wave for the Klein–Gordon–Zakharov system (1.1)–(1.3) and prove \( d_{\mathcal{M}} > 0 \) which was defined in Section 2. First we give the definitions of standing wave and ground state.

**Definition 3.1 (Standing wave).** If a pair of real functions \( (\phi, \psi) = (\phi(x), \psi(x)) \), \( x \in \mathbb{R}^N \), verifies the system

\[ \begin{cases} -\Delta \phi + c^2 \phi = -\phi \psi - |\phi|^2 \phi, & x \in \mathbb{R}^N, \\ \psi + |\phi|^2 = 0, & x \in \mathbb{R}^N, \end{cases} \]  
\hfill (E)

and

\[ (\phi, \psi) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \setminus \{(0, 0)\}, \]

then

\[ u(t, x) = \phi(x), \quad n(t, x) = \psi(x), \quad t \geq 0, \quad x \in \mathbb{R}^N, \]
satisfy (1.1)–(1.3), and are called standing wave solution (or standing wave) of the system (1.1)–(1.3).

**Definition 3.2 (Ground state).** A solution \( (\phi, \psi) \) of (E) is termed as a ground state (solution) if it has some minimal action \( J(\phi, \psi) \) (defined by (2.1)) among all positive solutions of (E).
Remark 3.1. Since the system (1.1)–(1.3) is equivalent to
\[
\begin{align*}
\frac{c^{-2}}{2}u_{tt} - \Delta u + c^2 u &= -nu - |u|^2 u, \\
\alpha^{-2}n_{tt} - \Delta n &= \Delta |n|^2,
\end{align*}
\tag{KGZ}
\]
Definitions 3.1 and 3.2 are consistent with the related definitions for single nonlinear Klein–Gordon equation. Moreover, from the physical viewpoint, an important role is played by the ground state solution of (E).

In the following, we give two lemmas.

Lemma 3.1. \( J(\phi, \psi) \) is bounded below on \( M \) and \( J(\phi, \psi) > 0 \) for all \( (\phi, \psi) \in M \).

Proof. From (2.1)–(2.4), we get on \( M \)
\[
J(\phi, \psi) = \frac{1}{3} \int |\nabla \phi|^2 \, dx + \frac{1}{3} c^2 \int |\phi|^2 \, dx + \frac{1}{6} \int |\psi|^2 \, dx - \frac{1}{6} \int |\phi|^4 \, dx \tag{3.1}
\]
and
\[
\int \psi |\phi|^2 \, dx < 0. \tag{3.2}
\]
By the inequality
\[
\left| \int \psi |\phi|^2 \, dx \right| \leq \frac{1}{2} \int |\psi|^2 \, dx + \frac{1}{2} \int |\phi|^4 \, dx, \tag{3.3}
\]
\( K(\phi, \psi) = 0 \) and \( (\phi, \psi) \neq (0, 0) \), it follows that
\[
2 \int |\nabla \phi|^2 \, dx + 2c^2 \int |\phi|^2 \, dx + 2 \int |\phi|^4 \, dx + \int |\psi|^2 \, dx = -3 \int \psi |\phi|^2 \, dx
\leq \frac{3}{2} \int |\psi|^2 \, dx + \frac{3}{2} \int |\phi|^4 \, dx
\]
which implies
\[
0 < 2 \int |\nabla \phi|^2 \, dx + 2 \int |\phi|^2 \, dx \leq \frac{1}{2} \int |\psi|^2 \, dx - \frac{1}{2} \int |\phi|^4 \, dx, \tag{3.4}
\]
that is,
\[
\int |\psi|^2 \, dx > \int |\phi|^4 \, dx. \tag{3.5}
\]
Hence if \( K(\phi, \psi) = 0 \) and \( (\phi, \psi) \neq (0, 0) \), then (3.5) is always true. By (3.1) and (3.5) we get \( J(\phi, \psi) > 0 \) for all \( (\phi, \psi) \in M \). □

Lemma 3.2. For \( (\phi, \psi) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \), \( (\phi, \psi) \neq (0, 0) \) and \( \lambda > 0 \), let \( \phi_\lambda(x) = \lambda \phi(x), \psi_\lambda(x) = \lambda \psi(x) \). Thus one of the following two conclusions holds:

(I) If \( K(\phi_\lambda, \psi_\lambda) \) has two different positive zero points \( \lambda_1 \) and \( \lambda_2 \) (depending on \( (\phi, \psi) \)) and \( \lambda_2 > \lambda_1 \), then \( K(\phi_{\lambda_1}, \psi_{\lambda_1}) = 0, K(\phi_{\lambda_2}, \psi_{\lambda_2}) = 0 \) and one gets the following eight possibilities.
(I-1) \[ K(\phi_1, \psi_1) > 0 \text{ for } \lambda \in (0, \lambda_1), \]
\[ K(\phi_1, \psi_1) < 0 \text{ for } \lambda \in (\lambda_1, \lambda_2), \]
\[ K(\phi_1, \psi_1) > 0 \text{ for } \lambda \in (\lambda_2, \infty), \]
\[ \forall \lambda \in (0, \lambda_2], \quad J(\phi_{\lambda_1}, \psi_{\lambda_1}) \geq J(\phi_1, \psi_1), \]
\[ \forall \lambda \in [\lambda_2, \infty), \quad J(\phi_{\lambda_2}, \psi_{\lambda_2}) \leq J(\phi_1, \psi_1). \]

(II) If \( K(\phi_1, \psi_1) \) has two same positive zero points \( \lambda^* \), then one gets \( K(\phi_{\lambda^*}, \psi_{\lambda^*}) = 0 \) and the following four possibilities hold.

(II) \[ K(\phi_2, \psi_2) > 0 \text{ for } \lambda \in (0, \lambda_1) \cup (\lambda_1, \lambda_2), \]
\[ K(\phi_2, \psi_2) > 0 \text{ for } \lambda \in (\lambda_2, \infty), \]
\[ \forall \lambda \in (0, \lambda_2], \quad J(\phi_{\lambda_1}, \psi_{\lambda_1}) \geq J(\phi_1, \psi_1), \]
\[ \forall \lambda \in [\lambda_2, \infty), \quad J(\phi_{\lambda_2}, \psi_{\lambda_2}) \geq J(\phi_1, \psi_1). \]

(III) \[ K(\phi_3, \psi_3) > 0 \text{ for } \lambda \in (0, \lambda_1) \cup (\lambda_1, \lambda_2), \]
\[ K(\phi_3, \psi_3) < 0 \text{ for } \lambda \in (\lambda_2, \infty), \]
\[ \forall \lambda \in (0, \lambda_2], \quad J(\phi_{\lambda_1}, \psi_{\lambda_1}) \leq J(\phi_1, \psi_1), \]
\[ \forall \lambda \in [\lambda_2, \infty), \quad J(\phi_{\lambda_2}, \psi_{\lambda_2}) \leq J(\phi_1, \psi_1). \]

(IV) \[ K(\phi_4, \psi_4) > 0 \text{ for } \lambda \in (0, \lambda_1) \cup (\lambda_1, \lambda_2), \]
\[ K(\phi_4, \psi_4) > 0 \text{ for } \lambda \in (\lambda_2, \infty), \]
\[ \forall \lambda \in (0, \lambda_2], \quad J(\phi_{\lambda_1}, \psi_{\lambda_1}) \leq J(\phi_1, \psi_1), \]
\[ \forall \lambda \in [\lambda_2, \infty), \quad J(\phi_{\lambda_2}, \psi_{\lambda_2}) \leq J(\phi_1, \psi_1). \]
(II-1) \( K(φ_λ, ψ_λ) < 0 \) for \( λ ∈ (0, λ^*) \),
\[ K(φ_λ, ψ_λ) > 0 \] for \( λ ∈ (λ^*, ∞) \),
\( ∀λ ∈ (0, ∞), \ J(φ_λ, ψ_λ) ≤ J(φ_λ, ψ_λ). \)

(II-2) \( K(φ_λ, ψ_λ) < 0 \) for \( λ ∈ (0, λ^*) \),
\( K(φ_λ, ψ_λ) < 0 \) for \( λ ∈ (λ^*, ∞) \),
\( ∀λ ∈ (0, λ^*), \ J(φ_λ, ψ_λ) ≤ J(φ_λ, ψ_λ). \)

(II-3) \( K(φ_λ, ψ_λ) > 0 \) for \( λ ∈ (0, λ^*) \),
\( K(φ_λ, ψ_λ) < 0 \) for \( λ ∈ (λ^*, ∞) \),
\( ∀λ ∈ (0, ∞), \ J(φ_λ, ψ_λ) ≥ J(φ_λ, ψ_λ). \)

(II-4) \( K(φ_λ, ψ_λ) > 0 \) for \( λ ∈ (0, λ^*) \),
\( K(φ_λ, ψ_λ) > 0 \) for \( λ ∈ (λ^*, ∞) \),
\( ∀λ ∈ (0, λ^*), \ J(φ_λ, ψ_λ) ≥ J(φ_λ, ψ_λ). \)

Proof. By (2.1) and (2.2), we have

\[ J(φ_λ, ψ_λ) = λ^2 \left( \int |∇φ|^2 dx + c^2 \int |φ|^2 dx \right) + \frac{1}{2} λ^4 \int |φ|^4 dx + \frac{1}{2} λ^2 \int |ψ|^2 dx + λ^3 \int |ψ||φ|^2 dx, \quad (3.6) \]

\[ K(φ_λ, ψ_λ) = 2λ^2 \left( \int |∇φ|^2 dx + c^2 \int |φ|^2 dx \right) + 2λ^3 \int |φ|^4 dx + λ^2 \int |ψ|^2 dx + 3λ^3 \int |ψ||φ|^2 dx. \quad (3.7) \]

From the definition of \( M \) (\( M \) is not an empty set which has been shown in Proposition 2.1), there must exist a \( γ > 0 \) such that \( K(φ_γ, ψ_γ) = 0 \) and \( (φ_γ, ψ_γ) \neq (0, 0) \). Thus by (3.7), one of the following two cases is true.

In the following, we first prove (I).

Proof of (I). If \( K(φ_λ, ψ_λ) \) has two positive zero points \( λ_1 \) and \( λ_2 \) with \( λ_2 > λ_1 \), then from (3.7) we obtain \( K(φ_{λ_1}, ψ_{λ_1}) = 0, K(φ_{λ_2}, ψ_{λ_2}) = 0 \) and \( (φ_{λ_1}, ψ_{λ_1}) \neq (0, 0), (φ_{λ_2}, ψ_{λ_2}) \neq (0, 0) \). First of all, we show (I-1) holds. By the expression (3.7), it follows that the possibility holds: \( K(φ_λ, ψ_λ) > 0 \) for \( λ ∈ (0, λ_1) \), \( K(φ_λ, ψ_λ) < 0 \) for \( λ ∈ (λ_1, λ_2) \) and \( K(φ_λ, ψ_λ) > 0 \) for \( λ ∈ (λ_2, ∞) \). Since

\[ \frac{d}{dλ} J(φ_λ, ψ_λ) = 2λ \left( \int |∇φ|^2 dx + c^2 \int |φ|^2 dx + \frac{1}{2} \int |ψ|^2 dx \right) + 2λ^3 \int |φ|^4 dx + 3λ^2 \int |ψ||φ|^2 dx \]

\[ = λ^{-1} K(φ_λ, ψ_λ), \quad (3.8) \]

noting that \( K(φ_{λ_1}, ψ_{λ_1}) = 0 \) and \( K(φ_{λ_2}, ψ_{λ_2}) = 0 \), it follows that for \( ∀λ ∈ (0, λ_2), \ J(φ_λ, ψ_λ) ≥ J(φ_λ, ψ_λ) \) and for \( ∀λ ∈ [λ_2, ∞), \ J(φ_λ, ψ_λ) ≤ J(φ_λ, ψ_λ). \)

Similarly, we can prove possibilities (I-2)–(I-8). □
Next we prove (II).

**Proof of (II).** If \( K(\phi_\lambda, \psi_\lambda) \) has two same positive zero points \( \lambda^* \), then from (3.7) one gets \( K(\phi^*_\lambda, \psi^*_\lambda) = 0 \) and \( (\phi^*_\lambda, \psi^*_\lambda) \neq (0, 0) \). We first show (II-1) holds. Applying (3.7), it follows that the possibility holds: \( K(\phi_\lambda, \psi_\lambda) < 0 \) for \( \lambda \in (0, \lambda^*) \), \( K(\phi_\lambda, \psi_\lambda) > 0 \) for \( \lambda \in (\lambda^*, \infty) \). Thus from (3.8), we get for \( \forall \lambda \in (0, \infty), J(\phi_\lambda, \psi_\lambda) \leq J(\phi_\lambda, \psi_\lambda) \).

Similarly, we can show possibilities (II-2)–(II-4). \( \square \)

So far, the proof of Lemma 3.2 is completed.

In the following, we give the existence of standing waves for the system (1.1)–(1.3) whose proof strongly depends upon Lemma 3.2.

**Remark 3.2.** In Lemma 3.2 and its proof, with the absence of \( |\phi|^4 \) in \( K(\phi, \psi) \), \( K(\phi_\lambda, \psi_\lambda) = 0 \) exists only one standing point \( \lambda^* \) which depends only on \( (\phi, \psi) \). In this case, action \( J(\phi_\lambda, \psi_\lambda) \) can be determined uniquely by the standing point \( \lambda^* \). However, with the presence of \( |\phi|^4 \) in \( K(\phi, \psi) \), owing to the interaction of multiple nonlinearities in \( K(\phi, \psi) \), \( K(\phi_\lambda, \psi_\lambda) = 0 \) exist two different positive standing points \( \lambda_1, \lambda_2 \) or two same positive standing points \( \lambda_1 = \lambda_2 = \lambda^* \) which depend only on \( (\phi, \psi) \). In this case, action \( J(\phi_\lambda, \psi_\lambda) \) cannot be determined uniquely by the standing points, but varies from the value of \( K(\phi_\lambda, \psi_\lambda) \) on the intervals \((0, \lambda_1), (\lambda_1, \lambda_2)\) and \((\lambda_2, \infty)\) or \((0, \lambda^*)\) and \((\lambda^*, \infty)\). Therefore, from the physical viewpoint, the above mathematical discussions can be viewed that: the more the standing points, the more complex the action becomes. In other words, the form of the functional \( K(\phi, \psi) \) decides directly the characteristic of action \( J(\phi, \psi) \).

**Theorem 3.1.** There exists \((P, Q) \in \mathcal{M}\) such that

\[
J(P, Q) = \inf_{(\phi, \psi) \in \mathcal{M}} J(\phi, \psi) = d^*_{\mathcal{M}}.
\]

**Proof.** Considering the constrained variational problem (2.4) and applying Lemma 3.2 we may let \( \{(\phi_n)_{\xi}, (\psi_n)_{\xi} \colon n \in \mathbb{N} \} \subset \mathcal{M} \) be a minimizing sequence for (2.4), then one gets

\[
\begin{align}
K((\phi_n)_{\xi}, (\psi_n)_{\xi}) &= 0, \\
J((\phi_n)_{\xi}, (\psi_n)_{\xi}) \to \inf_{(\phi, \psi) \in \mathcal{M}} J(\phi, \psi) \quad \text{as} \quad n \to \infty,
\end{align}
\]

where

\[
\begin{cases}
\xi = \lambda_1 \text{ or } \xi \in [\lambda_2, \infty), & \text{if (I-1) in Lemma 3.2 holds}, \\
\xi = \lambda_2 \text{ or } \xi \in (\lambda_2, \infty), & \text{if (I-2) in Lemma 3.2 holds}, \\
\xi = \lambda_2, & \text{if (I-3) in Lemma 3.2 holds}, \\
\xi = \lambda_1, & \text{if (I-4) in Lemma 3.2 holds}, \\
\xi \in (0, \infty), & \text{if (I-5) in Lemma 3.2 holds}, \\
\xi = \lambda_2 \text{ or } \xi \in (0, \lambda_2], & \text{if (I-6) in Lemma 3.2 holds}, \\
\xi \in (0, \infty), & \text{if (I-7) in Lemma 3.2 holds}, \\
\xi = \lambda_1 \text{ or } \xi \in (0, \lambda_1], & \text{if (I-8) in Lemma 3.2 holds}, \\
\xi \in (0, \infty), & \text{if (II-1) in Lemma 3.2 holds}, \\
\xi = \lambda^* \text{ or } \xi \in (0, \lambda^*), & \text{if (II-2) in Lemma 3.2 holds}, \\
\xi = \lambda^*, & \text{if (II-3) in Lemma 3.2 holds}, \\
\xi = \lambda^* \text{ or } \xi \in [\lambda^*, \infty), & \text{if (II-4) in Lemma 3.2 holds}.
\end{cases}
\]
Let \( \phi^* \) and \( \psi^* \) denote the Schwarz spherical rearrangements of functions \( \phi \) and \( \psi \), respectively. We recall that \( \phi^* \) and \( \psi^* \) are spherically symmetric, non-increasing (with respect to \( |x| \)) functions, and the symmetrization has the following properties:

\[
\int |\nabla \phi^*|^2 \, dx \leq \int |\nabla \phi|^2 \, dx, \quad \int |\nabla \psi^*|^2 \, dx \leq \int |\nabla \psi|^2 \, dx, \quad (3.10)
\]

\[
\int |\phi^*|^\sigma \, dx = \int |\phi|^\sigma \, dx, \quad \int |\psi^*|^\sigma \, dx = \int |\psi|^\sigma \, dx \quad \text{for} \quad \sigma > 1. \quad (3.11)
\]

Furthermore, it is evident that

\[
(\phi_\lambda)^* = (\phi^*)_\lambda, \quad (\psi_\lambda)^* = (\psi^*)_\lambda, \quad (3.12)
\]

where as in Lemma 3.2, \( \phi_\lambda(x) = \lambda \phi(x) \), \( \psi_\lambda(x) = \lambda \psi(x) \).

Now we consider the minimizing sequence \( \{((\phi_n)_\lambda, (\psi_n)_\lambda) : n \in \mathbb{N} \} \) and choose suitable \( \{((\phi_n)_\lambda)^*, ((\psi_n)_\lambda)^* \} \) such that

\[
K\{((\phi_n)_\lambda)^*)_{\eta_n}, ((\psi_n)_\lambda)^*)_{\eta_n} \} = 0, \quad (3.13)
\]

where

\[
\eta_n \in (0, \lambda_2] \quad \text{or} \quad \eta_n = \lambda_2, \quad \text{if (I-1) in Lemma 3.2 holds},
\]

\[
\eta_n \in (0, \lambda_2] \quad \text{or} \quad \eta_n = \lambda_2, \quad \text{if (I-2) in Lemma 3.2 holds},
\]

\[
\eta_n \in (0, \infty), \quad \text{if (I-3) in Lemma 3.2 holds},
\]

\[
\eta_n \in (0, \infty), \quad \text{if (I-4) in Lemma 3.2 holds},
\]

\[
\eta_n = \lambda_1, \quad \text{if (I-5) in Lemma 3.2 holds},
\]

\[
\eta_n = (\lambda_2, \infty) \quad \text{or} \quad \eta_n = \lambda_1, \quad \text{if (I-6) in Lemma 3.2 holds},
\]

\[
\eta_n = \lambda_2, \quad \text{if (I-7) in Lemma 3.2 holds},
\]

\[
\eta_n \in [\lambda_1, \infty) \quad \text{or} \quad \eta_n = \lambda_1, \quad \text{if (I-8) in Lemma 3.2 holds},
\]

\[
\eta_n = \lambda^*, \quad \text{if (II-1) in Lemma 3.2 holds},
\]

\[
\eta_n \in [\lambda^*, \infty) \quad \text{or} \quad \eta_n = \lambda^*, \quad \text{if (II-2) in Lemma 3.2 holds},
\]

\[
\eta_n \in (0, \infty), \quad \text{if (II-3) in Lemma 3.2 holds},
\]

\[
\eta_n \in (0, \lambda^*) \quad \text{or} \quad \eta_n = \lambda^*, \quad \text{if (II-4) in Lemma 3.2 holds}.
\]

Let

\[
P_n = \{((\phi_n)_\lambda)^*)_{\eta_n}, \quad Q_n = \{((\psi_n)_\lambda)^*)_{\eta_n} \}. \quad (3.15)
\]

From (3.12), we also have

\[
P_n = \{((\phi_n)_\lambda)^*)_{\eta_n}, \quad Q_n = \{((\psi_n)_\lambda)^*)_{\eta_n} \}^*
\]

and therefore by (3.1), (3.10) and (3.11), we have from \( K((\phi_n)_\lambda, (\psi_n)_\lambda) = 0 \) that

\[
J(P_n, Q_n) \leq J\{((\phi_n)_\lambda)^*)_{\eta_n}, ((\psi_n)_\lambda)^*)_{\eta_n} \} \leq J((\phi_n)_\lambda, (\psi_n)_\lambda). \quad (3.16)
\]

The right-hand side inequality in (3.16) is a consequence of Lemma 3.2. Thus \( \{P_n, Q_n\} : n \in \mathbb{N} \} \subset \mathcal{M} \) and by (3.16) we have

\[
J(P_n, Q_n) \leq J((\phi_n)_\lambda, (\psi_n)_\lambda).
\]

Therefore \( \{P_n, Q_n\} : n \in \mathbb{N} \} \) is also a minimizing sequence for (2.4).
From (3.1), (3.9) and Sobolev’s embedding theorem, one knows that \( \|P_n\|_{H^1(\mathbb{R}^N)} \) and \( \|Q_n\|_{L^2(\mathbb{R}^N)} \) are both bounded for all \( n \in \mathbb{N} \). Then there exists a subsequence
\[
\{(P_n)_k : k \in \mathbb{N}\} \subset \{P_n : n \in \mathbb{N}\}
\]
such that
\[
(P_n)_k \rightharpoonup P_\infty \text{ weakly in } H^1(\mathbb{R}^N) \text{ as } n \to \infty.
\]
Now for
\[
\{(Q_n)_k : k \in \mathbb{N}\} \subset \{Q_n : n \in \mathbb{N}\}
\]
there also exists a subsequence
\[
\{((Q_n)_k)_m : m \in \mathbb{N}\} \subset \{(Q_n)_k : k \in \mathbb{N}\}
\]
such that
\[
((Q_n)_k)_m \rightharpoonup Q_\infty \text{ weakly in } L^2(\mathbb{R}^N) \text{ as } n \to \infty.
\]
(3.17)

It is of course that
\[
((P_n)_k)_m \rightharpoonup P_\infty \text{ weakly in } H^1(\mathbb{R}^N) \text{ as } n \to \infty.
\]
(3.18)

Thus we extract a subsequence \(((P_n)_k)_m, ((Q_n)_k)_m : m \in \mathbb{N}\) from \{(P_n, Q_n) : n \in \mathbb{N}\} such that (3.17) and (3.18) hold. For simplicity, we still denote \(((P_n)_k)_m, ((Q_n)_k)_m : m \in \mathbb{N}\) by \{(P_n, Q_n) : n \in \mathbb{N}\}.

Note the compactness lemma in [19]: For \( 2 < \delta < N + 2 \) (when \( N = 2 \), \( N + 2 = \infty \)), the embedding \( H^1_{\text{radial}}(\mathbb{R}^N) \to L^\delta(\mathbb{R}^N) \) is compact, where
\[
H^1_{\text{radial}}(\mathbb{R}^N) = \{ f(x) \in H^1(\mathbb{R}^N), f(x) = f(|x|) \text{ is a function of } |x| \text{ alone}\}.
\]

Thus from (3.18), one gets
\[
P_n \to P_\infty \text{ strongly in } L^4(\mathbb{R}^N) \text{ as } n \to \infty.
\]
(3.19)

Now we assert that \( (P_\infty, Q_\infty) \neq (0, 0) \), which will be proved by contradiction. Otherwise, if \( (P_\infty, Q_\infty) \equiv (0, 0) \), then from (3.17) and (3.19) one obtains
\[
P_n \to 0 \text{ strongly in } L^4(\mathbb{R}^N) \text{ as } n \to \infty,
\]
\[
Q_n \to 0 \text{ weakly in } L^2(\mathbb{R}^N) \text{ as } n \to \infty.
\]
(3.20)

Moreover, from Hölder’s inequality
\[
\left| \int Q_n |P_n|^2 \, dx \right| \leq \left( \int |Q_n|^2 \, dx \right)^{\frac{1}{2}} \left( \int |P_n|^4 \, dx \right)^{\frac{1}{2}}
\]
and (3.20), we get
\[
\int Q_n |P_n|^2 \, dx \to 0 \text{ as } n \to \infty.
\]
(3.21)
Since \((P_n, Q_n) \in \mathcal{M}\), \(K(P_n, Q_n) = 0\) and (3.21) imply that
\[
2 \int \left| \nabla P_n \right|^2 \, dx + 2c^2 \int |P_n|^2 \, dx + 2 \int |P_n|^4 \, dx + \int |Q_n|^2 \, dx \to 0 \quad \text{as } n \to \infty
\]
which together with (3.20) yields that
\[
2 \int \left| \nabla P_n \right|^2 \, dx + 2c^2 \int |P_n|^2 \, dx + \int |Q_n|^2 \, dx \to 0 \quad \text{as } n \to \infty. \tag{3.22}
\]
On the other hand, from \((P_n, Q_n) \in \mathcal{M}\) it follows that
\[
2 \int \left| \nabla P_n \right|^2 \, dx + 2c^2 \int |P_n|^2 \, dx + \int |Q_n|^2 \, dx = -3 \int Q_n |P_n|^2 \, dx. \tag{3.23}
\]
Noting Young inequality
\[
\left| \int Q_n |P_n|^2 \, dx \right| \leq \frac{1}{3} \int |Q_n|^2 \, dx + \frac{3}{4} \int |P_n|^4 \, dx
\]
and (3.23), we get
\[
2 \int \left| \nabla P_n \right|^2 \, dx + 2c^2 \int |P_n|^2 \, dx + \int |Q_n|^2 \, dx \leq \int |Q_n|^2 \, dx + \frac{9}{4} \int |P_n|^4 \, dx,
\]
namely,
\[
2 \int \left| \nabla P_n \right|^2 \, dx + 2c^2 \int |P_n|^2 \, dx \leq \frac{9}{4} \int |P_n|^4 \, dx,
\]
which together with Sobolev’s embedding theorem implies that
\[
2 \int \left| \nabla P_n \right|^2 \, dx + 2c^2 \int |P_n|^2 \, dx \leq \frac{9}{4} \int |P_n|^4 \, dx \leq \frac{9}{4} C \left( \int \left| \nabla P_n \right|^2 \, dx + c^2 \int |P_n|^2 \, dx \right)^2 \leq C \left( 2 \int \left| \nabla P_n \right|^2 \, dx + 2c^2 \int |P_n|^2 \, dx \right)^2. \tag{3.24}
\]
From (3.24), we have
\[
2 \int \left| \nabla P_n \right|^2 \, dx + 2c^2 \int |P_n|^2 \, dx \geq C.
\]
Thus we get
\[
2 \int \left| \nabla P_n \right|^2 \, dx + 2c^2 \int |P_n|^2 \, dx + \int |Q_n|^2 \, dx \geq C,
\]
which is contradictory with (3.22). So \((P_\infty, Q_\infty) \neq (0, 0)\).

Now we take
\[
P = \tilde{P}_\xi = ((P_\infty)_\xi)_\eta, \quad Q = \tilde{Q}_\xi = ((Q_\infty)_\xi)_\eta \tag{3.25}
\]
with

\[
\begin{align*}
\xi &= \lambda_1 \text{ or } \xi \in [\lambda_2, \infty), \quad \text{if (I-1) in Lemma 3.2 holds,} \\
\xi &= \lambda_2 \text{ or } \xi \in (\lambda_2, \infty), \quad \text{if (I-2) in Lemma 3.2 holds,} \\
\xi &= \lambda_2, \quad \text{if (I-3) in Lemma 3.2 holds,} \\
\xi &= \lambda_1, \quad \text{if (I-4) in Lemma 3.2 holds,} \\
\xi \in (0, \infty), \quad \text{if (I-5) in Lemma 3.2 holds,} \\
\xi &= \lambda_2 \text{ or } \xi \in (0, \lambda_2], \quad \text{if (I-6) in Lemma 3.2 holds,} \\
\xi \in (0, \infty), \quad \text{if (I-7) in Lemma 3.2 holds,} \\
\xi &= \lambda_1 \text{ or } \xi \in (0, \lambda_1], \quad \text{if (I-8) in Lemma 3.2 holds,} \\
\xi \in (0, \infty), \quad \text{if (II-1) in Lemma 3.2 holds,} \\
\xi &= \lambda^* \text{ or } \xi \in (0, \lambda^*), \quad \text{if (II-2) in Lemma 3.2 holds,} \\
\xi &= \lambda^*, \quad \text{if (II-3) in Lemma 3.2 holds,} \\
\xi &= \lambda^* \text{ or } \xi \in [\lambda^*, \infty), \quad \text{if (II-4) in Lemma 3.2 holds}
\end{align*}
\]

such that

\[
K(\tilde{P}_\xi, \bar{Q}_\xi) = K\left((P_\infty)_\eta, ((Q_\infty)_\eta)\right) = 0, \quad (3.26)
\]

where

\[
\begin{align*}
\eta \in (0, \lambda_2] \text{ or } \eta = \lambda_2, \quad \text{if (I-1) in Lemma 3.2 holds,} \\
\eta \in (0, \lambda_2] \text{ or } \eta = \lambda_2, \quad \text{if (I-2) in Lemma 3.2 holds,} \\
\eta \in (0, \infty), \quad \text{if (I-3) in Lemma 3.2 holds,} \\
\eta \in (0, \infty), \quad \text{if (I-4) in Lemma 3.2 holds,} \\
\eta = \lambda_1, \quad \text{if (I-5) in Lemma 3.2 holds,} \\
\eta \in (\lambda_2, \infty) \text{ or } \eta = \lambda_1, \quad \text{if (I-6) in Lemma 3.2 holds,} \\
\eta = \lambda_2, \quad \text{if (I-7) in Lemma 3.2 holds,} \\
\eta \in [\lambda_1, \infty) \text{ or } \eta = \lambda_1, \quad \text{if (I-8) in Lemma 3.2 holds,} \\
\eta = \lambda^*, \quad \text{if (II-1) in Lemma 3.2 holds,} \\
\eta \in [\lambda^*, \infty) \text{ or } \eta = \lambda^*, \quad \text{if (II-2) in Lemma 3.2 holds,} \\
\eta \in (0, \infty), \quad \text{if (II-3) in Lemma 3.2 holds,} \\
\eta \in (0, \lambda^*) \text{ or } \eta = \lambda^*, \quad \text{if (II-4) in Lemma 3.2 holds}
\end{align*}
\]

By (3.17), (3.18) and (3.19), thus one gets

\[
\begin{align*}
\left( (P_n)_\xi \right)_\eta \rightarrow \tilde{P}_\xi & \quad \text{strongly in } L^4(\mathbb{R}^N) \text{ as } n \rightarrow \infty, \\
\left( (P_n)_\xi \right)_\eta \rightarrow \bar{P}_\xi & \quad \text{weakly in } H^1(\mathbb{R}^N) \text{ as } n \rightarrow \infty, \\
\left( (Q_n)_\xi \right)_\eta \rightarrow \bar{Q}_\xi & \quad \text{weakly in } L^2(\mathbb{R}^N) \text{ as } n \rightarrow \infty.
\end{align*} \quad (3.27)
\]

Since \( K((P_n)_\xi, (Q_n)_\xi) = 0 \), Lemma 3.2 shows that

\[
J[( (P_n)_\xi )_\eta, ( (Q_n)_\xi )_\eta ] \leq J[( P_n)_\xi, (Q_n)_\xi]. \quad (3.28)
\]

Hence, using (3.26) and (3.27) one has
As \((\tilde{P}_\xi, \tilde{Q}_\xi) \neq (0, 0)\) and \(K(\tilde{P}_\xi, \tilde{Q}_\xi) = 0\), one has \((\tilde{P}_\xi, \tilde{Q}_\xi) \in \mathcal{M}\). Therefore, from (3.25) and (3.29), \((P, Q) = (\tilde{P}_\xi, \tilde{Q}_\xi)\) solves the minimization problem

\[
J(P, Q) = \min_{(\phi, \psi) \in \mathcal{M}} J(\phi, \psi). \tag{3.30}
\]

From Lemma 3.1 and Theorem 3.1, we get an important conclusion—Proposition 3.1 which is key throughout this paper.

**Proposition 3.1.** \(d_{\mathcal{M}} > 0\).

At the end of this section, we get a conclusion on the existence of a standing wave for the Klein–Gordon–Zakharov system (1.1)–(1.3), which will be helpful to prove the modified instability of the standing wave for the system (1.1)–(1.3) in Section 5.

**Theorem 3.2.** Assume \((P, Q)\) is a solution of the problem (3.30). Then \(Q = -|P|^2\) and \((P, Q)\) is a ground state solution of the following system:

\[
\begin{cases}
-\Delta P + c^2 P = -P Q - |P|^2 P, & x \in \mathbb{R}^N, \\
Q + |P|^2 = 0, & x \in \mathbb{R}^N.
\end{cases} \tag{3.31}
\]

Moreover, \((u(t, x), n(t, x)) = (P(x), Q(x))\) is a standing wave solution for the system (1.1)–(1.3).

**Proof.** Since \((P, Q)\) is a solution of the problem (3.30), there exists a Lagrange multiplier \(\Lambda\) such that

\[
\delta_P [J + \Lambda K] = 0, \quad \delta_Q [J + \Lambda K] = 0, \tag{3.32}
\]

where \(\delta_P G\) denotes the variation of \(G(\phi, \psi)\) about \(\phi\). By the formula

\[
\delta_P G(\phi, \psi) = \frac{\partial}{\partial \eta} G(\phi + \eta \delta \phi, \psi)|_{\eta=0},
\]

we get

\[
\delta_P [J + \Lambda K] = (2 + 4 \Lambda) \int (-\Delta \phi \cdot \delta \phi + c^2 \phi \delta \phi) \, dx \\
+ (2 + 8 \Lambda) \int (|\phi|^2 \phi \delta \phi) \, dx + (2 + 6 \Lambda) \int \phi \psi \delta \phi \, dx, \tag{3.33}
\]

\[
\delta_Q [J + \Lambda K] = (1 + 2 \Lambda) \int \psi \delta \psi \, dx + (1 + 3 \Lambda) \int |\phi|^2 \delta \psi \, dx, \tag{3.34}
\]

where \(\delta \phi\) denotes the variation of \(\phi\). By (3.32) one has

\[
(2 + 4 \Lambda) \int (|\nabla P|^2 + c^2 |P|^2) \, dx + (2 + 8 \Lambda) \int |P|^4 \, dx + (2 + 6 \Lambda) \int |P|^2 Q \, dx = 0, \tag{3.35}
\]

\[
(1 + 2 \Lambda) \int |Q|^2 \, dx + (1 + 3 \Lambda) \int |P|^2 Q \, dx = 0. \tag{3.36}
\]
From $K(P, Q) = 0$ it follows that
\[
2 \int |\nabla P|^2 \, dx + 2c^2 \int |P|^2 \, dx + 2 \int |P|^4 \, dx + \int |Q|^2 \, dx + 3 \int Q|P|^2 \, dx = 0. \tag{3.37}
\]
By (3.35), (3.36) and (3.37), one gets $\Lambda = 0$.

Thus (3.35) and (3.36) yield that
\[
2 \int (\nabla P \nabla \bar{P}) \, dx + 2c^2 \int P \bar{P} \, dx + 2 \int |P|^2 P \, dx + 2 \int P \bar{P} Q \, dx = 0, \tag{3.38}
\]
\[
\int Q^2 \, dx + \int |P|^2 Q \, dx = 0. \tag{3.39}
\]
Thus (3.38) and (3.39) conclude that
\[
\begin{cases}
-\Delta P + c^2 P = -PQ - |P|^2 P, & x \in \mathbb{R}^N, \\
Q + |P|^2 = 0, & x \in \mathbb{R}^N,
\end{cases}
\]
that is,
\[
\begin{cases}
-\Delta P + c^2 P = 0, & x \in \mathbb{R}^N, \\
Q = -|P|^2, & x \in \mathbb{R}^N.
\end{cases}
\]
Putting $(u(t, x), n(t, x)) = (P(x), Q(x))$ into the system (1.1)–(1.3), we can obtain easily that $(P, Q)$ is a standing wave solution for the system (1.1)–(1.3).

This completes the proof of Theorem 3.2. \qed

4. Blowup and global existence

In this section, we discuss global existence and blowup for the solution to the Cauchy problem (1.1)–(1.4). Firstly, we give a sharp threshold of global existence and blowup for the solution $(u(t, x), n(t, x), V(t, x))$ to the Cauchy problem (1.1)–(1.4) in terms of the relationship between the initial energy $E(u_0, u_1, n_0, v_0)$ and $d_M$, namely

**Theorem 4.1.** Let $(u_0, u_1, n_0, v_0) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ and satisfy
\[
E(u_0, u_1, n_0, v_0) < d_M. \tag{4.1}
\]
Then:

1. If
\[
K(u_0, n_0) < 0, \tag{4.2}
\]
then the solution $(u(t, x), n(t, x), V(t, x))$ of the Cauchy problem (1.1)–(1.4) blows up in finite time, that is, there exists $T > 0$ such that
\[
\lim_{t \to T} \left( \|u\|^2_{L^2(\mathbb{R}^N)} + \|n\|^2_{H^{-1}(\mathbb{R}^N)} \right) = \infty.
\]
2. If
\[ K(u_0, n_0) > 0, \]  
then the solution \((u(t, x), n(t, x), V(t, x))\) of the Cauchy problem (1.1)–(1.4) exists globally on \(t \in [0, \infty)\). Moreover, \((u(t, x), n(t, x), V(t, x))\) satisfies
\[ c^{-2}\|u_t\|_{L^2}^2 + \frac{1}{2}\alpha^{-2}\|V\|_{L^2}^2 + \|u\|_{H_0^1}^2 + \frac{1}{2}\|n\|_{L^2}^2 < d_M, \]  
or
\[ c^{-2}\|u_1\|_{L^2}^2 + \frac{1}{2}\alpha^{-2}\|V\|_{L^2}^2 + \frac{1}{3}\|u\|_{H_0^1}^2 + \frac{1}{6}\|n\|_{L^2}^2 < 2d_M + \frac{1}{6}c^*d_M^2. \]  
Here and hereafter, for simplicity, we denote \(\|u\|_{L^p}^p = \int |u|^p \, dx\) and \(c^*\) is the positive constant which satisfies the Sobolev’s inequality
\[ \|u\|_{L^2}^4 \leq C\|u\|_{H_0^1}^4. \]

Remark 4.1. Unlike the Klein–Gordon–Zakharov system (1.5)–(1.6), for the Klein–Gordon–Zakharov system (1.1)–(1.3), we know that the initial energy \(\mathcal{E}(u_0, u_1, n_0, v_0)\) is always nonnegative from the inequality
\[ \int n_0|u_0|^2 \, dx \leq \frac{1}{2} \int |n_0|^2 \, dx + \int |u_0|^4 \, dx. \]
But \(d_M > 0\) (Proposition 3.1) can also guarantee the set
\[ \{(u_0, u_1, n_0, v_0) \in H^1(R^N) \times L^2(R^N) \times L^2(R^N) \times L^2(R^N) : \mathcal{E}(u_0, u_1, n_0, v_0) < d_M\} \]
is not empty.

From Theorem 4.1, we can obtain two corollaries based on the relationship between \(\int n_0|u_0|^2 \, dx\) and 0.
When \(\int n_0|u_0|^2 \, dx < 0\), we get

Corollary 4.1 (Small data criterion I). If \((u_0, u_1, n_0, v_0) \in H^1(R^N) \times L^2(R^N) \times L^2(R^N) \times L^2(R^N)\) and satisfies
\[ \|u_0\|_{H_0^1}^2 + \frac{1}{2}c^*\|u_0\|_{H_0^1}^4 + \frac{1}{2}\|n_0\|_{L^2}^2 + \frac{1}{2}\alpha^{-2}\|v_0\|_{L^2}^2 + c^{-2}\|u_1\|_{L^2}^2 < d_M, \]  
and
\[ \int n_0|u_0|^2 \, dx < 0, \]  
then the solution \((u(t, x), n(t, x), V(t, x))\) of the Cauchy problem (1.1)–(1.4) exists globally. Here, \(c^*\) is the same as that appeared in Theorem 4.1.
When \( \int n_0|u_0|^2 \, dx > 0 \), we get

**Corollary 4.2 (Small data criterion II).** If \((u_0, u_1, n_0, v_0) \in H^1(R^N) \times L^2(R^N) \times L^2(R^N) \times L^2(R^N)\) and satisfies

\[
\|u_0\|_{H^1}^2 + c^*\|u_0\|_{H^1}^4 + n_0\|\gamma\|_{L^2}^2 + \frac{1}{2}\|v_0\|_{L^2}^2 + c^*\|u_1\|_{L^2}^2 < d_M.
\]

then the solution \((u(t, x), n(t, x), V(t, x))\) of the Cauchy problem (1.1)–(1.4) exists globally. Here, \(c^*\) is the same as that appeared in Theorem 4.1.

**Remark 4.2.** In fact, on the one hand, Corollaries 4.1 and 4.2 give two sufficient conditions of global existence for the solution to the Cauchy problem (1.1)–(1.4). On the other hand, the two corollaries answer the question: how small are the initial data \((u_0, u_1, n_0, v_0)\), the solution to the Cauchy problem (1.1)–(1.4) exists globally?

**Remark 4.3.** For the Klein–Gordon–Zakharov system (1.1)–(1.3) with the absence of the high-degree nonlinearity \(-|u|^2u\), one can obtain the same result as Theorem 4.1 and using the result, one can also get that when the initial energy \(\mathcal{E}(u_0, u_1, n_0, v_0) < 0\), the solution of the Cauchy problem (1.1)–(1.4) blows up in finite time. The similar result cannot be obtained in the present paper on account of the presence of the high-degree nonlinearity \(-|u|^2u\) such that the initial energy \(\mathcal{E}(u_0, u_1, n_0, v_0)\) is always nonnegative from Remark 4.1. But from Corollaries 4.1 and 4.2, we get two results on the global existence with small data of the Cauchy problem for the Klein–Gordon–Zakharov system (1.1)–(1.3). This kind of results here are more meticulous than those appeared in the Klein–Gordon–Zakharov system without the high-degree nonlinearity \(-|u|^2u\).

Before we prove Theorem 4.1, Corollaries 4.1 and 4.2, we first give some preliminaries. Rewrite the energy functional \(\mathcal{E}\) as

\[
\mathcal{E}(\phi_1, \phi_2, \psi_1, \psi_2) = c^* \int |\phi_2|^2 \, dx + \frac{1}{2} \alpha^2 \int |\psi_2|^2 \, dx + J(\phi_1, \psi_1).
\]

Now we define a set \(S\) as

\[
S := \{(\phi_1, \phi_2, \psi_1, \psi_2) \in H^1(R^N) \times L^2(R^N) \times L^2(R^N) \times L^2(R^N) : \mathcal{E}(\phi_1, \phi_2, \psi_1, \psi_2) < d_M\}.
\]

Furthermore, we introduce two invariant sets for the Cauchy problem (1.1)–(1.4) as

\[
S_1 := \{(\phi_1, \phi_2, \psi_1, \psi_2) \in S \mid K(\phi_1, \psi_1) > 0\} \cup \{(0, \phi_2, 0, \psi_2) \in S\},
\]

\[
S_2 := \{(\phi_1, \phi_2, \psi_1, \psi_2) \in S \mid K(\phi_1, \psi_1) < 0\}.
\]

The fundamental property of the two sets \(S_1\) and \(S_2\) is the following.

**Proposition 4.1.** \(S_1\) and \(S_2\) are invariant regions under the solution flow generated by the Cauchy problem (1.1)–(1.4).
Proof. First we show $S_1$ is an invariant region. Let $(u_0, u_1, n_0, v_0) \in S_1$ and assume that there exists a $t^*$ such that $(u(t^*), u_t(t^*), n(t^*), V(t^*)) \notin S_1$. Then $(u(t^*), n(t^*)) \neq (0, 0)$ and $K(u(t^*), n(t^*)) \leq 0$, i.e., $(u(t^*), n(t^*)) \in M^-$. Let

$$s = \inf \{ 0 \leq t \leq t^* \mid (u(t), u_t(t), n(t), V(t)) \notin S_1 \},$$

(4.11)

then $K(u(t), n(t)) \geq 0$ for $0 \leq t < s$. Let $\{s_k\}$ be the minimizing sequence for problem (4.11), then $(u(s_k), n(s_k)) \in M^-$ and by weak lower semi-continuity of $K(u(\cdot), n(\cdot))$, we have

$$K(u(s), n(s)) \leq \lim_{k \to \infty} K(u(s_k), n(s_k)) \leq 0,$$

(4.12)

$$\{u(s), n(s)\} \neq (0, 0).$$

On the other hand, from (1.8), (1.9) and (4.10) it follows that

$$I(u(s), n(s)) = \lim_{t \to s^-} \inf I(u(t), n(t))$$

$$\leq \lim_{t \to s^-} \inf \left( I(u(t), n(t)) + \frac{1}{v + 1} K(u(t), n(t)) \right)$$

$$= \lim_{t \to s^-} \inf J(u(t), n(t))$$

$$\leq \lim_{t \to s^-} \inf \mathcal{E}(u(t), u_t(t), n(t), V(t))$$

$$< d_M,$$

which contradicts (4.12) from Proposition 2.1 and (2.7). Thus $S_1$ is invariant.

Next we show $S_2$ is also an invariant region.

Let $(u_0, u_1, n_0, v_0) \in S_2$ and assume that there exists a $\tau$ such that $(u(\tau), u_t(\tau), n(\tau), V(\tau)) \notin S_2$, from (1.8), (1.9), (2.5) and (4.10) it follows that $I(u(\tau), n(\tau)) \leq d_M$. Let

$$s = \inf \{ 0 \leq t \leq \tau \mid (u(t), u_t(t), n(t), V(t)) \notin S_2 \},$$

(4.13)

then $I(u(s), n(s)) \leq d_M$ and $I(u(t), n(t)) > d_M$ for all $0 < t < s$ from Proposition 2.1.

On the other hand, from (2.5) we have

$$K(u(s), n(s)) = \lim_{t \to s^-} \inf (v + 1) \left( J(u(t), n(t)) - I(u(t), n(t)) \right)$$

$$\leq \lim_{t \to s^-} \inf (v + 1) \left( \mathcal{E}(u(t), u_t(t), n(t), V(t)) - d_M \right)$$

$$\leq (v + 1) \left( \mathcal{E}(u_0, u_1, n_0, v_0) - d_M \right)$$

$$< 0,$$

which contradicts $I(u(s), n(s)) \leq d_M$ from Proposition 2.1 (if $K(\phi, \psi) < 0$, then $I(\phi, \psi) > d_M$). Therefore $S_2$ is also invariant. □

Now we begin to prove Theorem 4.1, Corollaries 4.1 and 4.2 by utilizing Payne and Sattinger’s potential well argument [15] and Levine’s concavity method [8] as well as introducing suitable dilation transformation.

First we prove Theorem 4.1.
Proof of Theorem 4.1. From (1.9), (2.1) and (4.1), it follows that

\[
J(u_0, n_0) \leq E(u_0, u_1, n_0, v_0) < d_M.
\]  

(4.14)

We first prove 1 of Theorem 4.1.

1. (4.2), (4.14) and Proposition 4.1 imply that \((u(t, x), u_t(t, x), n(t, x), V(t, x)) \in S_2\). Thus from (4.10) we get

\[
K(u(t, x), n(t, x)) < 0 \quad \text{for } t \in [0, T)
\]

and

\[
J(u(t, x), n(t, x)) < d_M.
\]

(4.15)

(4.16)

Since \((u(t, x), n(t, x), V(t, x))\) is a solution of the Cauchy problem (1.1)–(1.4) on \([0, T)\), by assumption (A1), we put

\[
F_{(c, \alpha)}(t) = 2c^{-2} \int |u|^2 dx + \alpha^{-2} \int |g|^2 dx.
\]

(4.17)

Thus we get

\[
F'_{(c, \alpha)}(t) = \int \left[ 2c^{-2}(u_t u^* + uu_t^*) + 2\alpha^{-2} gg_t^* \right] dx,
\]

(4.18)

where \(u^*\) is the complex conjugate of \(u\). By computing \(F''_{(c, \alpha)}(t)\) we obtain

\[
F''_{(c, \alpha)}(t) = 4c^{-2} \int |u_t|^2 dx + 4c^{-2} \text{Re} \int u_t u^* dx + 2\alpha^{-2} \int |g_t|^2 dx + 2\alpha^{-2} \int gg_t dx
\]

\[
= 4c^{-2} \int |u_t|^2 dx + 2\alpha^{-2} \int |\nabla|^2 dx - 4c^2 \int |u|^2 dx - 4 \int |\nabla u|^2 dx
\]

\[
- 4 \int |u|^4 dx - 4 \int n|u|^2 dx + 2 \int g(-\nabla n - \nabla |u|^2) dx
\]

\[
= 4c^{-2} \int |u_t|^2 dx + 2\alpha^{-2} \int |\nabla|^2 dx - 4c^2 \int |u|^2 dx - 4 \int |\nabla u|^2 dx
\]

\[
- 4 \int |u|^4 dx - 4 \int n|u|^2 dx + 2 \int \nabla \cdot g(n + |u|^2) dx
\]

\[
= 4c^{-2} \int |u_t|^2 dx + 2\alpha^{-2} \int |\nabla|^2 dx - 4c^2 \int |u|^2 dx - 4 \int |\nabla u|^2 dx
\]

\[
- 4 \int |u|^4 dx - 4 \int |u|^2 dx + 4 \int |u|^2 dx - 2 \int n|u|^2 dx
\]

\[
= 4c^{-2} \int |u_t|^2 dx + 2\alpha^{-2} \int |\nabla|^2 dx - 4c^2 \int |u|^2 dx - 4 \int |\nabla u|^2 dx
\]

\[
- 4 \int |u|^4 dx + 2 \int |u|^2 dx - 6 \int n|u|^2 dx
\]

\[
= 2 \left( 2c^{-2} \int |u_t|^2 dx + \alpha^{-2} \int |\nabla|^2 dx \right) - 2K(u, n).
\]

(4.19)
In view of (4.15), one has
\[ F''_{(c, \alpha)}(t) > 0 \quad \text{for} \ t \in [0, T). \] (4.20)

By (1.8)–(1.10), (4.17) and (4.19), one gets
\[
F''_{(c, \alpha)}(t) = 5 \left( 2c^{-2} \int |u_t|^2 \, dx + \alpha^{-2} \int |V|^2 \, dx \right) + 2c^2 \int |u|^2 \, dx + 2 \int |\nabla u|^2 \, dx \\
+ \int |n|^2 \, dx - \int |u|^4 \, dx - 6 \epsilon (u_0, u_1, n_0, v_0).
\] (4.21)

Because \( K(u, n) < 0 \), we have
\[
2c^2 \int |u|^2 \, dx + 2 \int |\nabla u|^2 \, dx + 2 \int |u|^4 \, dx + \int |n|^2 \, dx < -3 \int n|u|^2 \, dx \\
\leq \frac{3}{2} \int |u|^4 \, dx + \frac{3}{2} \int |n|^2 \, dx,
\]
which implies that
\[
0 < 2c^2 \int |u|^2 \, dx + 2 \int |\nabla u|^2 \, dx < \frac{1}{2} \int |n|^2 \, dx - \frac{1}{2} \int |u|^4 \, dx. \] (4.22)

Since \( F_{(c, \alpha)}(t) \) is a convex function of \( t \) by (4.20), it follows that if there exists a time \( t_1 \) such that \( F_{(c, \alpha)}(t) > 0 \), then \( F_{(c, \alpha)}(t) \) is increasing for all \( t > t_1 \) (within the interval of existence). Now we give a remark before we continue to prove 1 of Theorem 4.1.

**Remark 4.4.** If \( F_{(c, \alpha)}(t) \) is increasing for all \( t > t_1 \), then we can conclude that \( \int |u|^2 \, dx \) is increasing for all \( t > t_1 \). In fact, let
\[ G_c(t) = c^{-2} \int |u|^2 \, dx. \]

We get
\[
G'_c(t) = c^{-2} \int (u_t u^* + uu^*_t) \, dx, \\
G''_c(t) = 2c^2 \int |u_t|^2 \, dx - 2c^2 \int |u|^2 \, dx - 2 \int |\nabla u|^2 \, dx - 2 \int |u|^4 \, dx - 2 \int n|u|^2 \, dx \\
= 2c^2 \int |u_t|^2 \, dx - K(u, n) + \int n|u|^2 \, dx + \int |n|^2 \, dx \\
= 2c^2 \int |u_t|^2 \, dx - K(u, n) - \left( - \int n|u|^2 \, dx \right) + \int |n|^2 \, dx \\
\geq 2c^2 \int |u_t|^2 \, dx - K(u, n) + \int |n|^2 \, dx - \left( \frac{1}{2} \int |n|^2 \, dx + \frac{1}{2} \int |u|^4 \, dx \right) \\
= 2c^2 \int |u_t|^2 \, dx - K(u, n) + \frac{1}{2} \int |n|^2 \, dx - \frac{1}{2} \int |u|^4 \, dx.
By (4.22) and $K(u, n) < 0$, the above inequality implies $G'_c(t) > 0$. So $G'_c(t)$ is strictly increasing for all $t > 0$. Thus if we choose $(u_0, u_1)$ properly such that

$$G'_c(0) = c^{-2} \int (u_1u_0 + u_0u_1') dx \geq 0,$$

then for all $t > 0$, $G'_c(t) > 0$. Therefore, $G_c(t) = \int |u|^2 dx$ is strictly increasing for all $t > 0$. Without loss of generality, for simplicity, we omit the condition (R1) in the present paper and assume that if $F_{(c, \alpha)}(t)$ is increasing for all $t > t_1$, then $\int |u|^2 dx$ is increasing for all $t > t_1$.

Now we continue to our proof. From Remark 4.4 and (4.22), we get the quantity

$$2c^2 \int |u|^2 dx + 2 \int |\nabla u|^2 dx + \int |n|^2 dx - \int |u|^4 dx - 6\epsilon(u_0, u_1, n_0, v_0)$$

will eventually become positive and will remain positive thereafter. Thus form (4.21) and (4.22), we would have

$$F''_{(c, \alpha)}(t) \geq 5 \left(2c^{-2} \int |u_1|^2 dx + \alpha^{-2} \int |V|^2 dx \right).$$

In view of (4.17), (4.18) and (4.23), using Hölder’s inequality, we get

$$F_{(c, \alpha)}(t)F''_{(c, \alpha)}(t) \geq 5 \left(2c^{-2} \int |u|^2 dx + \alpha^{-2} \int |g|^2 dx \right) \left(2c^{-2} \int |u_1|^2 dx + \alpha^{-2} \int |V|^2 dx \right)
\geq 5 \left(2c^{-2} \int |u|^2 dx + \alpha^{-2} \int |g|^2 dx \right) \left(2c^{-2} \int |u_1|^2 dx + \alpha^{-2} \int |V|^2 dx \right)
\geq 5 \left(4c^{-2} \int |u_1|^2 dx \right)^2 + 2c^{-2} \alpha^{-2} \left(\int |uV| dx \right)^2
\geq 5 \left(4c^{-2} \int |u_1|^2 dx \right)^2 + 2c^{-2} \alpha^{-2} \left(\int |uV| dx \right)^2
\geq \frac{5}{4} \left(F_{(c, \alpha)}(t)\right)^2.
(4.24)$$

Since

$$(F_{(c, \alpha)}(t))'' = -\frac{1}{4} F_{(c, \alpha)}(t) F''_{(c, \alpha)}(t) F''_{(c, \alpha)}(t) - \frac{5}{4} (F'_{(c, \alpha)}(t))^2,$$

from (4.24) we get

$$(F_{(c, \alpha)}(t))'' \leq 0.
(4.26)$$

Therefore $F_{(c, \alpha)}(t)$ is concave for sufficiently large $t$ and there exists a finite time $T^*$ such that

$$\lim_{t \to T^*} F_{(c, \alpha)}(t) = 0.$$
In other words,
\[
\lim_{t \to T^*} F(c, \alpha)(t) = \infty.
\]

Thus one has \( T < \infty \) and
\[
\lim_{t \to T^*} \left( 2c^{-2} \int |u|^2 \, dx + \alpha^{-2} \int |g|^2 \, dx \right) = \infty.
\]

From assumption (A1), (1.1)–(1.3) and (1.10), we get
\[
\lim_{t \to T^*} \left( 2c^{-2} \int |u|^2 \, dx + \int \left| \alpha \nabla \right|^{-1} n \right) = \infty.
\]

The proof of 1 of Theorem 4.1 will be completed once we have shown that \( F'(c, \alpha)(t) > 0 \) for some \( t \).

We prove it by contradiction. Suppose
\[
F'(c, \alpha)(t) \leq 0 \quad \text{for all } t. \tag{4.27}
\]

Since \( F(c, \alpha)(t) \) is convex from (4.20), \( F(c, \alpha)(t) \) must tend to a finite, nonnegative limit \( L \) as \( t \to \infty \).

By Proposition 4.1, we assert \( L > 0 \). Therefore one has, as \( t \to \infty \), \( F(c, \alpha)(t) \to L > 0 \), \( F'(c, \alpha)(t) \to 0 \) and \( F''(c, \alpha)(t) \to 0 \). Thus (4.19) and \( K(u, n) < 0 \) yield
\[
\lim_{t \to \infty} 2 \left( 2c^{-2} \int |u|^2 \, dx + \alpha^{-2} \int |V|^2 \, dx \right) = 0 \tag{4.28}
\]

and
\[
\lim_{t \to \infty} K(u, n) = 0. \tag{4.29}
\]

Now for any fixed \( t > 0 \), because of \( K(u, n) < 0 \), there exists a \( 0 < \lambda < 1 \) such that \( K(\lambda^{\frac{1}{2}} u, \lambda n) = 0 \) and \( (\lambda^{\frac{1}{2}} u, \lambda n) \neq (0, 0) \). Therefore by (2.2) we obtain
\[
-3 \lambda^{2} \int n|u|^2 \, dx = \lambda \left( c^2 \int |u|^2 \, dx + \int |\nabla u|^2 \, dx \right) + \lambda^2 \int |u|^4 \, dx + \frac{1}{2} \lambda^2 \int |n|^2 \, dx \tag{4.30}
\]

and
\[
\int n|u|^2 \, dx < 0. \tag{4.31}
\]

Thus by (2.4), we get
\[
J(\lambda^{\frac{1}{2}} u, \lambda n) \geq d_M. \tag{4.32}
\]

From (4.30) and (4.31) it follows that
\[ J(\lambda^{1/2}u, \lambda n) = \lambda \left( c^2 \int |u|^2 \, dx + \int |\nabla u|^2 \, dx \right) + \frac{1}{2} \lambda^2 \int |u|^4 \, dx \]
\[ + \frac{1}{2} \lambda^2 \int |n|^2 \, dx + \frac{\lambda^2}{2} \int n|u|^2 \, dx \]
\[ = -\frac{3}{2} \lambda^2 \int n|u|^2 \, dx - \frac{1}{2} \lambda^2 \int |u|^4 \, dx + \frac{\lambda^2}{2} \int n|u|^2 \, dx \]
\[ = -\frac{1}{2} \lambda^2 \int n|u|^2 \, dx - \frac{1}{2} \lambda^2 \int |u|^4 \, dx. \tag{4.33} \]

So
\[ J(u, n) - J(\lambda^{1/2}u, \lambda n) = c^2 \int |u|^2 \, dx + \int |\nabla u|^2 \, dx \]
\[ + \frac{1}{2} \int |u|^4 \, dx + \frac{1}{2} \int |n|^2 \, dx + \int n|u|^2 \, dx \]
\[ + \frac{1}{2} \lambda^2 \int n|u|^2 \, dx + \frac{1}{2} \lambda^2 \int |u|^4 \, dx \]
\[ = c^2 \int |u|^2 \, dx + \int |\nabla u|^2 \, dx + \int |u|^4 \, dx \]
\[ + \frac{1}{2} \int |n|^2 \, dx + \int n|u|^2 \, dx - \frac{1}{2} \int |u|^4 \, dx \]
\[ + \frac{1}{2} \lambda^2 \int n|u|^2 \, dx + \frac{1}{2} \lambda^2 \int |u|^4 \, dx. \tag{4.34} \]

On the other hand, from (4.29) and (3.5) we get as \( t \to \infty \),
\[ \frac{1}{2} \int |u|^4 \, dx < \frac{1}{2} \int |n|^2 \, dx \tag{4.35} \]
which together with (2.2) and (4.29) yields
\[ \int |u|^4 \, dx \leq - \int n|u|^2 \, dx \leq \frac{1}{2} \int |u|^4 \, dx + \frac{1}{2} \int |n|^2 \, dx. \tag{4.36} \]
By (4.35) and (4.36) we get for \( 0 < \lambda < 1 \),
\[ \frac{1}{2} (\lambda^2 - 1) \int |u|^4 \, dx \geq \frac{1}{2} (\lambda^2 - 1) \left( - \int n|u|^2 \, dx \right). \]
So
\[ \frac{1}{2} \lambda^2 \int |u|^4 \, dx - \frac{1}{2} \int |u|^4 \, dx + \frac{1}{2} \lambda^2 \int n|u|^2 \, dx \geq \frac{1}{2} \int n|u|^2 \, dx. \tag{4.37} \]
Thus from (4.29), (4.34) and (4.37) we obtain as \( t \to \infty \),
\[ J(u, n) - J(\lambda^{1/2}u, \lambda n) \geq c^2 \int |u|^2 \, dx + \int |\nabla u|^2 \, dx + \int |u|^4 \, dx \]
\[ + \frac{1}{2} \int |n|^2 \, dx + \int n|u|^2 \, dx + \frac{1}{2} \int n|u|^2 \, dx \]
\[
= c^2 \int |u|^2 \, dx + \int |\nabla u|^2 \, dx + \int |u|^4 \, dx
\]
\[
+ \frac{1}{2} \int |n|^2 \, dx + \frac{3}{2} \int n|u|^2 \, dx
\]
\[
= \frac{1}{2} K(u, n).
\] & (4.38)

All in all, by (4.29), (4.32) and (4.38), we may conclude that as \( t \to \infty \),

\[
J(u, n) \geq J(\lambda^2 u, \lambda n) \geq d_M,
\] & (4.39)

which contradicts \( J(u, n) < d_M \) from (4.16). So the supposition (4.27) is not true. Thus \( F'(c, \alpha)(t) > 0 \) for some \( t \).

So far, we complete the proof of 1 of Theorem 4.1. Next we prove 2.

2. (4.3), (4.14) and Proposition 4.1 imply that \( (u, ut, n, V) \in S_1 \), namely,

(1) \( (u, n) = (0, 0), \mathcal{E}(0, ut, 0, V) < d_M \), or

(2) \( K(u, n) > 0, \mathcal{E}(u, ut, n, V) < d_M \) for \( t \in [0, T) \).

If (1) holds, the result is evidently true.

If (2) holds, then the following two cases hold:

(i) \( \int n|u|^2 \, dx \geq 0 \), or

(ii) \( \int n|u|^2 \, dx < 0 \) and

\[
\int n|u|^2 \, dx > -\frac{2}{3}c^2 \int |u|^2 \, dx - \frac{2}{3} \int |\nabla u|^2 \, dx - \frac{1}{3} \int n|^2 \, dx - \frac{2}{3} \int n|u|^4 \, dx.
\]

For the case (i) \( \int n|u|^2 \, dx \geq 0 \), from (1.8), (1.9) and (4.14) we obtain

\[
c^{-2} \int |u|^2 \, dx + \frac{1}{2} \alpha^{-2} \int |V|^2 \, dx + c^2 \int |u|^2 \, dx
\]
\[
+ \int |\nabla u|^2 \, dx + \frac{1}{2} \int |u|^4 \, dx + \frac{1}{2} \int n|^2 \, dx
\]
\[
\leq \mathcal{E}(u_0, u_1, n_0, v_0) < d_M.
\] & (4.40)

Thus we established the boundedness of \( u_t \) in \( L^2(\mathbb{R}^N) \), \( V \) in \( L^2(\mathbb{R}^N) \), \( u \) in \( H_1^1(\mathbb{R}^N) \) and \( n \) in \( L^2(\mathbb{R}^N) \) for \( t \in [0, T) \). Hence it must be \( T = \infty \). Then the solution \( (u, n, V) \) of the Cauchy problem (1.1)–(1.4) exists globally on \( t \in [0, \infty) \). Furthermore, (4.40) implies the estimate (4.4).

For the case (ii) \( \int n|u|^2 \, dx < 0 \) and

\[
\int n|u|^2 \, dx > -\frac{2}{3}c^2 \int |u|^2 \, dx - \frac{2}{3} \int |\nabla u|^2 \, dx
\]
\[
- \frac{1}{3} \int n|^2 \, dx - \frac{2}{3} \int |u|^4 \, dx.
\]
from (1.9), (1.10) and the inequality
\[-\int n|u|^2 \, dx \leq \frac{1}{2} \int |n|^2 \, dx + \frac{1}{2} \int |u|^4 \, dx,
\]
we get
\[
c^{-2} \int |u_t|^2 \, dx + c^2 \int |u|^2 \, dx + \int |\nabla u|^2 \, dx + \frac{1}{2} \int |u|^4 \, dx
+ \frac{1}{2} \int |n|^2 \, dx + \frac{1}{2} \alpha^{-2} \int |V|^2 \, dx - \left( -\int n|u|^2 \, dx \right)
\geq c^{-2} \int |u_t|^2 \, dx + c^2 \int |u|^2 \, dx + \int |\nabla u|^2 \, dx + \frac{1}{2} \alpha^{-2} \int |V|^2 \, dx.
\]
Thus by (1.9) and (4.14), we obtain
\[
c^{-2} \int |u_t|^2 \, dx + c^2 \int |u|^2 \, dx + \int |\nabla u|^2 \, dx + \frac{1}{2} \alpha^{-2} \int |V|^2 \, dx
\leq \mathcal{E}(u, u_t, n, V)
= \mathcal{E}(u_0, u_1, n_0, v_0) < d_M. \quad (4.41)
\]
On the other hand, from \(K(u, n) > 0\) and \(J(u, n) < d_M\), we get
\[
\int n|u|^2 \, dx > -\frac{2}{3} c^2 \int |u|^2 \, dx - \frac{2}{3} \int |\nabla u|^2 \, dx - \frac{1}{3} \int |n|^2 \, dx - \frac{2}{3} \int |u|^4 \, dx, \quad (4.42)
\]
and
\[
c^2 \int |u|^2 \, dx + \int |\nabla u|^2 \, dx + \frac{1}{2} \int |u|^4 \, dx + \frac{1}{2} \int |n|^2 \, dx + \int n|u|^2 \, dx < d_M. \quad (4.43)
\]
Thus from (4.42) and (4.43) it follows that
\[
\frac{1}{3} c^2 \int |u|^2 \, dx + \frac{1}{3} \int |\nabla u|^2 \, dx + \frac{1}{6} \int |n|^2 \, dx - \frac{1}{6} \int |u|^4 \, dx < d_M,
\]
namely,
\[
\frac{1}{3} c^2 \int |u|^2 \, dx + \frac{1}{3} \int |\nabla u|^2 \, dx + \frac{1}{6} \int |n|^2 \, dx < d_M + \frac{1}{6} \int |u|^4 \, dx
\]
which together with Sobolev’s inequality
\[
\int |u|^4 \, dx \leq c^* \left( c^2 \int |u|^2 \, dx + \int |\nabla u|^2 \, dx \right)^2
\]
yields that
\[
\frac{1}{3} c^2 \int |u|^2 \, dx + \frac{1}{3} \int |\nabla u|^2 \, dx + \frac{1}{6} \int |n|^2 \, dx < d_M + \frac{1}{6} c^* \left( c^2 \int |u|^2 \, dx + \int |\nabla u|^2 \, dx \right)^2. \quad (4.44)
\]
By (4.41), we obtain
\[
c^2 \int |u|^2\,dx + \int |\nabla u|^2\,dx < d_M. \tag{4.45}
\]
Thus it follows from (4.44) and (4.45) that
\[
\frac{1}{3} c^2 \int |u|^2\,dx + \frac{1}{3} \int |\nabla u|^2\,dx + \frac{1}{6} \int |n|^2\,dx < d_M + \frac{1}{6} c^* d_M^2. \tag{4.46}
\]
where in (4.44) and (4.46), \( c^* \) is the same as that appeared in Theorem 4.1.

Therefore by (4.41) and (4.46), we get
\[
c^{-2} \int |u_t|^2\,dx + \frac{4}{3} c^2 \int |u|^2\,dx + \frac{4}{3} \int |\nabla u|^2\,dx + \frac{1}{2} \alpha^{-2} \int |V|^2\,dx + \frac{1}{6} \int |n|^2\,dx
< 2d_M + \frac{1}{6} c^* d_M^2. \tag{4.47}
\]
Thus we established the boundedness of \( u_t \) in \( L^2(\mathbb{R}^N) \), \( V \) in \( L^2(\mathbb{R}^N) \), \( u \) in \( H^1_c(\mathbb{R}^N) \) and \( n \) in \( L^2(\mathbb{R}^N) \) for \( t \in [0, T) \). Hence it must be \( T = \infty \). Then the solution \((u, n, V)\) of the Cauchy problem (1.1)–(1.4) exists globally on \( t \in [0, \infty) \). Furthermore, (4.37) implies the estimate (4.5). From the discussions of the case (i) and the case (ii), we complete the proof of 2 of Theorem 4.1.

From the above arguments, the proof of Theorem 4.1 is completed. \( \square \)

Next, we show Corollaries 4.1 and 4.2 by introducing proper dilation transformation and applying the result in Theorem 4.1. First we show Corollary 4.1.

**Proof of Corollary 4.1.** By (1.9), (4.6), (4.7) and Sobolev’s inequality
\[
\int |u|^4\,dx \leq c^* \left( c^2 \int |u|^2\,dx + \int |\nabla u|^2\,dx \right)^2,
\]
we get \((u_0, n_0) \neq (0, 0)\) and
\[
\mathcal{E}(u_0, u_1, n_0, v_0) = c^{-2} \int |u_1|^2\,dx + \frac{1}{2} \alpha^{-2} \int |v_0|^2\,dx + c^2 \int |u_0|^2\,dx + \int |\nabla u_0|^2\,dx
+ \frac{1}{2} \int |u_0|^4\,dx + \frac{1}{2} \int |n_0|^2\,dx + \int n_0 |u_0|^2\,dx
\leq c^{-2} \int |u_1|^2\,dx + \frac{1}{2} \alpha^{-2} \int |v_0|^2\,dx + c^2 \int |u_0|^2\,dx
+ \int |\nabla u_0|^2\,dx + \frac{1}{2} c^* \left( c^2 \int |u_0|^2\,dx + \int |\nabla u_0|^2\,dx \right)^2
+ \frac{1}{2} \int |n_0|^2\,dx + \int n_0 |u_0|^2\,dx
< d_M. \tag{4.48}
\]
Now we show that
\[
K(u_0, n_0) > 0. \tag{4.49}
\]
In the following, we prove (4.49) by contradiction. If (4.49) is not true, one would have
\[ K(u_0, n_0) \leq 0. \]  
(4.50)

Thus there exists \( 0 < \lambda \leq 1 \) such that
\[ K(\lambda^{\frac{1}{2}}u_0, \lambda n_0) = 0 \quad \text{and} \quad (\lambda^{\frac{1}{2}}u_0, \lambda n_0) \neq (0, 0) \]
since \((u_0, n_0) \neq (0, 0)\). So \((\lambda^{\frac{1}{2}}u_0, \lambda n_0) \in M\). By (2.3) and (2.4) we get
\[ J(\lambda^{\frac{1}{2}}u_0, \lambda n_0) \geq d_M. \]  
(4.51)

On the other hand, for \( 0 < \lambda \leq 1 \), \((\lambda^{\frac{1}{2}}u_0, \lambda n_0)\) and \((u_1, v_0)\) still satisfy (4.6) and (4.7). It follows from (4.6), (4.7), Sobolev's inequality and \( 0 < \lambda \leq 1 \) that
\[ J(\lambda^{\frac{1}{2}}u_0, \lambda n_0) = \lambda \left( c^2 \int |u_0|^2 \, dx + \int |\nabla u_0|^2 \, dx \right) + \frac{1}{2} \lambda^2 \int |n_0|^2 \, dx \\
+ \frac{1}{2} \lambda^2 \int |u_0|^4 \, dx + \lambda^2 \int n_0|u_0|^2 \, dx \\
\leq c^2 \int |u_0|^2 \, dx + \int |\nabla u_0|^2 \, dx + \frac{1}{2} \int |n_0|^2 \, dx \\
+ \frac{1}{2} c^* \left( c^2 \int |u_0|^2 \, dx + \int |\nabla u_0|^2 \, dx \right)^2 \\
< d_M, \]  
(4.52)

which is contradictory to (4.51). Thus (4.49) is true. Therefore by 2 of Theorem 4.1, we get the conclusion of Corollary 4.1. \( \square \)

Now we show Corollary 4.2.

**Proof of Corollary 4.2.** From (1.9), (4.8) and (4.9), using Sobolev's inequality
\[ \int |u|^4 \, dx \leq c^* \left( c^2 \int |u|^2 \, dx + \int |\nabla u|^2 \, dx \right)^2, \]
where \( c^* \) is the same as that appeared in Theorem 4.1 and the inequality
\[ \left| \int n|u|^2 \, dx \right| \leq \frac{1}{2} \int |n|^2 \, dx + \frac{1}{2} \int |u|^4 \, dx, \]
we get
\[ \mathcal{E}(u_0, u_1, n_0, v_0) = c^{-2} \int |u_1|^2 \, dx + c^2 \int |u_0|^2 \, dx + \int |\nabla u_0|^2 \, dx + \frac{1}{2} \int |u_0|^4 \, dx \\
+ \frac{1}{2} \int |n_0|^2 \, dx + \frac{1}{2} \alpha^{-2} \int |v_0|^2 \, dx + \int n_0|u_0|^2 \, dx \\
\leq c^{-2} \int |u_1|^2 \, dx + c^2 \int |u_0|^2 \, dx + \int |\nabla u_0|^2 \, dx + \frac{1}{2} \int |n_0|^2 \, dx \]
Moreover, it follows from (4.9) that

$$K(u_0, n_0) = 2c^2 \int |u_0|^2 \, dx + 2 \int |\nabla u_0|^2 \, dx + 2 \int |u_0|^4 \, dx + 3 \int n_0 |u_0|^2 \, dx > 0.$$  

(5.2)

From (4.53) and (4.54), using 2 of Theorem 4.1, we get the conclusion of Corollary 4.2. □

5. Modified instability of standing wave

In this section, from Lemma 3.2 and Theorem 3.2, on the characterization of standing wave of the Klein–Gordon–Zakharov system (1.1)–(1.3) with minimal action, we can get a result on the modified instability of standing wave for the system (1.1)–(1.3).

**Theorem 5.1 (Modified instability of standing wave).** Let $Q = -|P|^2$ and $(P, Q)$ be a ground state solution of (3.31). Put

$$P = \bar{P}, \quad Q = \bar{Q},$$

(5.1)

where $\xi$ is taken by

$$
\begin{cases}
\xi = \lambda_1 \quad \text{or} \quad \xi \in [\lambda_2, \infty), & \text{if (I-1) in Lemma 3.2 holds,} \\
\xi = \lambda_2, & \text{if (I-3) in Lemma 3.2 holds,} \\
\xi = \lambda_1, & \text{if (I-4) in Lemma 3.2 holds,} \\
\xi \in (0, \lambda_1), & \text{if (I-5) in Lemma 3.2 holds,} \\
\xi = \lambda_2 \quad \text{or} \quad \xi \in (0, \lambda_2], & \text{if (I-6) in Lemma 3.2 holds,} \\
\xi \in (\lambda_2, \infty), & \text{if (I-7) in Lemma 3.2 holds,} \\
\xi = \lambda_1 \quad \text{or} \quad \xi \in (0, \lambda_1], & \text{if (I-8) in Lemma 3.2 holds,} \\
\xi \in (\lambda^*, \infty), & \text{if (II-1) in Lemma 3.2 holds,} \\
\xi = \lambda^* \quad \text{or} \quad \xi \in (0, \lambda^*), & \text{if (II-2) in Lemma 3.2 holds,} \\
\xi = \lambda^*, & \text{if (II-3) in Lemma 3.2 holds.}
\end{cases}
$$

Then for any $\varepsilon > 0$, there exists $(u_0, u_1, n_0, v_0) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ such that

$$
\|u_0 - \bar{P}\|_{H^1(\mathbb{R}^N)} < \varepsilon, \quad \|n_0 - \bar{Q}\|_{L^2(\mathbb{R}^N)} < \varepsilon
$$

(5.2)

and the following property holds:
The solution \((u, n, V)\) of the Cauchy problem for the Klein–Gordon–Zakharov system (1.1)–(1.3) corresponding to the initial data

\[
\begin{aligned}
  u(0, x) &= u_0(x), \quad u_t(0, x) = 0, \\
  n(0, x) &= n_0(x), \quad V(0, x) = 0,
\end{aligned}
\]

is defined on \(T \in (0, \infty)\) such that

\[
(u, n, u_t, V) \in C([0, T); \mathcal{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N))
\]

and

\[
limit_{t \to T} \left( c^{-2} \|u\|_{L^2(\mathbb{R}^N)}^2 + \alpha^{-2} \|n\|_{\mathcal{H}^{-1}(\mathbb{R}^N)}^2 \right) = \infty.
\]

**Remark 5.1.** For the Klein–Gordon–Zakharov system with the absence of the high-degree nonlinearity \(-|u|^2 u\), instability of standing wave for it has been studied by Gan and Zhang [4] as well as Ohta and Todorova [12]. But for the Klein–Gordon–Zakharov system with the presence of the high-degree nonlinearity \(-|u|^2 u\) such as the system (1.1)–(1.3), to our knowledge, no results are known on this problem. In the present paper, we get a modified instability result of standing wave for the system (1.1)–(1.3), which depends strongly on Lemma 3.2. In other words, to some extent, the form of the functional \(K(\phi, \psi)\) decides directly the characteristic of standing wave for the nonlinear wave equation or system.

**Proof of Theorem 5.1.** From the initial data (5.3), (1.9), (1.10) and (2.1), it follows that

\[
\mathcal{E}(u_0, u_1, n_0, v_0) = J(u_0, n_0).
\]

Now we take

\[
u_0(x) = \mu \bar{P}(x), \quad n_0(x) = \mu \bar{Q}(x),
\]

with \(\mu > 0\) and

\[
\begin{aligned}
  \mu &\in (\lambda_1, \lambda_2), \quad \text{if (I-1) in Lemma 3.2 holds}, \\
  \mu &\in (\lambda_2, \infty), \quad \text{if (I-3) in Lemma 3.2 holds}, \\
  \mu &\in (\lambda_1, \infty), \quad \text{if (I-4) in Lemma 3.2 holds}, \\
  \mu &\in (\lambda_1, \infty), \quad \text{if (I-5) in Lemma 3.2 holds}, \\
  \mu &\in (\lambda_2, \infty), \quad \text{if (I-6) in Lemma 3.2 holds}, \\
  \mu &\in \lambda_2, \quad \text{if (I-7) in Lemma 3.2 holds}, \\
  \mu &\in (\lambda_1, \infty), \quad \text{if (I-8) in Lemma 3.2 holds}, \\
  \mu &\in \lambda_*, \quad \text{if (II-1) in Lemma 3.2 holds}, \\
  \mu &\in (\lambda_*, \infty) \text{ or } \mu = \lambda^*, \quad \text{if (II-2) in Lemma 3.2 holds}, \\
  \mu &\in (\lambda_*, \infty), \quad \text{if (II-3) in Lemma 3.2 holds}.
\end{aligned}
\]

For any \(\varepsilon > 0\), one can always choose suitable \(\zeta\) and \(\mu\) such that

\[
\begin{aligned}
  \|u_0 - \bar{P}\|_{\mathcal{H}^1(\mathbb{R}^N)} &= |\mu - 1| \|\bar{P}\|_{\mathcal{H}^1(\mathbb{R}^N)} < \varepsilon, \\
  \|n_0 - \bar{Q}\|_{L^2(\mathbb{R}^N)} &= |\mu - 1| \|\bar{Q}\|_{L^2(\mathbb{R}^N)} < \varepsilon.
\end{aligned}
\]

(5.7)
From Lemma 3.2 and (5.6), it follows that

\[
\begin{align*}
K(u_0, n_0) &< K(\zeta \hat{P}, \zeta \hat{Q}) = 0, \\
J(u_0, n_0) &< J(\zeta \hat{P}, \zeta \hat{Q}) = d_M.
\end{align*}
\]  

(5.8)

Again, we note (5.5) and obtain

\[
\mathcal{E}(u_0, u_1, n_0, \nu_0) = J(u_0, n_0) < d_M.
\]  

(5.9)

In view of 1 of Theorem 4.1, we complete the proof of Theorem 5.1. □

Acknowledgments

This work is partially done when the first author visited the IMS of the Chinese University of Hong Kong. She would like to express her thanks for the hospitality of the IMS, especially, she would like to thank professor Zhouping Xin for his help and encouragements. Finally, the authors thank the referees for their useful comments and professor Changxing Miao for his valuable suggestions.

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