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Caratheodory Selections and the Scorza Dragoni Property

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1. INTRODUCTION

A Caratheodory selection of a set-valued map $F(x, t)$ is a function $f(x, t)$ which is continuous in the x variable, measurable in the t variable, defined whenever $F(x, t)$ is not empty and satisfies $f(x, t) \in F(x, t)$. The mapping F is defined on a product $X \times T$ of a metric space and a measure space, and the values of F are subsets of another metric space Y. A need for Caratheodory selections arises in the study of random fixed points, see, e.g., Itoh [8], and in game theory and economics, see Kim *et al.* [11].

When Y is a Banach space and F has convex values and is lower semicontinuous, existence of Caratheodory selections is provided by Castaing [2] and Fryszkowski [5], both under some compactness assumptions, and recently by Kim *et al.* [10] and Rybinski [16] for a more general case. The method in [2, 10, 16] is to modify Michael's construction of a continuous selection, [13], and make it depend measurably on the parameter t, yielding a Caratheodory selection. Our approach is different. We reduce the problem of finding a Caratheodory

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selection to a problem of finding a continuous selection for a lower semicontinuous mapping. This is done by utilizing a Scorza Dragoni type result with respect to lower semicontinuity. To this end we have to confine T to be a metric space, a restriction not made in [2, 5, 10, and 16]. In turn, the reduction enables us to employ available results for the existence of continuous selections, other than for convex-valued maps, and establish the analogous existence of Caratheodory selections.

We prove the Scorza Dragoni type result in Section 2, and derive the existence of Caratheodory selections in Section 3. A seemingly strong measurability condition is employed; it cannot be dropped, as is shown by a counterexample in the closing section.

2. A Scorza Dragoni Type Result

Recall that a set-valued map F, from a metric space Z into subsets of a metric space Y, is lower semicontinuous if $z_k \to z_0$ in Z and $y_0 \in F(z_0)$ then there are $y_k \in F(z_k)$, at least for k large enough, so that $y_k \to y_0$. We say that a set-valued map F, from a measurable space Z into subsets of Y, is measurable, if for every open set $A \subset Y$ the inverse $F^-(A) = \{z: F(z) \cap A \neq 0\}$ is a measurable set. For elaboration see Castaing and Valadier [3]. Note, however, that the measurability or the lower semicontinuity of F are not affected if the values $F(z)$ are replaced by other sets, provided they have the same closure.

Framework and assumptions (see Remark 2.5 for possible refinements). Let X be a separable complete metric space, equipped with its Borel structure. Let T be a separable complete metric space and let μ be a finite measure on T. Let Y be a separable metric space. We consider a set-valued map $F(x, t)$, defined on $X \times T$, with values being subsets, not necessarily closed and possibly empty, of Y. It is assumed that:

(i) $F(\cdot, t)$ is lower semicontinuous for each fixed t, and

(ii) $F(\cdot,\cdot)$ is measurable on $X \times T$.

The main result of this section is the following.

THEOREM 2.1. *Under conditions* (i) *and* (ii), *for every* $\varepsilon > 0$ *there exists a compact set* $K \subset T$ *with* $\mu(T \backslash K) < \varepsilon$, *and such that* $F(x, t)$ *is lower semicontinuous on* $X \times K$.

This is a Scorza Dragoni type result, originally proved in [17] for real functions and continuity rather than lower semicontinuity. Set-valued analogs were provided by Castaing [1; Theorem 4] for the closed-graph property, and by Himmelberg and Val Vleck [7; Theorem 4.1] for the lower semicontinuity property, as the case is in the present note. Fryszkowski [5] obtained a similar result under strong conditions (compactness of X, a Banach space Y, and convexity of the values) as a consequence of the existence of Caratheodory selections. Our result strengthens the one in [7] as we do not assume that $F(\cdot, t)$ has a closed graph and that Y is locally compact. The latter condition is essential in the Himmelberg–Van Vleck method. Our approach is along the lines that Castaing [1] and Jacobs [9] prove Lusin type results for set-valued maps. The proof of Theorem 2.1 follows several lemmas, some standard, and the proofs are outlined for completeness.

In the sequel, if $\rho(\cdot,\cdot)$ is a metric and B is a set in the metric space then $\rho(a, B)$ denotes the $\inf\{\rho(a, b)\colon b \in B\}$, and $\inf\varnothing = \infty$.

LEMMA 2.2 (Castaing [1] and Jacobs [9]). *Let $G(t)$ be a set-valued map defined on T, with values being closed subsets of a separable metric space M. If G is measurable then for every $\varepsilon > 0$ there exists a compact set $K \subset T$, with $\mu(T \backslash K) \leqslant \varepsilon$, and such that G restricted to K has a closed graph.*

Proof. Denote by $\rho(\cdot,\cdot)$ the metric on M and let $\{m_j\}$, $j = 1, 2, \ldots$, be a dense sequence in M. For each j the function $\rho(m_j, G(t))\colon T \to [0, \infty]$ is measurable (by [18; Theorem 4.2]). By the standard Lusin theorem there is a compact set $K_j \subset T$ with $\mu(T \backslash K_j) < \varepsilon 2^{-j}$, and such that $\rho(m_j, G(t))$ is continuous on $t \in K_j$. Let K be the intersection of K_j for $j = 1, 2, \ldots$. Then K is compact, $\mu(T \backslash K) < \varepsilon$ and $\rho(m_j, G(t))$ is continuous on K for all m_j. The latter property implies that the restriction of G to K has a closed graph. Indeed, let (t_k, n_k) converge in $K \times M$ to (t_0, n_0) and $n_k \in G(t_k)$ for $k = 1, 2, \ldots$. Let m_j be an element from the dense sequence with $\rho(m_j, n_0) < \delta$, for small δ. By the continuity of $\rho(m_j, G(t))$, and since the limsup of $\rho(m_j, G(t_k))$ is no greater than δ, it follows that for an element $n_\delta \in G(t_0)$ the inequality $\rho(m_j, n_\delta) \leqslant \delta$ holds, hence $p(n_0, n_\delta) \leqslant 2\delta$. Since δ is arbitrarily small and $G(t_0)$ is closed, it follows that $n_0 \in G(t_0)$. This completes the proof.

Recall that a function $g\colon Z \to (-\infty, \infty]$ is upper semicontinuous, if $z_k \to z_0$ in Z implies $g(z_0) \geqslant \limsup g(z_k)$. In the sequel, and throughout, $d(\cdot,\cdot)$ denote the metric on Y.

LEMMA 2.3. *Let $F(z)$ be a set-valued map defined on a metric space Z, with values being subsets (possibly empty) of Y. Let $\{\eta_j\}$ be a dense sequence in Y. Then:* (a) *if F is lower semicontinuous then for every $\eta \in Y$ the mapping $d(\eta, F(z))\colon Z \to [0, \infty]$ is upper semicontinuous, and* (b) *if for every j the mapping $d(\eta_j, F(z))\colon Z \to [0, \infty]$ is upper semicontinuous then F is lower semicontinuous.*

Proof. Let $z_k \to z_0$. If $F(z_0)$ is empty then $d(\eta, F(z_0)) = \infty$ and part (a) holds at z_0. If $F(z_0) \neq \emptyset$, then, given $\varepsilon > 0$, a point $y_0 \in F(z_0)$ can be chosen with $d(\eta, y_0) - d(\eta, F(z_0)) < \varepsilon$. If F is lower semicontinuous, then a sequence $y_k \to y_0$ exists, with $y_k \in F(z_k)$ for k large. It follows then that $\limsup d(\eta, F(z_k)) \leqslant d(\eta, y_0)$; since the latter is less than $d(\eta, F(z_0)) + \varepsilon$ and since ε is arbitrarily small, part (a) follows. For part (b) let $z_k \to z_0$ and $y_0 \in F(z_0)$. For a given $\varepsilon > 0$ choose an η_j in the dense sequence so that $d(\eta_j, y_0) < \varepsilon$. In particular, $d(\eta_j, F(z_0)) < \varepsilon$. The upper semicontinuity assumption implies that $\limsup d(\eta_j, F(z_k)) \leqslant \varepsilon$ as $k \to \infty$; hence $y_k \in F(z_k)$ exist with $d(\eta_j, y_k) \leqslant 2\varepsilon$, this for k large enough; in particular, $d(\eta_j, y_0) \leqslant 3\varepsilon$. Since ε is arbitrary, a simple diagonal procedure with $\varepsilon \to 0$ implies the lower semicontinuity and completes the proof.

LEMMA 2.4. *Let F be a set-valued map on $X \times T$, satisfying* (i) *and* (ii). *Let $\eta \in Y$ be fixed and consider the set-valued map S from T into subsets of $X \times [0, \infty]$, defined by*

$$S(t) = \{(x, r): 0 \leqslant r \leqslant d(\eta, F(x, t))\}.$$

Then S has closed values and is measurable with respect to the completion of μ on T. Furthermore, whenever $K \subset T$ is such that the restriction of S to K has a closed graph, then $d(\eta, F(x, t))$ is upper semicontinuous on $(x, t) \in X \times K$.

Proof. Closedness of $S(t)$ follows from the upper semicontinuity of $d(\eta, F(\cdot, t))$ (in the x-variable), which is implied by condition (i) and part (a) of Lemma 2.3. To check measurability consider first the set

$$D = \{(t, x, r): r \leqslant d(\eta, F(x, t))\}.$$

It is measurable in $T \times X \times [0, \infty]$, being the inverse image of $[0, \infty]$ by the mapping $h(t, x, r) = d(\eta, F(x, t)) - r$, and the measurability of the latter follows from the measurability of $F(x, t)$ in (x, t) (condition (ii)) and [18, Theorem 4.2]. Given now an open set $A \subset X \times [0, \infty]$, the inverse $S^-(A)$ is the projection on T of $D \cap (T \times A)$, and by the projection theorem (e.g., [3; Theorem III.2.3.]) the set $S^-(A)$ is measurable with respect to the completion of μ. (In employing the projection theorem we use the completeness of the metric space X.) This completes the first part of the lemma. To check the upper semicontinuity of $d(\eta, F(x, t))$ on $X \times K$ suppose the contrary, namely that $\limsup d(\eta, F(x_k, t_k)) > d(\eta, F(x_0, t_0))$, where (x_k, t_k) converge to (x_0, t_0) in $X \times K$. For a subsequence of indices (still denoted here by k) there are $r_k \leqslant F(x_k, t_k)$ which converge, say to $r_0 \in [0, \infty]$, and hence $r_0 > d(\eta, F(x_k, t_k)) + \varepsilon$ for some $\varepsilon > 0$, this for k large enough. However, $(x_k, r_k) \in S(t_k)$ while $(x_0, r_0) \notin S(t_0)$, which contradicts the assumption that S has a closed graph on K. This completes the proof.

Proof of Theorem 2.1. First we choose a dense sequence $\{\eta_j\}$ in Y. For a given j consider the set-valued mapping S_j from T to $X\times[0,\infty]$ given by

$$S_j(t)=\{(x,r)\colon 0\leqslant r\leqslant d(\eta_j,F(x,t))\}.$$

By Lemma 2.4 the multifunction S_j is measurable, and in view of Lemma 2.2 there is a compact set $K_j\subset T$ with $\mu(T\backslash K_j)<\varepsilon 2^{-j}$ such that S_j restricted to K_j has a closed graph. In light of the second part of Lemma 2.4 the function $d(\eta_j, F(x,t))$ is upper semicontinuous on $X\times K_j$. Let K be the intersection of all K_j. Then $\mu(T\backslash K)<\varepsilon$, and $d(\eta_j,F(x,t))$ is upper semicontinuous on $X\times K$ for all j. Since $\{\eta_j\}$ is a dense sequence it follows from Lemma 2.3 that F is lower semicontinuous on $X\times K$, and this is the desired conclusion.

Remark 2.5. The condition on T and X can be somewhat relaxed. The completeness of the metric space T was used just to ensure that given B measurable and $\varepsilon>0$, there exists a compact $K\subset B$ with $\mu(B\backslash K)<\varepsilon$; this is needed in the Lusin theorem. Therefore assuming instead that μ is Radon, or regular, is enough. The completeness of X was used in applying the projection theorem, it would be enough therefore to assume that X is an analytic set or a similar condition (consult with [18]).

3. Caratheodory Selections

The result of the previous section enables us to derive existence of Caratheodory selections in situations where lower semicontinuity implies existence of continuous selections. (Lower semicontinuity is a natural condition in the search for continuous selections; in fact, as pointed out by Michael [13, 14], the lower semicontinuity is even a necessary condition if the continuous selection is allowed to be prescribed at a given point.) This leads us to the following abstract definition.

Definition 3.1. Let F be a set-valued map defined on a space Z with values in Y. We say that F is a Michael mapping (M-mapping for short) if the restriction of F to $Z_1\cap\{z\colon F(z)\neq\emptyset\}$, with $Z_1\subset Z$, has a continuous selection whenever the restriction of F to Z_1 is lower semicontinuous.

Almost every property which guarantees existence of a continuous selection for a lower semicontinuous mapping, is a property which determines M-mappings. For example: If Y is a Banach space and Z metric then F is an M-mapping if its values are convex and closed; if Y is separable then the values need not be closed. These are Michael's theorems, [13]. The convexity can be replaced by decomposability in a function space, this is

Fryszkowski's result [6]. Properties leading to M-mappings may not use linearity at all, but may be topological in nature, see Michael [14] and references therein, see also Curtis [4].

We turn now to our main theorem. In the sequel the set-valued F is defined on $X \times T$, with values in Y as described in section 2. A Caratheodory selection of F is a point-valued function $f(x, t)$, defined for almost every t and when $F(x, t) \neq \emptyset$, and such that $f(\cdot, t)$ is continuous, $f(x, \cdot)$ is measurable, and $f(x, t) \in F(x, t)$.

THEOREM 3.2. *Suppose F satisfies* (i) *and* (ii) *and it is an M-mapping. Then F has a Caratheodory selection.*

Proof. Let K_j be compact subsets of T such that $\mu(T \setminus K_j) < 2^{-j}$ and such that F restricted to $X \times K_j$ is lower semicontinuous. Such K_j exist by Theorem 2.1. Since F is an M-mapping it follows that for each index j a continuous function $f_j(x, t)$ exists, which is defined for $(x, t) \in X \times K_j$ if $F(x, t) \neq \emptyset$ and $f_j(x, t) \in F(x, t)$. We define $f(x, t)$ to be equal to $f_j(x, t)$ if $t \in K_j$ and $t \notin K_i$ for $i < j$. Then $f(x, t)$ is defined whenever $F(x, t) \neq \emptyset$, except possibly for the set $T \setminus (\cup K_j)$, which is a set of measure zero. Clearly f is a Caratheodory selection.

Occasionally one is interested in more than the existence of one selection, but rather in the existence of a sequence of selections that exhaust the set-valued map. Specifically, if $F(z)$ has closed convex values and is lower semicontinuous into a separable Banach space, then Michael, [13], proved the existence of a sequence $f_k(z)$ of continuous selections so that $F(z) = \mathrm{cl}\{f_k(z)\}$ for all z. This was extended by Fryszkowski [5] to Caratheodory selections. The method developed in our paper enables to deduce the result for Caratheodory selections from the one of continuous selections, as follows.

DEFINITION 3.3. Let F be a set-valued map defined on a space Z with values in Y. We say that F is an SM-mapping if whenever $Z_1 \subset Z$ is such that the restriction of F to Z_1 is lower semicontinuous, then there is a sequence continuous selections $f_k(z)$ of $F(z)$, defined on $\{z : z \in Z, F(z) \neq \emptyset\}$ such that $\mathrm{cl}\, F(z) = \mathrm{cl}\{f_k(z)\}$.

THEOREM 3.4. *Suppose F satisfies* (i) *and* (ii) *and it is an SM-mapping. Then there exists a sequence $f_k(x, t)$ of Caratheodory selections of F such that* $\mathrm{cl}\, F(x, t) = \mathrm{cl}\{f_k(x, t)\}$.

The proof is along the lines of the proof of Theorem 3.2, we leave out the details.

4. On the Measurability

One may ask whether the joint measurability condition (ii) can be replaced by the weaker one as follows

(ii)′ $F(x, \cdot)$ is measurable for every fixed x.

We provide a counterexample, namely a mapping satisfying (i) and (ii)′ which does not posses a Caratheodory selection. In particular, (i) and (ii)′ do not imply the Scorza Dragoni property, nor do they imply condition (ii). (For relations between (i), (ii), and (ii)′ see Papageorgiou [15].)

We use the *continuum hypothesis*. The continuum is the cardinality of all pairs (f, B) such that: B is a Borel subset of $[0, 1] \times [0, 1]$, of Lebesgue measure 1 and $f: B \to [0, 1]$ is a Borel function (see [11]). et (f_α, B_α) be a well ordering of such pairs.

We proceed inductively. Suppose that for each $\gamma < \alpha$ a pair (x_γ, t_γ) was determined. We choose t_α to be different from all t_γ for $\gamma < \alpha$ and such that the section $\{x: (x, t_\alpha) \in B_\alpha\}$ is of Lebesgue measure 1. Such t_α exists since the cardinality of sections with Lebesgue measure 1 is the continuum, and only a denumerable number of t_γ were chosen previously. Now x_α is determined such that $(x_\alpha, t_\alpha) \in B_\alpha$ and $x_\alpha \neq x_\gamma$ for all $\gamma < \alpha$. Again, the existence of such x_α is guaranteed since the aforementioned section has a positive measure.

The counterexample consists of a set-valued map $F(x, t)$ defined on $[0, 1] \times [0, 1]$ with values being closed sets in $[0, 1]$. If $t \neq t_\alpha$ for all ordinals α then we set $F(x, t) = [0, 1]$. If $t = t_\alpha$ then we set $F(x, t_\alpha) = [0, 1]$ if $x \neq x_\alpha$ and $F(x_\alpha, t_\alpha) = \{g_\alpha\}$ with g_α an arbitrary number with the only condition $g_\alpha \neq f_\alpha(x_\alpha, t_\alpha)$.

For a fixed t, and for a fixed x, the set-valued mapping is identically equal to $[0, 1]$ except possibly at one point (x_α, t_α). Therefore $F(x, t)$ is lower semicontinuous in each of the variables separately, in particular (i) and (ii)′ hold. Yet, a Caratheodory selection does not exist, since for every measurable function $f(x, t)$, supposedly a selection, there is a Borel set B with full measure in $[0, 1] \times [0, 1]$ on which f is Borel and $f(x, t) \in F(x, t)$ if $(x, t) \in B$. Then $(f, B) = (f_\alpha, B_\alpha)$ for some index α, but then f cannot be a selection of F since $f(x_\alpha, t_\alpha) = g_\alpha \notin F(x_\alpha, t_\alpha)$. This concludes the example.

References

1. C. Castaing, Une nouvelle extension du théorème de Dragoni-Scorza. *C. R. Acad. Sci. Paris Ser. A* **271** (1970), 396–398.
2. C. Castaing, Sur l'existence des sections separement mesurables et separement continues d'une multi-application, *in* "Séminaire d'Analyse Convexe, Montpellier 1975, Ex. 14.

3. C. Castaing and M. Valadier, "Convex Analysis and Measurable Multifunctions," Lecture Notes in Mathematics Vol. 580, Springer-Verlag, Berlin, 1977.
4. D. W. Curtis, Applications of a selection theorem to hyperspace contractibility, *Canad. J. Math.* **37** (1985), 747–759.
5. A. Fryszkowski, Caratheodory type selections of set-valued maps of two variables, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **25** (1977), 41–46.
6. A. Fryszkowski, Continuous selections for a class of non-convex multivalued maps, *Studia Math.* **76** (1983), 163–174.
7. C. F. Himmelberg and F. S. Van Vleck, An extension of Brunovsky's Scorza Draconi type theorem for unbounded set-valued functions, *Math. Slovaca* **26** (1976), 47–52.
8. S. Itoh, Random fixed point theorems with applications to random differential equations in Banach spaces, *J. Math. Anal. Appl.* **62** (1969), 261–273.
9. M. I. Jacobs, Measurable multivalued mappings and Lusin's theorem, *Trans. Amer. Math. Soc.* **134** (1968), 471–481.
10. T. Kim, K. Prikry, and N. C. Yannelis, Caratheodory-type selections and random fixed point theorems, *J. Math. Anal. Appl.* **122** (1987), 393–407.
11. T. Kim, K. Prikry, and N. C. Yannelis, Equilibria in abstract economies with a measure space of agents and with an infinite dimensional strategy space, preprint, University of Minnesota, 1985.
12. K. Kuratowski, "Topology I," Academic Press, New York, 1966.
13. E. A. Michael, Continuous selections I. *Ann. of Math.* **63** (1956), 363–382.
14. E. A. Michael, A survey of continuous selections, *in* "Set-Valued Mappings, Selections and Topological Properties of 2^X" (W. Fleischman, Ed.), Lecture Notes in Math. Vol. 171, pp. 54–58, Springer-Verlag, Berlin, 1970.
15. N. S. Papageorgiou, On measurable multifunctions with applications to random multivalued equations, preprint, University of Illinois, Urbana, 1985.
16. L. Rybinski, On caratheodory type selections. *Fund. Math.* **125** (1985), 187–193.
17. G. Scorza Dragoni, Una theorema sulla funzioni continue rispetto ad una i misurable rispetto at ultra variable, *Rend. Sem. Mat. Univ. Padova* **17** (1948), 102–106.
18. D. H. Wagner, Survey of measurable selection theorems, *SIAM J. Control Optim.* **15** (1977), 859–903.