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On a Family of Linear Grammars

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INTRODUCTION

The aim of our work is to define a family of linear grammars which we shall call even linear grammars and to show that the languages generated by them have mathematical properties analogous to those of regular events.

In Section I we give the definition and the basic properties of these grammars. In Section II we introduce equivalence relations with a certain symmetry property and a family of devices which are interrelated in an analogous way as right invariant equivalence relations and finite automata. These devices turn out to be more powerful than finite automata. In Section III we shall prove that the languages acceptable by these devices are just the languages generable by even linear grammars. Consequently, the result obtained, which is by no means obvious, is that the even linear grammars are, among others, able to generate every regular event.

SECTION I

Let $\Sigma \equiv \{\sigma\}$ be a finite alphabet which we shall call terminal, and let $T_\Sigma = \{\varphi, \psi \cdots\}$ be the free semigroup with unity λ on Σ.

In the following, given an alphabet a, we shall denote with T_a the free semigroup with unity on a. We put also $\Sigma \cup \lambda = \Omega = \{\omega\}$.

DEFINITION 1. A context free grammar (on the alphabet Σ) is a system $\mathcal{G} \equiv (\Delta, \delta_0, P)$ where:

1. $\Delta \equiv \{\delta\}$ is a finite alphabet (auxiliary alphabet).
2. $\delta_0 \in \Delta$ (initial symbol).
3. P is a finite set of productions of the form:

$$\delta \to x(x \in T_v, v = \Sigma \cup \Delta, x \neq \delta).$$

A string $y \in T_v$ directly generates $z \in T_v(y \Rightarrow z)$ if there are $u \in T_v$,

$v \in T_v$ such that $y = u\delta v$, $x = uwv$ and $(\delta \rightarrow w) \in P$; y generates $z(y \overset{*}{\Rightarrow} z)$ if there exists a sequence of strings $z_0, \cdots z_r (z_i \in T_v)$ such that $y = z_0$, $z = z_r$, $z_i \Rightarrow z_{i+1} (i = 0, \cdots (r-1))$.

DEFINITION 2. The language generated by a context free grammar $\mathcal{G} \equiv (\Delta, \delta_0, P)$, is the set $\mathcal{L}(\mathcal{G}) = \{\varphi \in T_\Sigma \mid \delta_0 \overset{*}{\Rightarrow} \varphi\}$. Two grammars $\mathcal{G}$ and $\bar{\mathcal{G}}$ are called equivalent if $\mathcal{L}(\mathcal{G}) = \mathcal{L}(\bar{\mathcal{G}})$.

DEFINITION 3. An even linear grammar (ELG) is a context free grammar having only productions of the form:

$$\delta \rightarrow \varphi$$

$$\delta \rightarrow \varphi' \tilde{\delta} \varphi''$$

with the condition $|\varphi'| = |\varphi''|$.[1]

The ELG's are a subclass of linear grammars, which are defined in the same way but with the condition $|\varphi'| = |\varphi''|$ relaxed (Chomsky and Schützenberger, 1963).

LEMMA 1. *Given an* ELG $\mathcal{G}$ *there exists an* ELG $\bar{\mathcal{G}}$ *equivalent to* $\mathcal{G}$ *and having only productions of the form:*

$$\delta \rightarrow \varphi$$

$$\delta \rightarrow \varphi' \tilde{\delta} \varphi''$$

with $|\varphi'| = |\varphi''| > 0$.

The proof can easily be obtained by a slight modification of the proof of lemma 4.2. of Bar-Hillel, Perles, and Shamir (1961).

THEOREM 1. *Given an* ELG $\mathcal{G} \equiv (\Delta, \delta_0, P)$ *there exists an* ELG $\bar{\mathcal{G}}$ *equivalent to* $\mathcal{G}$ *and having only productions of the form:*

$$\begin{aligned} &\delta \rightarrow \omega \\ &\delta \rightarrow \sigma' \tilde{\delta} \sigma''. \end{aligned} \tag{1}$$

PROOF. From Lemma 1 we may assume that P does not contain productions $\delta \rightarrow \tilde{\delta}$. Let $(\delta \rightarrow x)$ be a production of P not having form (1), so that we may rewrite it as $\delta \rightarrow \sigma' x_1 \sigma''$, where $|x_1| = (|x| - 2)$ and $x_1 \notin \Delta$. We define a grammar $\mathcal{G}^{(1)} \equiv (\Delta^{(1)}, \delta_0^{(1)}, P^{(1)})$, where $\Delta^{(1)} = \Delta \cup \delta^{(1)}$, $\delta^{(1)}$ being a new symbol, $\delta_0^{(1)} = \delta_0$, and $P^{(1)}$ is obtained from P with this change: we substitute $(\delta \rightarrow x)$ with

$$\delta \rightarrow \sigma' \delta^{(1)} \sigma'', \qquad \delta^{(1)} \rightarrow x_1 .$$

[1] With standard notation $|\varphi|$ is the length of the word φ.

Clearly $\mathcal{L}(\mathcal{G}) = \mathcal{L}(\mathcal{G}^{(1)})$ and by a finite number of the above described steps we prove the theorem.

We now give a theorem which is in many respects analogous to the theorem about the construction of a deterministic automaton from a given nondeterministic one. (See definition 11. and theorem 11 of Rabin and Scott (1959).)

THEOREM 2. *Given an* ELG $\mathcal{G} = (\Delta, \delta_0, P)$ *there exists an* ELG $\bar{\mathcal{G}} = (\bar{\Delta}, \bar{\delta}_0, \bar{P})$ *equivalent to* $\mathcal{G}$ *such that:*

(a) $\forall\bar{\delta}[\bar{\delta} \in \bar{\Delta}\ \&\ \bar{\delta} \neq \bar{\delta}_0]\bar{P}$ *contains only productions of the form* (1).

(b) *If* $(\bar{\delta} \rightarrow x) \in \bar{P}$ *and* $(\bar{\bar{\delta}} \rightarrow y) \in \bar{P}$ *with* $x = y$ *then* $\bar{\delta} = \bar{\bar{\delta}}$.

(c) *If* $(\bar{\delta}_0 \rightarrow x) \in \bar{P}$ *then* $x \in \bar{\Delta}$.

PROOF: From Theorem 1 we may assume that P has only productions of the form (1). Let $\bar{\delta}_0$ be a new symbol and let $\Theta \equiv \{\theta\}$ be the set of all the subsets of Δ. We put $\bar{\Delta} = \{\bar{\delta}\} = \Theta \cup \bar{\delta}_0$ and we define $\bar{P}$ as the set of the following rules:

(a) $\theta \rightarrow \sigma'\tilde{\theta}\sigma''$, i.e., $[\delta_{\alpha_1} \cdots \delta_{\alpha_r}] \rightarrow \sigma'[\tilde{\delta}_{\beta_1} \cdots \tilde{\delta}_{\beta_q}]\sigma''$ if and only if:

$$\begin{array}{ll} \forall\delta \in \theta & \exists\tilde{\delta}[\tilde{\delta} \in \tilde{\theta}\ \&\ (\delta \rightarrow \sigma'\tilde{\delta}\sigma'') \in P] \\ \forall\tilde{\delta} \in \tilde{\theta} & \theta \supset \{\delta \mid (\delta \rightarrow \sigma'\tilde{\delta}\sigma'') \in P\}. \end{array}$$

To be more explicit, given a set $\tilde{\theta} = [\tilde{\delta}_{\beta_1} \cdots \tilde{\delta}_{\beta_q}]$ and a pair (σ', σ''), we construct the set formed by all the symbols $\delta_{\alpha_1}, \cdots \delta_{\alpha_r}$ such that for each $\delta_{\alpha_i} \in \theta$ there exists at least one $\tilde{\delta}_{\beta_j} \in \tilde{\theta}$ and a rule in P of the form $\delta_{\alpha_i} \rightarrow \sigma'\tilde{\delta}_{\beta_j}\sigma''$.

We note that $\tilde{\theta}$ may also contain some $\tilde{\delta}$ such that any rule of the form $\delta \rightarrow \sigma'\tilde{\delta}\sigma''$ does not exist in P.

If θ is not empty, we put in $\bar{P}$ the rule $\theta \rightarrow \sigma'\tilde{\theta}\sigma''$.

(b) $\theta \rightarrow \omega$, i.e., $[\delta_{\alpha_1} \cdots \delta_{\alpha_t}] \rightarrow \omega$ if and only if: $\theta = \{\delta \mid (\delta \rightarrow \omega) \in P\}$.

(c) $\bar{\delta}_0 \rightarrow \theta$ if and only if $\delta_0 \in \theta$. With this definition $\bar{\mathcal{G}}$ has the required form; we shall show that $\mathcal{L}(\bar{\mathcal{G}}) = \mathcal{L}(\mathcal{G})$.

$\mathcal{L}(\mathcal{G}) \subset \mathcal{L}(\bar{\mathcal{G}})$. In fact let

$$\varphi = \sigma_p' \cdots \sigma_1'\omega\sigma_1'' \cdots \sigma_p'' \in \mathcal{L}(\mathcal{G}).$$

This means that there exists a sequence of symbols of Δ, $\delta^{(1)}, \delta^{(2)}, \cdots \delta^{(p+1)}$ such that $(\delta^{(1)} \rightarrow \omega) \in P$, $(\delta^{(i+1)} \rightarrow \sigma_i'\delta^{(i)}\sigma_i'') \in P (i = 1, 2, \cdots p)$ and $\delta^{(p+1)} = \delta_0$. Consider the sequence $\bar{\delta}^{(1)}, \bar{\delta}^{(2)} \cdots$ of symbols of $\bar{\Delta}$ so constructed: $\bar{\delta}^{(1)}$ is the symbol such that $(\bar{\delta}^{(1)} \rightarrow \omega) \in \bar{P}$,[2] $(\bar{\delta}^{(i+1)} \rightarrow \sigma_i'\bar{\delta}^{(i)}\sigma_i'') \in \bar{P}$. We shall show that we obtain in this way a sequence $\bar{\delta}^{(1)}, \bar{\delta}^{(2)}, \cdots \bar{\delta}^{(p+1)}$ where $\bar{\delta}^{(p+1)} \ni \delta_0$, which implies that $\varphi \in \mathcal{L}(\bar{\mathcal{G}})$. In

[2] Note that this rule exists in $\bar{P}$ since $(\delta^{(1)} \rightarrow \omega) \in P$.

fact suppose that $\bar{\delta}^{(i)} \ni \delta^{(i)}$, which is true for $i = 1$, we prove that $\bar{\delta}^{(i+1)} \ni \delta^{(i+1)}$. This follows immediately since $(\delta^{(i+1)} \to \sigma_i' \delta^{(i)} \sigma_i'') \in P$. Conversely it can be easily shown with a similar technique that $\mathcal{L}(\bar{\mathcal{G}}) \subset \mathcal{L}(\mathcal{G})$ q.e.d.

SECTION II

DEFINITION 4. An equivalence relation R on the free semigroup T is called a quasi-congruence if, whenever $\varphi R \psi$, then $\overset{l}{\forall}(\sigma', \sigma'')\ \sigma'\varphi\sigma'' R \sigma'\psi\sigma''$.

Clearly if $\varphi R \psi$, then $\forall(\varphi', \varphi'')$ with $|\varphi'| = |\varphi''|$ $\varphi'\varphi\varphi'' R \varphi'\psi\varphi''$.[3] If R has a finite number of equivalence classes we call it of finite index.

II.1. Every congruence relation is obviously a quasi-congruence.

DEFINITION 5. Given a set $H \subset T_\Sigma$ we define the following relation C_H:

$$\varphi\ C_H\ \psi \text{ if and only if } \begin{cases} \overset{l}{\forall}(\varphi', \varphi'') \\ \varphi'\varphi\varphi'' \in H \text{ implies } \varphi'\psi\varphi'' \in H \text{ and conversely.} \end{cases}$$

II.2. C_H is an equivalence relation.

II.3. C_H is a quasi-congruence.

PROOF: In fact from $\varphi C_H \psi$ we have:

$$\overset{l}{\forall}(\varphi', \varphi'')\ \ \varphi'\varphi\varphi'' \in H \rightleftarrows \varphi'\psi\varphi'' \in H$$

and putting $\varphi' = \bar{\varphi}'\sigma'$, $\varphi'' = \sigma''\bar{\varphi}''$:

$$\overset{l}{\forall}(\bar{\varphi}', \bar{\varphi}'')\ \ \{\forall(\sigma', \sigma'')\ \ \bar{\varphi}'\sigma'\varphi\sigma''\bar{\varphi}'' \in H \rightleftarrows \bar{\varphi}'\sigma'\psi\sigma''\bar{\varphi}'' \in H\}.$$

Consequently:

$$\forall(\sigma', \sigma'')\ \ \sigma'\varphi\sigma''\ C_H \sigma'\psi\sigma'' \qquad \text{q.e.d.}$$

II.4. H is saturated by C_H (i.e., H is the union of some equivalence classes of C_H).

PROOF. Putting in Definition 5 $\varphi' = \varphi'' = \lambda$ we have

$$\varphi \in H \rightleftarrows \psi \in H.$$

THEOREM 3. *Every quasi-congruence R which saturates H is a refinement of* C_H.

[3] In the following we shall write $\overset{l}{\forall}(\varphi', \varphi'')$ with the meaning $\forall(\varphi', \varphi'')$ with $|\varphi'| = |\varphi''|$.

PROOF. If $\varphi R \psi$ then $\overset{l}{\forall}(\varphi', \varphi'')\ \varphi'\varphi\varphi'' R \varphi'\psi\varphi''$ and since H is saturated by R we have:

$$\overset{l}{\forall}(\varphi', \varphi'')\ \varphi'\varphi\varphi'' \in H \rightleftarrows \varphi'\psi\varphi'' \in H, \quad \text{i.e., } \varphi C_H \psi \text{ q.e.d.}$$

DEFINITION 6. A two sided finite automaton (TFA) over the alphabet $\Sigma \equiv \{\sigma_1 \cdots \sigma_r\}$ is a system $B = (S, F, M, S_0, f)$ where:

(a) S is a finite nonempty set (the states of B).

(b) F is a subset of S (the final states of B).

(c) M is a mapping of $\Sigma \times S \times \Sigma \xrightarrow{\text{in}} S$.

(d) S_0 is a subset of S with at most $r + 1$ elements.

(e) f is a mapping of $\Omega \xrightarrow{\text{on}} S_0$.

We extend M from $\Sigma \times S \times \Sigma$ to $\bigcup_{n \geqq 0} (\Sigma^n \times S \times \Sigma^n)$ in the following way:

$$\forall s \in S \begin{cases} M(\lambda, s, \lambda) = s \\ M(\sigma'\varphi', s, \varphi''\sigma'') = M(\sigma', M(\varphi', s, \varphi''), \sigma'') \\ \text{with } |\varphi'| = |\varphi''|. \end{cases}$$

DEFINITION 7. The set of tapes accepted by a TFA B, in symbols $\mathfrak{L}(B)$, is the collection of all tapes φ such that $M(\varphi', f(\omega_\varphi), \varphi'') \in F$, where in a standard way we have decomposed φ as follows:

$$\varphi = \varphi'\omega_\varphi\varphi'' \quad \text{with} \quad |\varphi'| = |\varphi''|.$$

Below we shall use this decomposition, without writing it explicitly, whenever no confusion occurs. In analogy with finite automata we call all the sets acceptable by some TFA quasi-regular events.

THEOREM 4. *Let L be a set of tapes; there exists a TFA B which accepts L, if, and only if, there exists a quasi-congruence R of finite index which saturates L.*

PROOF. Let $\mathfrak{L}(B) = L$; we define the relation R:

$$\varphi R \psi \rightleftarrows M(\varphi', f(\omega_\varphi), \varphi'') = M(\psi', f(\omega_\psi), \psi'').$$

Clearly R is an equivalence relation of finite index; we shall show that it is a quasi-congruence: in fact let $\varphi R \psi$ then $\forall(\sigma', \sigma'')$ we have:

$$M(\sigma'\varphi', f(\omega_\varphi), \varphi''\sigma'') = M(\sigma', M(\varphi', f(\omega_\varphi), \varphi''), \sigma'')$$

$$= M(\sigma', M(\psi', f(\omega_\psi), \psi''), \sigma'') = M(\sigma'\psi', f(\omega_\psi), \psi''\sigma'')$$

i.e.,

$$\forall(\sigma', \sigma'') \ \sigma'\varphi\sigma''R\sigma'\psi\sigma'', \qquad \text{q.e.d.}$$

In this way we have associated to each $s \in S$ a class of R, formed by all the tapes φ such that

$$M(\varphi', f(\omega_\varphi), \varphi'') = s.$$

Clearly L is the union of all the classes of R associated to all the $s \in F$.

Conversely, let L be saturated by a quasi-congruence of finite index R, and let $[\varphi]$ be the class of R containing φ. We define

$$B' = (\{[\varphi]\}, \{[\varphi] \forall\varphi \in L\}, M, \{[\omega] \forall\omega \in \Omega\}, f)$$

where:

(a) $M(\sigma', [\varphi], \sigma'') = [\sigma'\varphi\sigma'']$.

(b) $f(\omega) = [\omega]$.

M is well defined since R is a quasi-congruence. B' accepts L; in fact let $\varphi = \varphi'\omega_\varphi\varphi''$. We have:

$$M(\varphi', f(\omega_\varphi), \varphi'') = M(\varphi', [\omega_\varphi], \varphi'') = [\varphi].$$

So φ is accepted if and only if $\varphi \in L$.

LEMMA 2. *The family of regular events is properly contained in the family of quasi-regular events.*

The inclusion follows from II.1, Theorem 4, and the theorem of Myhill (Rabin and Scott, 1959) about finite automata.

To show that the inclusion is proper we observe that, putting $\Sigma = (0, 1)$, the event $E = \{0^n 10^n\}$ $(n = 0, 1 \cdots)$, is saturated by the quasi-congruence of finite index having as classes E and $(T_\Sigma - E)$, while it is not a regular event (Rabin and Scott, 1959). In other words a device which reads the tape beginning from its center is more powerful than one which reads it beginning from one end. Note however that a TFA is a finite memory device with, furthermore, a very simple infinite operation, which is to find the center of the tape.

From Theorems 3 and 4 it follows that, as for finite automata, it is possible to associate to the "minimal" TFA which accepts a set L a particular quasi-congruence, namely, C_L (Rabin and Scott, 1959).

SECTION III

THEOREM 5. *Given a set $H \subset T_\Sigma$ which is saturated by a quasi-congruence of finite index C there exists an* ELG $\mathcal{G}$ *such that* $\mathcal{L}(\mathcal{G}) = H$, *and conversely.*

PROOF: Let $C_i (i = 1, \cdots n)$ be the classes of C so that

$$H = \bigcup_{1}^{p}{}_{\alpha} C_{i_\alpha} \qquad (p \leqq n).$$

We define $\mathcal{G} = (\Delta, \delta_0, P)$ in this way:

(a) $\Delta = \{\delta\} = \{C_i\} \cup \delta_0$ where δ_0 is a new symbol.

(b)

$$P = \begin{cases} C_i \to \sigma' C \sigma'' \rightleftarrows \sigma' C \sigma'' \subset C_i , \\ C_i \to \omega \rightleftarrows \omega \in C_i , \\ \delta_0 \to C_i \rightleftarrows C_i \subset H. \end{cases}$$

Clearly $\mathcal{G}$ is an ELG and it is not difficult to see that $\mathcal{L}(\mathcal{G}) = H$.

Conversely, let $\mathcal{G} = (\Delta, \delta_0, P)$ be an ELG; from Theorem 2 we may assume that $\mathcal{G}$ has the particular form stated there.

We define a new grammar $\bar{\mathcal{G}} = (\bar{\Delta}, \bar{\delta}_0, \bar{P})$ as follows:

(a) $\bar{\Delta} = \{\bar{\delta}\} = \Delta \cup \bar{\delta}_1$, where $\bar{\delta}_1$ is a new symbol.

(b) $\bar{\delta}_0 = \delta_0$.

(c) $\bar{P}$ contains P and the new productions:

$$\begin{array}{ll} \bar{\delta}_1 \to \omega & \text{if and only if } \exists\!\!\!/ (\delta \to \omega) \in P, \\ \bar{\delta}_1 \to \sigma' \delta \sigma'' & \text{if and only if } \exists\!\!\!/ (\delta' \to \sigma' \delta \sigma'') \in P, \\ \bar{\delta}_1 \to \sigma' \bar{\delta}_1 \sigma'' & \text{for } \forall (\sigma', \sigma''). \end{array}$$

Clearly $\bar{\mathcal{G}}$ still has the particular form requested from Theorem 2. and $\mathcal{L}(\mathcal{G}) = \mathcal{L}(\bar{\mathcal{G}})$.

Consider now the relation C:

$$\varphi C \psi \rightleftarrows \{\exists \bar{\delta} \neq \bar{\delta}_0 \ \ [\bar{\delta} \overset{*}{\Rightarrow} \varphi \ \& \ \bar{\delta} \overset{*}{\Rightarrow} \psi]\}.$$

(a) C is an equivalence relation; symmetry and reflexivity are immediate. To prove transitivity we have to show that if $\bar{\delta}_i \overset{*}{\Rightarrow} \varphi$ and $\bar{\delta}_j \overset{*}{\Rightarrow} \varphi$, then $\bar{\delta}_i = \bar{\delta}_j$. In fact suppose that $\bar{\delta}_i \neq \bar{\delta}_j$; let $\varphi = \sigma_1' \cdots \sigma_{m-1}' \omega \sigma_{m-1}'' \cdots \sigma_1''$, from condition (b) of Theorem 2 $\exists (\bar{\delta}_{i_1}, \bar{\delta}_{j_1})$ with $\bar{\delta}_{i_1} \neq \bar{\delta}_{j_1}$ such that:

$$\bar{\delta}_i \to \sigma_1' \bar{\delta}_{i_1} \sigma_1'',$$

$$\bar{\delta}_j \to \sigma_1' \bar{\delta}_{j_1} \sigma_1''.$$

By finite induction we prove that $\exists (\bar{\delta}_{i_m}, \bar{\delta}_{j_m})$ with $\bar{\delta}_{i_m} \neq \bar{\delta}_{j_m}$ such that:

$$\bar{\delta}_{i_m} \to \omega,$$

$$\bar{\delta}_{j_m} \rightarrow \omega.$$

This contradicts condition (b) of Theorem 2 and so $\bar{\delta}_i = \bar{\delta}_j$. C is therefore an equivalence relation of finite index, and the generic class of C, say C_i, is such that:

$$C_i = \{\varphi \mid \bar{\delta}_i \stackrel{*}{\Rightarrow} \varphi \ \& \ \bar{\delta}_i \neq \bar{\delta}_0\}.$$

From this we have immediately that $\mathcal{L}(\bar{\mathcal{G}}) = \mathcal{L}(\mathcal{G})$ is the union of those classes C_i such that for the associated symbol $\bar{\delta}_i$ there exists in $\bar{P}$ the production $(\bar{\delta}_0 \rightarrow \bar{\delta}_i)$.

We finally show that C is a quasi-congruence. Let C_i be a class of C, and $\bar{\delta}_i$ the corresponding symbol. For every (σ', σ'') there exists in $\bar{\Delta}$ one and only one $\bar{\delta}_j$ such that $(\bar{\delta}_j \rightarrow \sigma' \bar{\delta}_i \sigma'') \in \bar{P}$. Consequently, calling C_j the class associated to $\bar{\delta}_j$ we have:

$$\sigma' C_i \sigma'' \subset C_j . \qquad \text{q.e.d.}$$

LEMMA 3. *The family of languages generated by the* ELG's *is therefore, from Theorems* 4 *and* 5, *the family of quasi-regular events.*

LEMMA 4. *The family of languages generated by the* ELG's *contains the family of languages generated by the one-sided linear grammars* (Chomsky and Schützenberger, 1963), *i.e., given a one-sided linear grammar there exists an* ELG *equivalent to it.*

We will now state an immediate consequence of Lemma 4, which is nonobvious in itself (Stearns and Hartmanis, 1963).

Let H be a regular event; define $R(H)$ as the set of all "right halves" of strings of H, i.e.:

$$R(H) = \{\varphi'' \mid \exists \varphi \in H \ \& \ \varphi = \varphi' \omega_\varphi \varphi''\}.$$

We shall show that $R(H)$ is a regular event. In fact let $\mathcal{G} = (\Delta, \delta_0, P)$ be a ELG of the form stated in Theorem 1, which generates H. We define the grammar $\bar{\mathcal{G}} = (\bar{\Delta}, \bar{\delta}_0, \bar{P})$, where $\bar{\Delta} = \Delta$, $\bar{\delta}_0 = \delta_0$ and P contains all the productions:

$$\bar{\delta}_i \rightarrow \bar{\delta}_j \sigma'' \quad \text{if and only if } \exists \sigma' \text{ such that } \quad (\delta_i \rightarrow \sigma' \delta_j \sigma'') \in P,$$

$$\bar{\delta}_i \rightarrow \lambda \quad \text{if and only if } \exists \omega \text{ such that } \quad (\delta_i \rightarrow \omega) \in P.$$

$\bar{\mathcal{G}}$ is a one-sided linear grammar and it may be easily seen that $\mathcal{L}(\bar{\mathcal{G}}) = R(H)$. In the same way it can be shown that the set of all "left halves" of H, $L(H)$, is a regular event. From our paper it is clear that quasi-regular events behave very analogously to regular events; in a future paper we shall study their closure properties in order to obtain a "Kleene Theorem" (Rabin and Scott, 1959), for this family.

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