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# On the uniform exponential stability of a wide class of linear time-delay systems

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**Abstract**

This paper deals with the global uniform exponential stability independent of delay of time-delay linear and time-invariant systems subject to point and distributed delays for the initial conditions being continuous real functions except possibly on a set of zero measure of bounded discontinuities. It is assumed that the delay-free system as well as an auxiliary one are globally uniformly exponentially stable and globally uniform exponential stability independent of delay, respectively. *The auxiliary system is typically a part of the overall dynamics of the delayed system but not necessarily the isolated undelayed dynamics as usually assumed in the literature.* Since there is a great freedom in setting such an auxiliary system, the obtained stability conditions are very useful in a wide class of practical applications.


*Keywords:* Time-delay systems; Stability; Stabilization

## 1. Introduction

The stability and feedback stabilization of time-delay systems subject to constant point and distributed delays as well as time-varying ones has received important attention in the last years (see, for instance, [1,2,4–8,10,11,13]). A key point is that a system exhibiting stability in the absence of delays may loose that property for small delays and, in contrast, a stable delayed system may loose the property in the absence of delay (see, for instance,

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[1,4,6]). *This paper deals with the global uniform exponential stability independent of the delays (g.u.e.s.i.d.) of a class of homogeneous time-delay systems subject to combined point and distributed delays as well as integro-differential Volterra-type delayed dynamics.* The global stability is investigated for any function of initial conditions being everywhere continuous on its definition domain, a real interval $[-h, 0]$, where $h$ is the maximum delay in the system, except possibly on a set of zero measure where the function possess bounded discontinuities. Necessary and sufficient global uniform stability conditions independent of the delays are obtained if the delay-free system is globally uniformly exponentially stable (g.u.e.s.) and an auxiliary system is g.u.e.s.i.d. The obtained results are then applied to a number of particular cases of interest by setting different auxiliary systems including the standard delay-free one. The mathematical proofs are based on conditions which guarantee that a linear operator in a Banach space is compact within a domain that contains the closed complex right-half plane provided that another one defined for the auxiliary system is also compact within a (not necessarily identical) domain that contains the closed complex right-half plane. The auxiliary system may be a delay-free one or, in general, any particular parametrization of the whole system under study where a part of the delayed dynamics is deleted. Some sufficient conditions for the system to be g.u.e.s. dependent on delay are also obtained by using the same mathematical outlines. Extensions are given for forced systems under impulsive inputs and also by considering the closed-loop stabilization of time-delay systems of the given class.

**Notation.** (a) For the delayed system, $T:[0,\infty)\to L(X)$ is the inverse Laplace transform of the resolvent mapping $\hat{T}(s)$, which is holomorphic where it exists, with $X$ being the real Banach space of $n$-vector real functions endowed with the supremum norm on their definition domain. $\hat{T}^{-1}(s)$ takes the form $(\hat{T}_{JM}^{-1}(s) - \Delta\hat{T}_{JM}(s))$, where $\hat{T}_{JM}(s)$ is defined similarly as $\hat{T}(s)$ for the auxiliary system, whose delay-free dynamics is defined by a square $n$-matrix $M$, and $\Delta\hat{T}_{JM}(s) = \hat{T}_{JM}^{-1}(s) - \hat{T}^{-1}(s)$. For all complex $s$ such that $\hat{T}_{JM}(s)$ exists, $\hat{T}(s) = (I - \hat{T}_{JM}(s)\Delta\hat{T}_{JM}(s))^{-1}\hat{T}_{JM}(s) = \hat{\hat{T}}_{JM}^{-1}(s)\hat{T}_{JM}(s)$.

The subindex $J = (J_1, J_2, J_3)$ denotes a triple for sets of indices referred to the particular subsets of real constants describing point delays ($J_1$), infinitely distributed Volterra-type delays ($J_2$) and finitely distributed delays ($J_3$) of the system which are also present in the auxiliary system. For instance, $1 \in J_1 \Rightarrow h_1 > 0$ is a point delay of the time-delay system which is also present in the auxiliary system and so on. Also, $\mathrm{Card}(J_1) \leqslant m$, $\mathrm{Card}(J_2) \leqslant m' + 1$, $\mathrm{Card}(J_3) \leqslant m''$. If a pure convolution Volterra-type dynamics $\int_0^t d\alpha_0(\tau) A_{\alpha_0} x(t-\tau)$ is present then it is described by a fictitious delay $h_0' = 0$. If such a term is not present then $\mathrm{Card}(J_2) \leqslant m'$. The remaining infinitely distributed delays give contributions $\int_0^t d\alpha_i(\tau) A_{\alpha_i} x(t-\tau-h_i')$ with finite real constants $h_i' > 0$ with $i = 1, 2, \ldots, m'$ to $\dot{x}(t)$ which are point delays under the integral symbol. It is said that the delays are infinitely distributed because of the contribution of the delayed dynamics is made under an integral over $[0, \infty)$ as $t \to \infty$, i.e., $x(t-\tau-h_i')$ acts on the dynamics of $x(t)$ from $\tau = 0$ to $\tau = t$ for finite $t$ and as $t \to \infty$.

(b) $\hat{T}'^{-1}(s,\varphi) = \hat{T}_{JM}'^{-1}(s,\varphi) - \Delta\hat{T}_{JM}'(s,\varphi)$ is a complex operator-valued function with domain in $\mathbf{C} \times [-\pi,\pi]^{m+m'+m''+1} \subset \mathbf{C} \times \mathbf{R}^{m+m'+1}$ with $[-\pi,\pi]^{m+m'+m''+1}$ being the cross product of $[-\pi,\pi]$ by itself $(m+m'+m''+1)$-times and $\varphi^T=(\varphi_1,\varphi_2,\dots,\varphi_{m+m'+m''+1})$ and $\varphi_{m+1}=0$ (since $h'_0=0$) and range in $\mathbf{C}^n$.

(c) $N(s,\hat h)=N_{JM}(s,\hat h)[I-N_{JM}^{-1}(s,\hat h)\Delta N_{JM}(s,\hat h)]=N_{JM}(s,\hat h)$. $\tilde N_{JM}(s,\hat h)$ is an operator-valued function with domain in $\mathbf{C}\times\mathbf{R}^{m+m'+m''+1}$, where $\hat h=(\hat h_1'^T,\hat h_2'^T)^T$ and $h'_0=0$. For $s=j\omega$ and any $\hat h$ with $\varphi_{m+1}=0$ and remaining components $\varphi_i$ in $[-\pi,\pi]$ whose values depend on or $h_i$ $(i\leqslant m)$ or $h'_i$ $(i\geqslant m'+2)$. Similarly,

$$
\begin{aligned}
&N_{JM}(j\omega,\hat h)=\hat T'^{-1}_{JM}(j\omega,\varphi)=\hat T'^{-1}_{JM}(j\omega),\\
&\Delta N_{JM}(j\omega,\hat h)=\Delta\hat T'_{JM}(j\omega,\varphi)=\Delta\hat T_{JM}(j\omega),\\
&\tilde N_{JM}(j\omega,\hat h)=\hat{\tilde T}'^{-1}_{JM}(j\omega,\varphi)=\hat{\tilde T}'^{-1}_{JM}(j\omega),\\
&\tilde N(j\omega,\hat h)=\hat{\tilde T}'^{-1}(j\omega,\varphi)=\hat{\tilde T}^{-1}(j\omega)
\end{aligned}
$$

for the above $\hat h$ and $\varphi$. Note that $\hat T$, $\hat T'$ and $N^{-1}$ are distinct mathematical objects but, however, they take identical values for all pure imaginary $s=j\omega$ and a corresponding $\varphi_i\in[-\pi,\pi]$ such that $e^{-j\omega_i}=e^{\pm j\varphi_i}$ with $\varphi_{m+1}=h'_0=0$. The same applies for the related objects referred to the auxiliary system.

## 2. Problem statement

Consider the following linear and time-invariant impulsive system with delays:

$$
\begin{aligned}
\dot x(t)={}&\sum_{i=0}^{m}A_i x(t-h_i)+\sum_{i=0}^{m'}\int_0^t d\alpha_i(\tau)A_{\alpha_i}x(t-\tau-h'_i)\\
&+\sum_{i=m'+1}^{m'+m''}\int_{t-h_i}^{t} d\alpha_i(t-\tau)A_{\alpha_i}x(\tau)+\sum_{i\in\mathbf{I}}b_i\delta(t-t_i),
\end{aligned}
\tag{1}
$$

where $A_0$ and $A_i$, $A_{\alpha_k}$ $(i=1,2,\dots,m,\ k=0,1,\dots,m'+m'')$ belong to the spaces of unbounded and bounded operators, respectively, on a Banach space of $n$-vector real functions $x\in X$ endowed with the supremum norm where the vectors of point and distributed constant delays are $\hat h=(0,h_1,h_2,\dots,h_m)^T$ and $\hat h'=(\hat h_1'^T|\hat h_2'^T)^T=(0,h'_1,h'_2,\dots,h'_{m'}|h'_{m'+1},h'_{m'+2},\dots,h'_{m'+m''})^T$, respectively, with $h_i\geqslant 0$ and $h'_k\geqslant 0$ $(i=1,2,\dots,m'+m'')$ and $h_0=h'_0=0$, $A_0\equiv A$, $A_{\alpha_0}\equiv A_\alpha$ and $\alpha_0(\cdot)\equiv\alpha(\cdot)$. The functions $\alpha_i:[0,\infty)\to\mathbf{R}$ and $\alpha_k:[0,h'_k]\to\mathbf{R}$ are continuously differentiable real functions within their definition domains except possibly on sets of zero measure where the time-derivatives have bounded discontinuities. All or some of the $\alpha_i(\cdot)$ and $\alpha_k(\cdot)$ may be alternatively matrix functions $\alpha_i:[0,t]\to\mathbf{R}^{n\times n}$ for $t\in\mathbf{R}_+$ and $\alpha_i:[0,h'_k]\to\mathbf{R}^{n\times n}$. We will not do any explicit difference between both possibilities in the notation for the sake of simplicity. The impulsive input $v(t)=\sum_{i\in\mathbf{I}}b_i\delta(t-t_i)$ is built with the finite or infinite sequence of Dirac $\delta(t-t_i)$ impulses at the sequence of time instants $\{t_i,\ i\in\mathbf{I}\}$ with $t_{t+1}>t_i$ for some totally ordered proper or improper numerable subset $\mathbf{I}\subseteq\mathbf{N}$. If $\mathrm{Card}(\mathbf{I})=p<\infty$ then $v(t):=\sum_{i=1}^{p}b_i\delta(t-t_i)$ and $\mathbf{I}:=\{i\in\mathbf{N}\colon i\leqslant p\}$. Note that the system (1) is very general since it includes point-delayed dynamics, like, for instance, in typical war/peace models or the so-called Minorski' s problem appearing when controlling the lateral dynamics of a ship. It also includes real constants $h_i'$ $(i = 0, 1, \ldots, m')$, with $h_0' = 0$, associated with infinitely distributed delayed contributions to the dynamics through integrals, related to $\alpha_i(\cdot)$, $i = 0, 1, \ldots, m'$. Such delays are relevant, for instance, in viscoelastic fluids, electrodynamics and population growth [1,6,7]. In particular, an integro-differential Volterra's type term is also included through $h_0' = 0$. Apart from those delays, the action of finite distributed delays characterized by real constants $h_i'$ $(i = 0, 1, \ldots, m' + m'')$ is also included in (1). That kind of delays is well known, for instance, in econometric models related to production rate [7]. Finally, the impulsive input $v(t) = \sum_{i \in \mathbf{I}} b_i \delta(t - t_i)$ generates bounded discontinuities of the solution trajectory $x(t)$ at $t = t_i$ $(i \in \mathbf{I})$, see, for instance, [8–10]. The following technical hypothesis are made.

(H1) All the operators $A_k$ $(0 \leqslant k \leqslant m)$, $A_{\alpha_k}$ $(0 \leqslant k \leqslant m' + m'')$ are in $L(X) := L(X, X)$, the set of linear operators on $X$, of dual $X^*$, and $h_k$ and $h_\ell'$ $(k = 1, 2, \ldots, m$, $\ell = 0, 1, \ldots, m' + m'')$ are nonnegative constants with

$$h_0 = h_0' = 0 \quad \text{and} \quad h = \mathrm{Max}\Big(\operatorname*{Max}_{1 \leqslant i \leqslant m}(h_i), \operatorname*{Max}_{1 \leqslant i \leqslant m' + m''}(h_i')\Big).$$

(H2) The initial conditions of (1) are real $n$-vector functions $\phi \in C(h)$, where $C_e(h) := \{\phi = \phi_1 + \phi_2\colon\ \phi_1 \in C(h),\ \phi_2 \in B^0(h)\}$ with $C(h) := \{C^0([-h, 0]; X)\}$; i.e., the set of continuous mappings from $[-h, 0]$ into the Banach space $X$ with norm $\bar{\phi} := |\phi| = \mathrm{Sup}\{\|\phi(t)\|\colon -h \leqslant t \leqslant 0\}$; $\|\cdot\|$ denoting the Euclidean norm of vectors in $\mathbf{R}^n$ and matrices in $\mathbf{R}^{n \times n}$, and $B^0(h) := \{\phi : [-h, 0] \to X\}$ is the set of real bounded vector functions on $X$ endowed with the supremum norm having support of zero measure. Roughly speaking, $\phi \in B^0(h)$ if and only if it is almost everywhere zero except at isolated points of within $[-h, 0]$ where it is bounded. Thus, $\phi \in C_e(h)$ if and only if it is almost everywhere continuous in $[-h, 0]$ except possibly on a set of zero measure of bounded discontinuities. $C_e(h)$ is also endowed with the supremum norm since $\phi = \phi_1 + \phi_2$, some $\phi_1 \in C(h)$, $\phi_2 \in B^0(h)$ for each $\phi \in C_e(h)$. In the following, the supremum norms on $L(X)$ are also denoted with $|\cdot|$.

(H3) The linear operators $A_{\alpha_i} \in L(X)$, with abbreviated notation $A_{\alpha_0} = A_\alpha$, are closed and densely defined linear operators with respective domain and range $D(A_{\alpha_i})$ and $R(A_{\alpha_i}) \subset X$ $(i = 0, 1, \ldots, m' + m'')$. The functions $\alpha_i \in C^0([0, \infty); \mathbf{R}) \cap BV_{\mathrm{loc}}(\mathbf{R}_+)$ $(i = 0, 1, \ldots, m')$ and $\alpha_i \in C^0([-h, 0); \mathbf{R})$ $(i = 0, 1, \ldots, m' + m'')$ being everywhere differentiable with possibly bounded discontinuities on subsets of zero measure of their definition domains with $\int_0^\infty e^{vt} |d\alpha_i(t)| < \infty$ some nonnegative real constant $v$ $(i = 0, 1, \ldots, m')$. If $\alpha_i(\cdot)$ is a matrix function $\alpha_i : [0, \infty) \times X^* \to L(X, X^*)$ then it is in $C^0([0, \infty); \mathbf{R}^{n \times n}) \cap BV_{\mathrm{loc}}(\mathbf{R}_+^{n \times n})$ with $\int_0^\infty e^{vt} |d\alpha_i(t)| < \infty$ and its entries being everywhere time-differentiable with possibly bounded discontinuities on a subset of zero measure of their definition domains.

The integrability of the $\alpha_i(\cdot)$-functions (or matrix functions) on $[t - h_i', t]$, $m' + 1 \leqslant i \leqslant m' + m''$, follows since their definition domain is bounded. The technical hypothesis (H1)–(H3) guarantee the existence and uniqueness of the solution of the homogeneous system (1) (i.e., $v \equiv 0$) for each initial condition $\phi \in C_e(h)$. Take Laplace transforms in (1) by using the convolution theorem and the relations $d\alpha(\tau) = \dot{\alpha}(\tau)\, d\tau$. It follows that $d\hat{\alpha}_i(s) = s\hat{\alpha}_i(s) - \alpha_i(0)$, where $\hat{f}(s)$ denotes the Laplace transform of $f(t)$. Thus, one gets from (1),

$$\hat{x}(s) = \hat{T}(s)\left(x(0^+) + \sum_{i\in\mathbf{I}} b_i e^{-t_i s}\right), \tag{2}$$

where

$$\begin{aligned}\hat{T}(s) = \Bigg[ & s\left(I - \hat{\alpha}(s)A_\alpha - \sum_{i=1}^{m'} \hat{\alpha}_i(s)A_{\alpha_i} e^{-h_i' s} - \sum_{i=m'+1}^{m'+m''} \hat{\alpha}_i(s)A_{\alpha_i}\left(1 - e^{-h_i' s}\right)\right) \\ & - A - \sum_{i=1}^{m} A_i e^{-h_i s} + \alpha(0)A_\alpha + \sum_{i=1}^{m} \alpha_i(0)A_{\alpha_i} e^{-h_i' s} \\ & + \sum_{i=m'+1}^{m'+m''} \alpha_i(0)A_{\alpha_i}\left(1 - e^{-h_i' s}\right)\Bigg]^{-1}.\end{aligned} \tag{3}$$

Note that (1) is guaranteed to be g.u.e.s.i.d. if and only if $\hat{T}(s)$ exists within some region including properly the right-complex plane. In other words, if it is compact for $\operatorname{Re} s > -\alpha_0$, for some constant $\alpha_0 \in \mathbf{R}_+$, since then all the entries of its Laplace transform $T(t)$ decay with exponential rate on $[0, \infty)$ for $\phi \in C_e(h)$ and then $|x(t)|$ decays with exponential rate on $\mathbf{R}_+$. *The unique solution of the homogeneous* (1) *for each* $\phi \in C_e(h)$ *may be equivalently written in infinitely many cases by first rewriting* (1) *by considering different 'auxiliary' reference homogeneous systems plus additional terms considered as forcing actions. The next arrangements lead to conditions guaranteeing that the homogeneous system* (1) *is g.u.e.s.i.d. if it is g.u.e.s. in the absence of delay* (*i.e., for* $h = 0$). *Through this arrangement, it is not necessarily requested for* $\dot{z}(t) = Az(t)$, *which is in fact one of the possible auxiliary homogeneous systems for* (1), *to be g.u.e.s.i.d. for any* $\phi \in C_e(h)$. This is the main underlying idea focused on in this paper compared to previous results [1,5–8]. Thus, note that (1) may be written compactly as

$$\dot{x}(t) = Lx_t + v(t) = L_{JM}x_t + \left(\bar{L}_{JM}x_t + v(t)\right), \tag{4}$$

where $L = L_{JM} + \bar{L}_{JM}$ is a linear operator in $L(X)$ defined by $Lx_t$ equalizing by the unforced right-hand side of (1) where $x_t$ denotes the string $x : [t-h, t] \to X$ of the solution to (1) for $\phi \in C_e(h)$ for all $t \geqslant 0$; and $L_{JM}$ and $\bar{L}_{JM}$ are also linear operators in $L(X)$ which define a nonunique additive decomposition of $L$ that depends on $M$, an $n$-square arbitrary real matrix, and $J$, a triple $J = (J_1, J_2, J_3)$ of indices $J_i$ ($i = 1, 2, 3$). The $M$-matrix and the $J$-triple define the subsequent g.u.e.s.i.d. auxiliary system. That property is the starting point to derive conditions for the current delayed system (1) to be g.u.e.s.i.d. as well. The auxiliary system is

$$\begin{aligned}\dot{z}(t) = L_{JM}z_t = Mz(t) + (A - M)z(t) + \sum_{i\in J_1} A_i x(t - h_i) & \\ + \sum_{i\in J_2} \int_0^t d\alpha_i(\tau) A_\alpha x(t - h_i' - \tau) & \\ + \sum_{i\in J_3} \int_{t-h_i'}^t d\alpha_i(t - \tau) A_{\alpha_i} x(\tau) &\end{aligned} \tag{5}$$

subject to initial conditions $z(t)=\phi(t)$ for $t\in[-h,0]$ with $\phi\in C_e(h)$, some given matrix $M\in\mathbf{R}^{n\times n}$; and

$$\begin{aligned}
J_1 &= \big\{i\in\mathbf{N}\colon\ 1\leqslant i\leqslant m \text{ and } h_i \text{ is a point delay in } \dot z(t)=L_{JM}z_t\big\},\\
J_2 &= \big\{i\in\mathbf{N}\colon\ 1\leqslant i\leqslant m' \text{ and } h_i' \text{ is a constant defining an infinitely distributed delay}\\
&\qquad \text{in } \dot z(t)=L_{JM}z_t\big\},\\
J_3 &= \big\{i\in\mathbf{N}\colon\ m'+1\leqslant i\leqslant m'+m'' \text{ and } h_i' \text{ is a finitely distributed delay}\\
&\qquad \text{in } \dot z(t)=L_{JM}z_t\big\}
\end{aligned}\tag{6}$$

are respective proper or improper subsets of $N_1=\{1,2,\dots,m\}$, $N_2=\{0,1,\dots,m'\}$ and $N_3=\{m'+1,m'+2,\dots,m'+m''\}$ that define the $J$-triple. $\bar J_i=N_i/J_i$ denotes the complement of $J_i$ in $N_i$ $(i=1,2,3)$. Then $i\in J_1$ if and only if the point delay $h_i$ is explicit in the auxiliary system (5) and $i\in J_{2,3}$ if and only if the distributed delay $h_i'$ is explicit in (5). In particular, $J_i=\Phi$ (the empty set) for some $i\in\{1,2,3\}$ if there is no delay of the corresponding class in (5). Thus, (1) may be compactly rewritten as

$$\dot x(t)=Lx_t+v(t)=L_{JM}x_t+\big(\bar L_{JM}x_t+v(t)\big)\tag{7a}$$

with $x(t)=\phi(t)$ for $t\in[-h,0)$, $\phi\in C_e(h)$, where

$$\begin{aligned}
Lx_t &= Ax(t)+\sum_{i=1}^{m}A_ix(t-h_i)+\sum_{i=0}^{m}\int_0^t d\alpha_i(\tau)A_{\alpha_i}x(t-\tau-h_i')\\
&\quad+\sum_{i=m'+1}^{m'+m''}\int_{t-h_i'}^{t} d\alpha_i(\tau)A_{\alpha_i}x(\tau),
\end{aligned}\tag{7b}$$

$$\begin{aligned}
L_{JM}x_t &= Mx(t)+\sum_{i\in J_1}A_ix(t-h_i)+\sum_{i\in J_2}\int_0^t d\alpha_i(\tau)A_{\alpha_i}x(t-\tau-h_i')\\
&\quad+\sum_{i\in J_3}\int_{t-h_i'}^{t} d\alpha_i(t-\tau)A_{\alpha_i}x(\tau),\quad t\geqslant 0,
\end{aligned}\tag{7c}$$

$$\begin{aligned}
\bar L_{JM}x_t &= (L-L_{JM})x(t)\\
&= (M-A)x_t+\sum_{i\in\bar J_1}A_ix(t-h_i)+\sum_{i\in\bar J_2}\int_0^t d\alpha_i(\tau)A_{\alpha_i}x(t-\tau-h_i')\\
&\quad+\sum_{i\in\bar J_3}\int_{t-h_i'}^{t} d\alpha_i(t-\tau)A_{\alpha_i}x(\tau),\quad t\geqslant 0.
\end{aligned}\tag{7d}$$

In view of (7), the unique solution of (1) for any $\phi\in C_e(h)$ is

$$\begin{aligned} x(t,\phi) &= T(t)\phi(0^+) + \sum_{i=1}^{m} \int_{-h_i}^{0} T(t-\tau)\phi(\tau)\,d\tau \\ &\quad + \sum_{i=1}^{m'+m''} \int_{-h_i'}^{0} T(t-\tau)\phi(\tau)\,d\tau + \sum_{i\in\mathbf{I}} T(t-t_i)b_i U(t-t_i) \qquad \text{(8a)} \\ &= T_{JM}(t)\phi(0^+) + \sum_{i\in J_1} \int_{-h_i}^{0} T_{JM}(t-\tau)\phi(\tau)\,d\tau \\ &\quad + \sum_{i\in J_2\cup J_3} \int_{-h_i'}^{0} T_{JM}(t-\tau)\phi(\tau)\,d\tau \\ &\quad + \int_0^t T_{JM}(t-\tau)\Bigg[(A-M)x(\tau) + \sum_{i\in \bar{J}_1} A_i x(\tau-h_i) \\ &\quad + \sum_{i\in \bar{J}_2} \int_0^{\tau} T_{JM}(\tau')\dot{\alpha}_i(\tau')A_{\alpha_i} x(\tau-\tau'-h_i')\,d\tau' \\ &\quad + \sum_{i\in \bar{J}_3} \int_{\tau-h_i'}^{\tau} T_{JM}(\tau-\tau')\dot{\alpha}_i(\tau-\tau')A_{\alpha_i} x(\tau')\,d\tau'\Bigg]\,d\tau \\ &\quad + \sum_{i\in\mathbf{I}} T(t-t_i)b_i U(t-t_i), \qquad \text{(8b)} \end{aligned}$$

where $T(t)$ satisfies $\dot{T}(t) = LT_t$ for $t > 0$ with $T(0) = I$ (the $n$-identity matrix) and $T(t) = 0$ for $t < 0$ with $T(t)$ being the inverse Laplace transform of $\hat{T}^{-1}(s)$, $\hat{T}(s)$ defined in (3), and $T_{JM}(t)$ satisfies $\dot{T}_{JM}(t) = L_{JM}(T_{JM})_t$ for $t > 0$ with $T_{JM}(0) = I$ and $T_{JM}(t) = 0$ for $t < 0$. $U(t) = 1(t)$ is the unity Heaviside function. Thus, $T_{JM}(t)$ is the inverse Laplace transform of the holomorphic (where it exists) mapping $\hat{T}_{JM}(s)$ with

$$\begin{aligned} \hat{T}_{JM}(s) &= \Bigg[ s\Bigg(I - \sum_{i\in J_2} \hat{\alpha}_i(s)A_{\alpha_i}e^{-h_i's} - \sum_{i\in J_3} \hat{\alpha}_i(s)A_{\alpha_i}(1-e^{-h_i's})\Bigg) \\ &\quad - M - \sum_{i\in J_1} A_i e^{-h_i s} + \sum_{i\in J_2} \alpha_i(0^+)A_{\alpha_i} \\ &\quad + \sum_{i\in J_3} \alpha_i(0^+)A_{\alpha_i}(1-e^{-h_i's}) \Bigg]^{-1}. \qquad \text{(9)} \end{aligned}$$

Note that $T(t)$ ($\equiv T_{JM}(t)$ if $J = (N_1, N_2, N_3)$) and $T_{JM}(t)$ for any $J$-triple are $C_0$-semigroups on $C_e(h)$ of operators of $L(X)$. In particular, if $J_i = \Phi$ ($i = 1, 2, 3$) then $L_{JM}z_t = Mz(t)$ and $T_{JM}(t) = e^{At}$ is an analytic semigroup if $J_1$ and $J_3$ are empty and

$J_2 = \{0\}$ (i.e., $h_0' = 0$) is the unique contribution to a Volterra-type integral term then $L_{JM} z_t = Mz(t) + \int_0^t d\alpha(\tau) A_\alpha x(t-\tau)$ and $T_{JM}(t)$ is a transition operator if $\hat{T}_{JM}(s)$ is compact for $\mathrm{Re}\, s > -\gamma_{JM}$ ($\gamma_{JM} \in \mathbf{R}_+$).

**Remark 1.** Note that the compactness of the operator-valued functions $\hat{T}(s)$ and $\hat{T}_{JM}(s)$ for all $\mathrm{Re}\, s > -\gamma$ and $\mathrm{Re}\, s > -\gamma_{JM}$, some $\gamma \in \mathbf{R}_+$ and $\gamma_{JM} \in \mathbf{R}_+$, respectively, if $\dot{x}(t) = Lx_t$, $\dot{z}(t) = Lx_t$, respectively, are g.u.e.s.i.d for all $\phi \in C_e(h)$ holds directly if they are bounded provided that $X$ is considered as a Hilbert space endowed with the usual inner product norm. The stability properties of the operator-valued function $T : [0, \infty) \to L(X)$ are independent of the use of any of both alternative formal characterizations. Thus, if $X$ is a Hilbert space, then there exist dense injective mappings $X \to X^*$ (dual of $X$) $\to X^{**}$ (dual of $X^*$) $\equiv X$, instead of the generic result which may include in some cases proper inclusion $X^{**} \supset X \neq X^{**}$ so that $X$ is a reflexive linear space and any operator in $L(X^{**}, X)$ ($\equiv L(X,X) = L(X)$ in this case) is compact if and only if it is completely continuous (i.e, if it maps any weakly convergent sequence into a strongly convergent one with respect to the norm topology). Thus, $\hat{T}(s)$ is compact (or completely continuous) where it exists since $(\hat{T})^* \hat{T}$ is bounded for $\mathrm{Re}\, s > -\gamma$. The same property holds for any $\hat{T}_{JM}$ for $\mathrm{Re}\, s > -\gamma_{JM}$.

Note that $\hat{T}(s) = [\hat{T}_{JM}(s) - \Delta\hat{T}_{JM}(s)]^{-1} = \hat{T}_{JM}(s)[I - \hat{T}_{JM}^{-1}(s)\Delta\hat{T}_{JM}(s)]^{-1}$ in the definition domain of $\hat{T}_{JM}$ for any auxiliary system defined from some given $J$-triple. The following special cases are of interest.

*Case* 1. The auxiliary system is delay-free. $J = (J_1, J_2, J_3)$ with $J_i = \Phi$ $(i = 1, 2, 3)$ so that the auxiliary system is $\dot{z}(t) = Mz(t)$. This is the case usually treated in the literature (see, for instance, [1,7]). Thus, $\bar{J}_i = N_i$ $(i = 1, 2, 3)$ and $T_{JM}(t) = e^{Mt}$ is an analytic semigroup.

*Case* 2. The auxiliary system is subject to delay-free dynamics and all point delays. $J_1 = N_1$ and $J_2 \cup J_3 = \Phi$ so that $\bar{J}_1 = \Phi$ and $\bar{J}_i = N_i$ $(i = 2, 3)$. Then, $\dot{z}(t) = Mz(t) + \sum_{i=1}^m A_i z(t - h_i)$ with initial conditions $z(t) = \phi(t)$, $\phi \in C_e(h)$, for $t \in [\max_{1 \leqslant i \leqslant 0}(-h_i), 0]$ so that $\dot{T}_{JM}(t) = MT_{JM}(t) + \sum_{i=1}^m A_i T_{JM}(t - h_i)$ with $T_{JM}(0) = I$ and $T_{JM}(t) = 0$ $(t < 0)$ yields a unique solution

$$T_{JM}(t) = e^{Mt}\left(I + \sum_{i=1}^m \int_{h_i}^t e^{-M\tau} A_i T_{JM}(\tau - h_i)\, d\tau\right) \quad \text{for } t \geqslant 0.$$

*Case* 3. The auxiliary system is subject to delay-free dynamics and Volterra-integral type dynamics. $J_1 \cup J_3 = \Phi$, $\bar{J}_i = N_i$ and $\bar{J}_2 = \{1, 2, \ldots, m'\}$. Thus, $\dot{z}(t) = Mz(t) + \int_0^t d\alpha(\tau) A_\alpha z(t - \tau)$. In particular, $T_{JM}(t)$ is ensured to be a transition operator with $|T_{JM}(t)| \leqslant Ke^{-\rho t}$ for some positive real constants $K \geqslant 1$ and $\rho$ and all $t \geqslant 0$ (see, for instance, [3,6]), if $\hat{T}_{JM}^{-1}(s) = [s(I - \hat{\alpha}(s)A_\alpha) + \alpha(0)A_\alpha - M]^{-1}$ is compact for $\mathrm{Re}\, s > -\rho$, any real constant $\rho < \gamma_{JM}$, and

$$\left|\frac{d^i(\hat{T}_{JM}^{-1}(s))}{ds^i}\right| < \left|\frac{K}{(s+\rho)^{i-1}}\right| \quad (\text{for } i = 1, 2, 3).$$

*Case* 4. The auxiliary system has delay-free dynamics and all the infinitely distributed delays. Now, $J = (J_1, J_2, J_3)$ with $J_i = \Phi$ $(i = 1, 3)$ and $J_2 = N_2$ so that $\bar{J}_i = N_i$ $(i = 1, 3)$ and $\bar{J}_2 = \Phi$, which leads to

$$\dot{z}(t) = Mz(t) + \sum_{i=0}^{m} \int_0^t d\alpha_i(\tau) z(t - \tau - h_i')$$

under initial conditions $\phi \in C_e(h)$. Thus, one gets

$$\dot{T}_{JM}(t) = MT_{JM}(t) + \sum_{i=0}^{m} \int_0^t d\alpha_i(\tau) A_{\alpha_i} T_{JM}(t - \tau - h_i')$$

for $t > 0$ with $T_{JM}(0) = I$; $T_{JM}(t) = 0$ for $t < 0$, whose unique solution for all $t > 0$ is

$$T_{JM}(t) = e^{Mt}\left(I + \sum_{i=0}^{m} \int_0^t \int_0^\tau e^{-M\tau} d\alpha_i(\tau) A_{\alpha_i} T_{JM}(\tau - \tau' - h_i')\, d\tau'\right).$$

*Case* 5. The auxiliary system has delay-free dynamics and all the finitely distributed delays. Now, $J = (J_1, J_2, J_3)$ with $J_i = \Phi$ $(i = 1, 2)$ and $J_3 = N_3$ so that $\bar{J}_i = N_i$ $(i = 1, 2)$ and $\bar{J}_3 = \Phi$. Under the same initial conditions as in the above case, one gets

$$\dot{z}(t) = Mz(t) + \sum_{i=m'+1}^{m'+m''} \int_{t-h_i}^{t} d\alpha_i(t - \tau) A_{\alpha_i} z(\tau) \quad \text{for } t > 0$$

which is also satisfied by the transition operator of the auxiliary system whose unique solution under the same initial conditions as in Case 4 is

$$T_{JM}(t) = e^{Mt}\left(I + \sum_{i=m'+1}^{m'+m''} \int_0^t \int_{\tau-h_i'}^{\tau} e^{-M\tau} d\alpha_i(\tau - \tau') A_{\alpha_i} T_{JM}(\tau')\, d\tau'\right).$$

## 3. Uniform stability of the homogeneous system

**Theorem 1.** *Assume that* (1) *is g.u.e.s. for* $\hat{h} = 0$ *and that* $\dot{z}(t) = L_{JM} z_t$ *is g.u.e.s.i.d. for all* $\phi \in C_e(h)$. *Thus, the homogeneous equation* (1), $\dot{x}(t) = L_{JM} x_t$ *is g.u.e.s.i.d.* (*i.e., for all* $\phi \in C_e(h)$) *if and only if the operator-valued function*

$$\hat{\hat{T}}_{JM}^{-1}(j\omega, \varphi) = \left(I - \hat{T}_{JM}'(j\omega, \varphi) \Delta \hat{T}_{JM}'(j\omega, \varphi)\right)^{-1} \tag{10}$$

*exists for all real* $\omega \in (0, \infty)$ *and all* $\varphi_{ik_i} \in [-\pi, \pi]$ $(i = 1, 2, 3)$ $(\varphi_{21} = 0)$; $k_i = 1, 2, \ldots, p$ *with* $p = m$ *if* $i = 1$, $p = m' + 1$ *if* $i = 2$ *and* $p = m''$ *if* $i = 3$.

**Proof.** First note that the argument $\omega = 0$ for the above operator-valued function is excluded from the conditions since (1) is g.u.e.s. for $\hat{h} = 0$. The system (1) is g.u.e.s.i.d. if and only if $N^{-1}(s, \hat{h})$ exists for $\operatorname{Re} s > -\gamma$ (some $\gamma \in \mathbf{R}_+$) for any sets of delays.

Since $N_{JM}^{-1}(s)(s,\hat h)N(s,\hat h) = [I - N_{JM}^{-1}(s,\hat h)\Delta N_{JM}(s,\hat h)]$ and $N_{JM}(s,\hat h)$ has an inverse $\operatorname{Re} s > -\gamma_{JM}$ (some $\gamma_{JM} \in \mathbf{R}_+$) for all the sets of delays explicit in the auxiliary system (5), $N^{-1}(s,\hat h)$ exists for a pair $(s,\hat h)$ if and only if $\tilde N^{-1}(s,\hat h)$ exists for $(s,\hat h)$, where

$$\tilde N_{JM}(s,\hat h) = N_{JM}^{-1}(s,\hat h)N(s,\hat h) = \left[I - N_{JM}^{-1}(s,\hat h)\Delta N_{JM}(s,\hat h)\right].$$

*Proof of necessity*. The rank condition cannot fail for $\omega = 0$ since the system (1) is g.u.e.s. and then $N(j_0,\hat h)$ is full rank for any set of delays. Assume that

$$\operatorname{rank}\left[\hat{\tilde T}'_{JM}(j\omega,\varphi)\right] < n \quad \text{for some } \omega \neq 0;$$

then $\operatorname{rank}[\tilde N_{JM}(j\omega,\hat h)] < n$, and the set of delays $h_i = \varphi_i/\omega$, where $\varphi_i$ is the $i$th component of $\varphi$. This is a contradiction and necessity follows.

*Proof of sufficiency*. Since $\hat T(s)$, $\hat T_{JM}(s)$ (and $N^{-1}(s,\hat h)$) are compact wherever they exist, any possible singularities of $\hat T(s)$ and $\hat T_{JM}(s)$ are poles [1,3]. Since $N_{JM}(s,\hat h)$ has an inverse for $\operatorname{Re} s > -\gamma_{JM}$, then, if $\tilde N_{JM}(s,\hat h)$ has an inverse in $\operatorname{Re} s > -\gamma'_{JM}$, it follows that the operator-valued function $N(s,\hat h)$ has an inverse in $\operatorname{Re} s > \operatorname{Min}(-\gamma_{JM}, -\gamma'_{JM})$ by construction of $N(s,\hat h)$. $\hat T(s)$ has a pole with $\operatorname{Re} s_0 > \operatorname{Min}(-\gamma_{JM}, -\gamma'_{JM})$ if and only if the operator-valued function $N(s,\hat h)$ has an eigenvalue one at $s = s_0$ so that $N_{JM}^{-1}(s_0,\hat h)$ and $\tilde N_{JM}(s_0,\hat h)$ and $N(s_0,\hat h)$ are not full rank. Define the vector function of delays $f(\hat h) := \operatorname{Sup}(\operatorname{Re} s\colon\ \tilde N_{JM}(s,\hat h)$ has an eigenvalue one). This function is continuous on its definition domain $\mathbf{R}_+^0$. Since the delay-free system (1) is g.u.e.s. then $f(0) < 0$ so that (1) is not (is) g.u.e.s.i.d. if and only if $f(\hat h_0) > 0$ for some vector of delays $\hat h_0$ with $h_{0,m+1} = 0$ and $h_{0,i} > 0$ for $i \neq m+1$ (if and only if $f(\hat h) < 0$ for all $\hat h$ with $h_{0,m+1} = 0$ and $h_{0,i} > 0$ for $i \neq m+1$). Furthermore, there is a domain properly included in $\mathbf{R}_+^0$ such that $\hat{\tilde T}'_{JM}(j\omega,\varphi_0)$ has an eigenvalue one if $f(\hat h_0) = 0$ since $\tilde N_{JM}(j\omega,\hat h_0)$ is not full rank for some real $\omega$, where $\varphi_{0i} = \omega h_{0i}$ $(i = 1,2,\dots,m+m'+1)$ with $h_{0,m+1} = 0$. But then, from the definition of $\hat T'(j\omega,\varphi_0)$, there always exists $[-\pi,\pi]$ such that the ranks of $\hat T(j\omega,\varphi_0)$ and $\hat T'_{JM}(j\omega,\varphi_0)$ are less than $n$ and the result has been proved. Note that the test for negative $\omega$ is unnecessary since eventual complex poles appear in conjugate pairs. □

Theorem 1 is now used to obtain stability results for the special cases of auxiliary systems in Section 2.

*Corollaries*

Assume, in the following corollaries, that $\dot x(t) = Lx_t$ is g.u.e.s. for $\hat h = 0$ for all $\phi \in C_e(h)$.

**Corollary 1** (Auxiliary system involving delay-free dynamics). *If $M$ is strictly Hurwitzian then $\dot x(t) = Lx_t$ is g.u.e.s. for all $\hat h \in [0,\infty)$, i.e., g.u.e.s.i.d. if and only if the operator-valued function* (10) *exists, where*

$$\hat T'_{JM}(j\omega) = (j\omega I - M)^{-1},$$

$$\Delta\hat T'_{JM}(j\omega,\varphi) = M - A + \sum_{i=1}^{m} A_i e^{j\varphi_{1i}}$$

$$+ j\omega\left(\sum_{i=0}^{m'}\hat{\alpha}_i(j\omega)A_{\alpha_i}e^{j\varphi_{2i}} + \sum_{i=m'+1}^{m'+m''}\hat{\alpha}_i(j\omega)A_{\alpha_i}(1-e^{j\varphi_{3i}})\right)$$
$$+ \left(\sum_{i=0}^{m'}\alpha_i(0)A_{\alpha_i}e^{j\varphi_{2i}} + \sum_{i=m'+1}^{m'+m''}\alpha_i(0)A_{\alpha_i}(1-e^{j\varphi_{3i}})\right).$$

**Corollary 2** (Auxiliary system involving delay-free and point-delayed dynamics). *If $\dot{z}(t) = L_{JM}z_t = L_{JM}z_t \equiv Mz(t) + \sum_{i=1}^{m} A_i z(t-h_i)$ is g.u.e.s.i.d. for some given real square n-matrix $M$, then $\dot{x}(t) = Lx_t$ is g.u.e.s.i.d. if and only if* (10) *exists, where*

$$\hat{T}'_{JM}(j\omega,\varphi) = \left[j\omega\bar{I} - M - \sum_{i=1}^{m} A_i e^{j\varphi_{1i}}\right]^{-1},$$
$$\Delta\hat{T}'_{JM}(j\omega,\varphi) = M - A$$
$$+ j\omega\left(\sum_{i=0}^{m'}\hat{\alpha}_i(j\omega)A_{\alpha_i}e^{j\varphi_{2i}} + \sum_{i=m'+1}^{m'+m''}\hat{\alpha}_i(j\omega)A_{\alpha_i}(1-e^{j\varphi_{3i}})\right)$$
$$+ \left(\sum_{i=0}^{m'}\alpha_i(0)A_{\alpha_i}e^{j\varphi_{2i}} + \sum_{i=m'+1}^{m'+m''}\alpha_i(0)A_{\alpha_i}(1-e^{j\varphi_{3i}})\right).$$

**Corollary 3** (Auxiliary system involving delay-free and convolution Volterra-type dynamics). *If $\dot{z}(t) = L_{JM}z_t \equiv Mz(t) + \int_0^t d\alpha(\tau)A_\alpha z(t-\tau)$ is g.u.e.s. for all bounded $z(0) \in \mathbf{R}^n$ for some given real square n-matrix $M$, then $\dot{x}(t) = Lx_t$ is g.u.e.s.i.d. if and only if* (10) *exists, where*

$$\hat{T}'_{JM}(j\omega,\varphi) = \left[j\omega\big(I - \hat{\alpha}(j\omega)A_\alpha\big) - M + \alpha(0)A_\alpha\right]^{-1},$$
$$\Delta\hat{T}'_{JM}(j\omega,\varphi) = M - A + \sum_{i=1}^{m} A_i e^{j\varphi_{1i}}$$
$$+ j\omega\left(\sum_{i=1}^{m'}\hat{\alpha}_i(j\omega)A_{\alpha_i}e^{j\varphi_{2i}} + \sum_{i=m'+1}^{m'+m''}\hat{\alpha}_i(j\omega)A_{\alpha_i}(1-e^{j\varphi_{3i}})\right)$$
$$+ \left(\sum_{i=1}^{m'}\alpha_i(0)A_{\alpha_i}e^{j\varphi_{2i}} + \sum_{i=m'+1}^{m'+m''}\alpha_i(0)A_{\alpha_i}(1-e^{j\varphi_{3i}})\right).$$

**Corollary 4** (Auxiliary system involving delay-free and infinitely-distributed delayed dynamics). *If $\dot{z}(t) = L_{JM}z_t \equiv Mz(t) + \sum_{i=0}^{m'}\int_0^t d\alpha_i(\tau)A_{\alpha_i}z(t-\tau-h'_i)$ is g.u.e.s. for all bounded $\phi \in C_e(h)$ for some given real square n-matrix $M$, then $\dot{x}(t) = L_{x_t}$ is g.u.e.s.i.d. if and only if* (10) *exists, where*

$$\hat{T}'_{JM}(j\omega,\varphi) = \left[j\omega\left(I - \sum_{i=0}^{m'}\hat{\alpha}_i(j\omega)A_{\alpha_i}e^{j\varphi_{2i}}\right) - M\right]^{-1},$$

$$\Delta\hat{T}'_{JM}(j\omega,\varphi) = M - A + \sum_{i=1}^{m} A_i e^{j\varphi_{1i}} + j\omega\left(\sum_{i=m'+1}^{m'+m''} \hat{\alpha}_i(j\omega)A_{\alpha_i}\big(1 - e^{j\varphi_{3i}}\big)\right) + \left(\sum_{i=m'+1}^{m'+m''} \alpha_i(0)A_{\alpha_i}\big(1 - e^{j\varphi_{3i}}\big)\right).$$

**Corollary 5** (Auxiliary system involving delay-free and finitely-distributed delayed dynamics). *If* $\dot{z}(t) = L_{JM}z_t \equiv Mz(t) + \sum_{i=m'+1}^{m'+m''} \int_{t-h'_i}^{t} d\alpha_i(t-\tau)A_{\alpha_i}z(\tau)$ *is g.u.e.s. for all bounded* $\phi \in C_e(h)$ *for some given real square* $n$*-matrix* $M$*, then* $\dot{x}(t) = Lx_t$ *is g.u.e.s.i.d. for all bounded* $\phi \in C_e(h)$ *if and only if* (10) *exists, where*

$$\hat{T}'_{JM}(j\omega,\varphi) = \left[ j\omega\left(I - \sum_{i=m'+1}^{m'+m''} \hat{\alpha}_i(j\omega)A_{\alpha_i}\big(1 - e^{j\varphi_{3i}}\big)\right) - M\right]^{-1},$$

$$\Delta\hat{T}'_{JM}(j\omega,\varphi) = M - A + \sum_{i=1}^{m} A_i e^{j\varphi_{1i}} + j\omega\left(\sum_{i=0}^{m} \hat{\alpha}_i(j\omega)A_{\alpha_i}e^{j\varphi_{2i}}\right) + \left(\sum_{i=0}^{m} \alpha_i(0)A_{\alpha_i}e^{j\varphi_{2i}} + \sum_{i=m'+1}^{m'+m''} \alpha_i(0)A_{\alpha_i}\big(1 - e^{j\varphi_{3i}}\big)\right).$$

The global uniform exponential stability of (1) may be investigated provided that each group of delayed dynamics (like, for instance, all point delays, infinitely distributed delays or finitely distributed ones) is successively introduced in the system as addressed as follows. Note, for instance, that the system with combined delay-free and point-delayed dynamics $\dot{z}(t) = L_{JM}z_t \equiv Az(t)\sum_{i=1}^{m} A_i z(t-h_i)$ is g.u.e.s.i.d. for all bounded $\phi \in C_e(h)$ if and only if $(I - (j\omega I - A)^{-1}\sum_{i=1}^{m} A_i e^{j\varphi_{1i}})^{-1}$ exists for all $\omega \in (0,\infty)$ and all $\varphi_{1i} \in [-\pi,\pi]$, $i = 1,2,\ldots,m$, provided that $A$ is strictly Hurwitzian (i.e., if the undelayed auxiliary system is g.u.e.s. so that Corollary 1 holds).

**Corollary 6** (Delay-free, point-delayed and infinitely distributed-delayed dynamics). *If* $\dot{z}(t) = L_{JM}z_t \equiv Az(t) + \sum_{i=1}^{m} A_i z(t-h_i) + \sum_{i=0}^{m'} \int_0^t d\alpha_i(\tau)A_{\alpha_i}z(t-\tau-h'_i)$ *is g.u.e.s.i.d. for all bounded* $\phi \in C_e(h)$ *if and only if*

$$\left[ I - (j\omega I - A)^{-1}\left(\sum_{i=1}^{m} A_i e^{j\varphi_{1i}} + \sum_{i=0}^{m} \big(j\omega\hat{\alpha}_i(j\omega) - \alpha_i(0)\big)e^{j\varphi_{2i}}\right)\right]^{-1}$$

*exists for all* $\omega \in (0,\infty)$ *and all* $\varphi_{ki} \in [-\pi,\pi]$, $k_i \in N_i$, $i = 1,2$, *provided that* $A$ *is strictly Hurwitzian* (*i.e., provided that the undelayed auxiliary system is g.u.e.s. so that Corollary* 1 *holds with* $M = A$). *It is also true that* $\dot{z}(t) = L_{JM}z_t$ *is g.u.e.s.i.d. for all bounded* $\phi \in C_e(h)$ *if and only if*

$$\left(I - \left(j\omega I - A - \sum_{i=1}^{m} A_i e^{j\varphi_{1i}}\right)^{-1}\right)\left[\sum_{i=0}^{m'} \big(j\omega\hat{\alpha}_i(j\omega) - \alpha_i(0)\big)e^{j\varphi_{2i}}\right]^{-1}$$

*exists for all* $\omega \in (0, \infty)$, $\varphi_{21} = 0$ *and all* $\varphi_{ki} \in [-\pi, \pi]$, $k_i \in N_i$, $i = 1, 2$, *provided that* $A$ *is strictly Hurwitzian* (*i.e., provided that the auxiliary system with both undelayed and point delayed dynamics is g.u.e.s. so that Corollary* 2 *holds with* $M = A$).

We might proceed in that way by giving conditions that ensure that each added group of delays maintains the uniform stability independent of delay provided that it was g.u.e.s.i.d. before adding those delays. It is also interesting to derive conditions for losing or ensuring uniform stability dependent of delay as follows.

**Theorem 2.** *Assume that* (1) *is g.u.e.s. for* $\hat{h} = 0$. *Thus,* $\dot{x}(t) = Lx_t$ *is not* (*respectively, is*) *g.u.e.s. for all sets of delays that satisfy simultaneously* $h_i = \varphi_{1i}/\omega$, $h'_k = \varphi_{2k}/\omega$ *and* $h'_\ell = \varphi_{3k}/\omega$ *with* $\varphi_{21} = 0$ *for some* $\omega \in \mathbf{R}_+$, $\varphi_{ik} \in [-\pi, \pi]$, $i = 1, 2, \ldots, m$, $k = 1, 2, \ldots, m'$ *and* $\ell = m' + 1, m' + 2, \ldots, m' + m''$ *provided that* (10) *does not exist* (*respectively, exists*).

**Proof** (Outline). It follows directly since for such sets of delays, the proof of Theorem 1 fails since there is some pole of $\hat{T}(s)$, so that it is not holomorphic, on $\mathrm{Re}\, s \geqslant 0$ since $\hat{T}'(j\omega, \varphi)$ has not an inverse for some $\omega \in \mathbf{R}_+$ and $\varphi = (\varphi_1^T, \varphi_2^T, \varphi_3^T)^T$ of components in the real interval $[-\pi, \pi]$. □

## 4. Uniform stability under impulsive forcing terms

The stability under impulsive forcing terms in (1) may be formulated under a direct extension of the basic results of Section 3 as follows.

**Theorem 3.** *Assume that* $\dot{x}(t) = Lx_t$ *is g.u.e.s.i.d., which holds if Theorem* 1 *holds with* $\hat{T}'(j\omega, \varphi) = [I, -\hat{T}'_{JM}(j\omega, \varphi)\Delta\hat{T}'_{JM}(j\omega, \varphi)]^{-1}\hat{T}'_{JM}(j\omega, \varphi)$ *existing within some appropriate domain with* $\hat{T}'_{JM}(j\omega, \varphi)$ *defining any g.u.e.s.i.d. auxiliary system defined for some* $J$*-triple, and thus* $\Delta\hat{T}'_{JM}(j\omega, \varphi) = \hat{T}'^{-1}_{JM}(j\omega, \varphi) - \hat{T}'^{-1}(j\omega, \varphi)$. *Assume also that the forcing impulsive vector function* $v : [0, \infty) \to \mathbf{R}^n$ *satisfies* $|b_i| \leqslant K_i e^{-i\rho}$ *with* $t_{i+1} - t_i \geqslant T_{\min} \geqslant (\rho - \rho')/\gamma$, *some real constant* $\rho' \in (0, \rho)$ *and* $K_i \in \mathbf{R}_+$ *being bounded constants for all* $i \in \mathbf{I}$. *Thus, the solution of* (1), $x(t, \phi)$ *is bounded on* $\mathbf{R}_+$ *and* $x(t, \phi) \to 0$ *exponentially as* $t \to \infty$ *for any* $\phi \in C_e(h)$.

**Proof.** Let $x_0(t, \phi)$ the unique solution of the homogeneous $\dot{x}(t) = Lx_t$ for $t \geqslant 0$ for any given $\phi \in C_e(h)$. Thus, the unique solution $x(t, \phi)$ for $t \geqslant 0$ for identical $\phi \in C_e(h)$ of the forced $\dot{x}(t) = Lx_t + u(t)$, with $v(t) = \sum_{i \in \mathbf{I}} b_i e^{-(t-t_i)}$, is bounded $\mathbf{R}_+$ on and satisfies

$$\|x(t, \phi) - x_0(t, \phi)\| \leqslant \left\| \sum_{i \in \mathbf{I}} T(t - t_i) b_i \right\| \leqslant \left| \sum_{i \in \mathbf{I}} K e^{-\gamma(t - t_i)} b_i \right|$$

since $\|T(t)\| \leqslant K e^{-\gamma t}$ ($\gamma \in \mathbf{R}_+$). If $\mathrm{Card}(\mathbf{I}) < \infty$ then $x(t)$ is bounded and $x(t) \to 0$ exponentially as $t \to \infty$ if $x_0(t) \to 0$ exponentially as $t \to \infty$. It only remains to consider the case when $\mathrm{Card}(\mathbf{I}) = \infty$. Since $|b_i| \leqslant K_i e^{-j\rho}$, $\gamma t_i - i\rho \geqslant -i\rho'$ for some real $\rho' \in (0, \rho)$

since $t_i = \sum_{k=1}^{i} T_k \geqslant i T_{\min} \geqslant i(\rho - \rho')/\gamma$ with $T_i \geqslant T_{\min} = (\rho - \rho')/\gamma$, $\forall i \in \mathbf{I}$. Thus, since $\sum_{i=0}^{\infty} e^{-i\rho} < \infty$,

$$\left\| x(t_k, \phi) - x_0(t_k, \phi) \right\| \leqslant K\bar{K} \sum_{i=0}^{k} e^{-i\varphi} e^{-\gamma(t_k - t_i)} \leqslant K\bar{K} \sum_{i=0}^{\infty} e^{-\gamma t_k} e^{-i(\rho' - \rho)}$$
$$\leqslant K\bar{K} e^{-\gamma t_k} \frac{\varepsilon}{1-\varepsilon} \to 0$$

as $t_k \to \infty$ and $x \in L_\infty([0,\infty); \mathbf{R}^n)$, with $\varepsilon = e^{-(\rho-\rho')} < 1$, $\forall \phi \in L_\infty([0,\infty); \mathbf{R}^n)$. Then, it is also exponentially continuous over $\mathbf{I}$. Since the solution $x(t,\phi)$ of (1) is continuous over the finite intervals of nonzero measures $[t_k, t_{k+1})$, $k \in \mathbf{I}$, it cannot diverge within such intervals. Thus, $x(t,\phi)$ is bounded and converges exponentially to zero as $t \to \infty$. □

**Theorem 4.** *Assume that the homogeneous* (1) *is g.u.e.s.i.d. for all $\phi \in C_e(h)$. Thus, it is g.u.e.s.i.d. for all $\phi \in C_e(h)$ and any impulsive $v(t) = \sum_{i \in \mathbf{I}} b_i \delta(t - t_i)$ with* Card($\mathbf{I}$) *being finite or infinite if any of the subsequent conditions hold for all $i\,(\geqslant i_0) \in \mathbf{I}$, some arbitrary $i_0 \in \mathbf{N}$,*

(i) $\displaystyle \operatorname*{Min}_{i \in \mathbf{I}} (t_{\ell+1} - t_\ell) \geqslant T_{\min} > \frac{1}{i\gamma} \sum_{k=1}^{i} \ln\left(\|I + B_k\|\right)$;

(ii) *If* $T_0 = t_1$ *and* $T_i = t_{t+1} - t_i$, $\forall i \in I$, *then* $\displaystyle \sum_{k=1}^{i} T_k \geqslant \sum_{k=1}^{i} \ln\left(\|I + B_k\|\right)^{1/\gamma}$;

(iii) $T_i \geqslant \ln\left(\|I + B_i\|\right)^{1/\gamma}$ *for all* $i = i_0, i_0 + 1, \ldots, \mathrm{Card}(\mathbf{I}) - 1$.

**Proof.** Let $T_i$ be $T_i = t_{i+1} - t_i$, $\forall i \in \mathbf{I}$, and $\tau \in [0, T_i)$. Thus, from (1),

$$x(t_{i+1} + \tau, \phi) = T(\tau) x\left(t_i^+\right) + \int_{-h}^{0} T(\tau - \tau') x(t_{i+1} + \tau, \varphi)\, d\tau',$$

$$x\left(t_i^+, \phi\right) = (I + B_i) x\left(t_i^-, \phi\right) = (I + B_i)\left[ T(t_i)\phi(0^+) + \sum_{k=1}^{i-1} T(t_i - t_k) B_k x\left(t_k^-, \phi\right) \right]$$
$$= (I + B_i)\left[ T(t_i - t_{i-1}) x\left(t_{i-1}^+, \phi\right) \right.$$
$$\left. + \int_{-h}^{0} T(t_i - t_{i-1} - \tau') x(t_{i-1} + \tau', \phi)\, d\tau' \right].$$

Taking Euclidean norms in the above relations, one gets

$$\left\| x\left(t_i^+, \phi\right) \right\| \leqslant \frac{\gamma + 1 - e^{-\gamma h}}{\gamma} \left\| (I + B_i) T(t_i - t_{i-1}) \right\| \operatorname*{Sup}_{0 \leqslant \tau \leqslant t_i^-} \left( \left\| x(\tau, \phi) \right\| \right)$$

$$\leqslant Ke^{-\gamma(t_i-t_{i-1})}\|I+B_i\| \operatorname*{Sup}_{t_{i-1-h}^-\leqslant\tau\leqslant t_1^-}\big(\|x(\tau,\phi)\|\big),$$

$$\|x(t_i+\tau,\phi)\| \leqslant \|T(\tau)\|\left[\|x(t_i^+,\phi)\| + \int_{-h}^{0}\|R(\tau')\|\,\|x(t_i+\tau',\phi)\|\,d\tau'\right]$$
$$\leqslant \frac{\gamma+1-e^{-\gamma h}}{\gamma}\|T(\tau)\| \operatorname*{Sup}_{t_i^++\tau-h\leqslant\tau'\leqslant t_{i+1}^+}\big(\|x(\tau',\phi)\|\big)$$
$$\leqslant Ke^{-\gamma t} \operatorname*{Sup}_{t_i^++\tau-h\leqslant\tau'\leqslant t_{i+1}^+}\big(\|x(\tau',\phi)\|\big)$$

with $R:[0,\infty)\to\mathbf{R}^{n\times n}$ being a matrix function that defines the factored representation $T(t-\tau)=T(t)R(\tau)$ so that $\|R(\tau)\|\leqslant e^{-\gamma\tau}$ since $\|T(t-\tau)\|\leqslant Ke^{-\gamma(t-\tau)}$ for all $t\geqslant\tau\geqslant 0$. Since $\int_{-h}^{0}\|R(\tau)\|\,d\tau\leqslant(1-e^{-\gamma h})/\gamma$ then

$$\left\|x(t_i^+,\phi)+\int_{-h}^{0}R(\tau')x(t_i^++\tau',\phi)\,d\tau'\right\|$$
$$\leqslant \|x(t_i^+,\phi)\| + \frac{1}{\gamma}\operatorname*{Sup}_{t_i^--h\leqslant\tau\leqslant t_i^+}\big(\|x(\tau,\phi)\|\big)$$
$$\leqslant \frac{\gamma+1-e^{-\gamma h}}{\gamma}\operatorname*{Sup}_{t_i^--h\leqslant\tau\leqslant t_i^+}\big(\|x(\tau,\phi)\|\big). \tag{11}$$

The recursive use of (11) for all while relating $x(t_1,\phi)$ to initial conditions $\phi:[-h,0]\to\mathbf{R}^n$ of supreme norm $\bar\phi$ leads to

$$\operatorname*{Sup}_{t_i^+\leqslant\tau\leqslant t_{i+1}^-}\big(\|x(\tau,\phi)\|\big) \leqslant \frac{\gamma+1-e^{-\gamma h}}{\gamma}\|T(\tau)\|\prod_{k=1}^{i}\big(\|I+B_k\|\big)\|T(t_k-t_{i+1})\|\bar\phi$$
$$\leqslant Ke^{-\gamma(\tau+t_i)}\prod_{k=1}^{i}\big(\|I+B_k\|\big)\bar\phi.$$

A sufficient condition for global uniform exponential stability independent of delay is $e^{-i\gamma T_{\min}}\prod_{k=1}^{i}(\|I+B_{k+\ell}\|)<1$ for any finite integer $i\geqslant 0$ provided that $t_{i+1}-t_i\geqslant T_{\min}$ for any integer $i\geqslant\ell$. Thus, it follows that (1) is g.u.e.s.i.d. under (i). It is proved that (1) is g.u.e.s.i.d. under (ii) by replacing $e^{-i\gamma T_{\min}}\to e^{-t_i}=e^{-\sum_{k=1}^{i-1}T_k}$. The fact that (1) is g.u.e.s.i.d. under (iii) is direct since the fulfillment of (iii) guarantees that of (ii). □

## 5. Closed-loop uniform exponential stability under linear feedback

In the subsequent study, consider the unforced (1). The discussion is limited to the case of delay-free combined point-delayed dynamics in (1); i.e., $m'=m''=0$. The extension to the general case is direct. The auxiliary system is $\dot{z}(t) = Mz(t)$, i.e., $J_i = \Phi$ $(i = 1, 2, 3)$ with $M$ strictly Hurwitzian. Thus, (1) is g.u.e.s.i.d. From Corollary 1, (1) is g.u.e.s.i.d. if and only if

$$\hat{T}(j\omega) = \left(I - (j\omega I - M)^{-1}\left[M - A + \sum_{i=1}^{m} A_i x(t - h_i)\right]\right)^{-1} (j\omega I - M)^{-1}$$

exists for all $\omega \in \mathbf{R}_+$ and is g.u.e.s. for $\hat{h} = 0$, i.e., for any bounded $x(0) = \phi(0) \in \mathbf{R}^n$. Note that $(j\omega I - M)^{-1}$ exists for all $\omega \in \mathbf{R}_+$ since $M$ is strictly Hurwitzian. Consider the set $H_\infty(X) = \{x : \mathbf{C}_+^0 \to X\colon \operatorname{Sup}_{\operatorname{Re} s>0}(\|x(s)\|) < \infty\}$, where $\mathbf{C}_+^0$ is the complex open right-hand side half plane. A similar $H_\infty$-space is defined for the set of linear operators on $X$ by replacing $X \to L(X, X)$. Note that $\hat{T} \in H_\infty(L(X, X))$ where it exists. Simple calculations for $H_\infty$-norms yield

$$\begin{aligned}\gamma_M &:= \left\|(j\omega I - M)^{-1}\right\|_\infty \\ &= \operatorname{Max}\left\{\gamma \in \mathbf{R}_+\colon\ H_M := \begin{bmatrix} M & 1/\gamma_M^2 \\ -I & -M^T \end{bmatrix}\right. \\ &\qquad\qquad \left.\text{has an eigenvalue on the imaginary axis}\right\}.\end{aligned}$$

$\hat{T}(j\omega)$ exists for all $\omega \in \mathbf{R}_+^0$ if $1 > \gamma_M[\|M - A\|_2 + \sum_{i=1}^{m} \|A_i\|_2]$ since

$$\operatorname{Sup}_{\omega\in\mathbf{R}_+^0}\left\{\left\|M - A + \sum_{i=1}^{m} A_i e^{-j\omega h_i}\right\|_2\right\} \leqslant \|M - A\|_2 + \sum_{i=1}^{m} \|A_i\|_2,$$
$$\text{where } \mathbf{R}_+^0 := \mathbf{R}_+ \cup \{0\}$$

and $\|\cdot\|_2$ denotes the $l_2$-matrix norm for each $\omega \in \mathbf{R}_+$. Now, consider the following feedback system:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_i x(t - h_i) + Bu(t) + \sum_{i=1}^{m'} B_i u(t - h_i') \tag{12}$$

with $B, B_j \in \mathbf{R}^{q\times n}$ $(i = 1, 2, \ldots, m')$, where the control function $u : [0, \infty) \to \mathbf{R}^q$ is continuous and has range $\mathbf{U}$, i.e., $u \in C^{(0)}([0, \infty); \mathbf{U})$ while being generated from the control law

$$u(t) = KCx(t) + \sum_{i=1}^{m} K_i Cx(t - h_i) + \sum_{i=1}^{m'} K_i' u(t - h_i') \tag{13}$$

with real matrices $K, K_i \in \mathbf{R}^{q\times r}$, $C \in \mathbf{R}^{r\times n}$. It is assumed that $y$ is an $r$-measurable output signal $y : [0, \infty) \to \mathbf{R}^r$ defined by $y(t) = Cx(t)$ for all $t \geqslant 0$. Taking Laplace transforms in (12) with zero initial conditions with $s = j\omega$, one gets directly the closed-loop relations

$$\left(j\omega I - A - \sum_{i=1}^{m} A_i e^{-j\omega h_i}\right)\hat{x}(j\omega) = B_0 + \sum_{i=1}^{m'} B_i e^{-j\omega h_i'} \hat{u}(j\omega), \tag{14}$$

$$\left(I-\sum_{i=1}^{m}K_i'e^{-j\omega h_i'}\right)\hat{u}(j\omega)=\sum_{i=0}^{m}K_iCe^{-j\omega h_i}\hat{x}(j\omega). \tag{15}$$

The substitution of (15) into (14) yields $\hat{S}_c(j\omega)\hat{x}(j\omega)=0$ with

$$\hat{S}_c(j\omega)=(j\omega I-M)\hat{T}_c(j\omega),$$
$$\hat{T}_c(j\omega)=I-(j\omega I-M)^{-1}\Bigg\{M-A-BKC+\sum_{i=1}^{m}(A_i+BK_iC)e^{-j\omega h_i}$$
$$+\left(\sum_{i=1}^{m'}B_ie^{-j\omega h_i'}\right)S_u^{-1}(j\omega)+\left(KC+\sum_{i=1}^{m}K_iCe^{-j\omega h_i}\right)\Bigg\}$$

with $S_u(j\omega)=I-\sum_{i=1}^{m'}K_i'e^{-jh_i\omega}$. The closed-loop system is g.u.e.s.i.d. if $\hat{T}_c^{-1}(j\omega)$ exists provided that $S_u^{-1}(j\omega)$ for all $\omega\in\mathbf{R}_+$. The following result holds.

**Theorem 5.** *The closed-loop system is g.u.e.s.i.d. if the following conditions are fulfilled*:

(a) *$M$ is strictly Hurwitzian so that the auxiliary system $\dot{z}(t)=Mz(t)$ is g.u.e.s.*;
(b) $\sum_{i=1}^{m}\|K_i'\|<1$;
(c) *The closed-loop system is g.u.e.s. in the absence of delays, i.e., for $h_i=0$, $h_k'$ ($i=1,2,\ldots,m$, $k=1,2,\ldots,m'$)*;
(d)
$$1>\gamma_M\Bigg\{\|M-A-BKC\|_2+\sum_{i=1}^{m}\|A_i+BK_iC\|_2$$
$$+\sum_{i=1}^{m'}\|B_i\|_2\sum_{i=1}^{m}\|K_i\|_2\|C\|_2\frac{1}{1-\|\sum_{i=1}^{m}\|K_i'\|_2\|}\Bigg\}. \tag{16}$$

Extensions of the above results in this section to the presence of distributed delays are direct. Assume, for instance, that the state (or only the output) is available for measurement, i.e., $C=I$ (or $C\neq I$), and that there are distributed delays in the system. Thus, the control law (15) may be generalized to

$$u(t)=KCx(t)+\sum_{i=1}^{m}K_iCx(t-h_i)+\sum_{i=0}^{m}\int_0^t d\alpha_i(\tau)K_{\alpha_i}x(t-\tau-h_i')$$
$$+\sum_{i=m'+1}^{m'+m''}\int_{t-h_i'}^{t}d\alpha_i(t-\tau)K_{\alpha_i}x(\tau). \tag{17}$$

Define

$$\beta=\mathop{\mathrm{Max}}_{0\leqslant i\leqslant m'+m''}\left(\beta_j\in\mathbf{R}_+:\ \int_0^\infty|\alpha_i(\tau)|\,d\tau<\infty\right).$$

Thus, $\hat{\alpha}_i(s) = \int_0^\infty \alpha_i(\tau)e^{-s\tau}\,d\tau \leqslant \int_0^\infty |\alpha_i(\tau)|e^{\beta\tau}\,d\tau < \infty$ for $\operatorname{Re} s \geqslant -\beta$ $(i = 0, 1, \ldots, m' + m'')$, it follows that $|\alpha_i(t)| \leqslant Ke^{(\varepsilon-\beta)t}$, $\forall t \geqslant 0$ and any real constant $\varepsilon > 0$, so that

$$\operatorname*{Max}_{0\leqslant i\leqslant m'+m''}\big(\big|\hat{\alpha}_i(s)\big|\big) \leqslant \frac{K}{|s+\beta-\varepsilon|} < \frac{K}{|s+\beta|} \quad \text{for } \operatorname{Re} s < -\gamma.$$

Then,

$$\big\|s\hat{\alpha}_i(s) - \alpha_i(0)\big\| \leqslant K + \operatorname*{Max}_{0\leqslant i\leqslant m'+m''}\big(\big|\alpha_i(0)\big|\big) \leqslant 2K. \tag{18}$$

Thus, the condition (d) of Theorem 5 becomes after substituting (17) into (1), via (18), and obtaining a relation in Laplace transforms for the closed-loop system description:

$$1 > \gamma_M \Bigg\{ \|M - A - BKC\|_2 + \sum_{i=1}^{m} \|A_i + BK_iC\|_2 + 2K\Bigg[\sum_{i=0}^{m'} \|A_{\alpha_i} + BK_{\alpha_i}C\|_2 + 2\sum_{i=m'+1}^{m'+m''} \|A_{\alpha_i} + BK_{\alpha_i}C\|_2\Bigg]\Bigg\}. \tag{19}$$

Very similar considerations as for point-delays (Theorem 5) may be used for the case when $(A, B, C)$ is controllable and observable or for $(A, B)$ being controllable and $C = I$ with $r = n$ and for that when $(A, B, C)$ is stabilizable and detectable but the triples $(A_i, B, C)$ and $(A_{\alpha_k}, B, C)$ $(i = 1, 2, \ldots, m,\ k = 0, 1, \ldots, m' + m'')$ are controllable and observable or, if $C = I$, then the pairs $(A_i, B)$ and $(A_{\alpha_k}, B)$ $(i = 1, 2, \ldots, m,\ k = 0, 1, \ldots, m' + m'')$ are all controllable.

## 6. Examples

**Example 1.** The simple first-order system $\dot{x}(t) = -ax(t) + a_1x(t-h)$ with $x(0) = x_0$. If $a > 0$ then Theorem 5 yields $\alpha_a = \|(s-a)^{-1}\|_\infty = 1/|a|$ and the system is g.u.e.s.i.d. if $1 > \gamma_a|a_1|\operatorname{Sup}_{\omega\in\mathbf{R}_+^0}(|e^{-jh\omega}|) = \gamma_a|a_1|$ provided that the auxiliary system $\dot{z}(t) = -az(t)$ with $z(0) = z_0$ is g.u.e.s., i.e., $a > 0$. Thus, the system is g.u.e.s.i.d. if $a > |a_1| > 0$. The same conclusion is obtained by applying Gronwall's lemma [9] as follows. Compute the solution to the system differential equation to obtain

$$\big|g(t)\big| \leqslant e^{-ah}|x_0| + \int_0^h \big|e^{a(\tau-h)}\phi(\tau-h)\big|\,d\tau + |a_1|\int_h^t \big|g(\tau-h)\big|\,d\tau$$

so that $\|x(t,\phi)\| \leqslant \nu(\phi)e^{-(a+|a_1|)t}$ for all $t \geqslant 0$, where

$$\nu = \|x_0\|e^{-ah} + \left|\frac{a_1}{a}\right|\big|e^{ah} - 1\big| \operatorname*{Sup}_{-h\leqslant\tau\leqslant 0}\big(\|\phi(\tau)\|\big).$$

Thus, exponential stability follows for $a > |a_1| > 0$. Assume, for instance, that $a < 0$ so that the auxiliary system is unstable. Thus, use the delay-free control law $u(t) = kx(t)$ with $k > -a$. Thus, the above results hold by replacing $a \to k - |a|$ so that the closed-loop uniform exponential stability independent of delay is ensured if $k > |a| + |a_1|$ still from Theorem 5. Note that Theorem 1 holds with $\hat{T}(s) = (s + a - a_1 e^{-hs})^{-1}$, $\hat{T}_{JM}^{-1} = (s + a)^{-1}$ and $M = a$.

**Example 2.** Consider the multiple point-delay $n$th-order system under an impulsive forcing term

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_i x(t-h) + \sum_{i \in \mathbf{I}} b_i \delta(t - t_i).$$

Thus, the unique solution for any admissible $n$-vector real function of initial conditions is

$$\begin{aligned} x(t) &= e^{A(t-t_i)} x(t_i^+) + \sum_{i=1}^{m} \int_{t_i}^{t} e^{A(t-\tau)} A_i x(\tau - h_i)\, d\tau \\ &= T(t)\phi(0^+) + \int_{-h}^{0} T(t-\tau)\phi(\tau)\, d\tau + \sum_{i \in \mathbf{I}} T(t - t_i) U(t - t_i) b_i \end{aligned}$$

for $t \in (t_i, t_{i+1})$; $x(t_{i+1}^+) = x(t_{i+1}^-) + b_{i+1}$ with $h = \mathrm{Max}_{1 \leqslant i \leqslant m}(h_i)$ with $T(t)$ satisfying $\dot{T}(t) = AT(t) + \sum_{i=1}^{m} A_i T(t-h)$ with $T(0) = I$ and $T(t) = 0$ for $t < 0$. Several special situations are now discussed.

(a) $(A, B)$ is stabilizable [12] with stability abscissa is $\mathrm{Min}(-\vartheta, -\vartheta') < 0$, where $(-\vartheta) < 0$ is obtained from the relocated closed-loop controllable poles through the controller gain matrix $K$ and $(-\vartheta') < 0$ is the stability abscissa of the uncontrollable open-loop stable (since the system is stabilizable) poles which cannot be relocated through feedback. Thus, the delayed system is g.u.e.s.i.d. if $\mathrm{Max}(|\vartheta|^{\mu}, |\vartheta'|^{\mu'}) > \sum_{i=1}^{m} \|A_i\|_2$.

(b) Assume that the impulsive input is nonzero. If there is a finite number of impulses then the above conditions of uniform stability still remain valid. If there is an infinite number of impulses $b_i = B_i x(t_i^-)$ then the global uniform stability independent of delay is preserved if all the time intervals in-between consecutive impulses satisfy the constraint

$$T_{\min} \geqslant \operatorname*{Sup}_{k \in \mathbf{I}} \left( \frac{1}{i\gamma'} \right) \sum_{k=1}^{i} \|I + B_k\| \quad \text{with } \gamma' = \gamma_A^{-1} - \sum_{i=1}^{m} \|A_i\|_2$$

from Theorems 4(ii) and 5 provided that $\gamma_A := \|(sI - A)^{-1}\|_\infty > \sum_{i=1}^{m} \|A_i\|_2$.

(c) If in case (b), $A$ is not strictly Hurwitzian and a closed-loop stabilization is performed, then $\gamma_A$ is replaced by the appropriate gain $\gamma_M$ for $M = A + BK$. If all the $b_i$ ($i \in \mathbf{I}$) converge exponentially to zero while being state-independent, Theorem 4 may be used instead of Theorem 5.

(d) Now, assume that an auxiliary system $\dot{z}(t) = Az(t) + A_1 z(t - h_1)$ is g.u.e.s.i.d. for all $\phi \in C_e(h_1)$ and any delay $h_1$. A sufficient condition is $|\vartheta_a|^{\mu_A} > |A_1|_2$ with $\gamma_A^{-1} = |\vartheta_A|^{\mu_A}$, where $(-\vartheta_A) < 0$ is the stability abscissa of $A$ provided that $A$ is strictly Hurwitzian with the dominant eigenvalue being of multiplicity $\mu_A$. Define the $H_\infty$-norm

$$\gamma_{\text{aux}} := \left\|(sI - A - A_1 e^{-hs})^{-1}\right\|_\infty \leqslant \left\|(sl - A)^{-1}\right\|_\infty \left\|I - (sI - A)^{-1} A_1 e^{-hs}\right\|_\infty \leqslant \frac{A}{1 - \gamma_A \|A_1\|_2}, \tag{20}$$

provided that $\|A_1\|_2 < \gamma_A^{-1} = |\vartheta_A|^{\mu_A}$. Then, a sufficient condition for the current system to be g.u.e.s.i.d. when no impulsive input is injected is that $1 < \gamma_{\text{aux}} \sum_{i=2}^{m} \|A_i\|_2$ which is guaranteed if $1 > \gamma_A \sum_{i=2}^{m} \|A_i\|_2/(1 - \gamma_A \|A_1\|_2)$ provided that $\|A_1\|_2 < \gamma_A^{-1} = |\vartheta_A|^{\mu_A}$. If $A + A_1 e^{-hs}$ has stable eigenvalues but $A$ is not strictly Hurwitzian then $\gamma_{\text{aux}}$ is finite but it cannot be calculated from sufficiency-type conditions for stability using (20). However, the system is still g.u.e.s.i.d. if $1 > \gamma_{\text{aux}} \sum_{i=2}^{m} \|A_i\|_2$.

(e) Now, assume that in case (d) there is an impulsive input consisting of infinitely many impulses. Thus, the current system is g.u.e.s.i.d. if the impulses occur at consecutive times being not less than $\text{Sup}_{k \in \mathbf{I}}(1/(i\gamma'_{\text{aux}})) \sum_{k=1}^{i} \|I + B_k\|$, except possibly on a set of zero measure, with $\gamma'^{-1} \to \gamma'^{-1}_{\text{aux}} = (1 - \gamma_A \|A_1\|_2)/\gamma_A - \sum_{i=2}^{m} \|A_i\|_2$, provided that $\|A_1\|_2 < \gamma_A^{-1}$ and $\sum_{i=2}^{m} \|A_i\|_2 < \gamma_A^{-1}(1 - \gamma_A \|A_1\|_2)$, from Theorem 4(ii) and Theorem 5.

**Example 3.** Consider the second-order scalar functional equation $\ddot{x}(t) = -a\dot{x}(t) + bx(t-h)$ decomposed as

$$x(t) = x_1(t), \qquad \dot{x}_1(t) = x_2(t),$$

$$\dot{x}_2(t) = -ax_2(t) + bx_1(t) - \int_{-h}^{0} bx_2(t+\tau)\,d\tau.$$

Using the Lyapunov functional candidate $V(x_{1t}, x_{2t}) = x_{2t}^2 - bx_{1t}^2 + \xi \int_{-h}^{0} \int_{t+s}^{t} x_{2t}^2(\tau)\,d\tau\,ds$ [7], the system is proved to be globally asymptotically stable dependent of the delay if $(-2a/h + \xi - b) < 0$ for some real constant $\xi > 0$ if $x_i(t)$, for $i = 1$ or 2, is nonzero for some subinterval of nonzero measure of $[t-h, t]$ any $t \geqslant t_0$ (some finite $t_0 \in R_+^0$). This holds for all $h > 0$ if $\text{Min}(a, b) \geqslant 0$ and $a$ and $b$ are not simultaneously zero. A general necessary condition is that $h < -a/b$. A necessary condition for exponential stability for $h = 0$ is that $a > 0$ and $b < 0$. As a result, if $a > 0$ and $b < 0$ then the system is globally uniformly asymptotically stable if $h \in [0, a/|b|]$. Decompose the system equation into two first-order differential equations as follows:

$$\dot{x}(t) = Ax(t) + A_1\bigl(x(t) - x(t-h)\bigr), \tag{21}$$

where $x(t) = (x_1(t), x_2(t))^T$, and

$$A = \begin{bmatrix} 0 & 1 \\ b & -a \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 0 \\ -b & 0 \end{bmatrix}.$$

The system is g.u.e.s.i.d. from Theorem 5 if $a > 0$, $b < 0$ and $a > 2|b| + \sqrt{a^2 - 4|b|}$ since the stability condition for $a \geqslant 2|b|$ is $(a - \sqrt{a^2 - 4b}) > 2|b|$ and for $a < 2|b|$.

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## References

[1] R.F. Datko, Time-delayed perturbations and robust stability, in: Differential Equations, Dynamical Systems and Control Science, in: Lecture Notes in Pure ands Applied Mathematics, Vol. 152, Dekker, New York, 1994, pp. 457–468.
[2] G. Da Prato, M. Iannelli, Linear integro-differential equations in Banach space, Rend. Sem. Mat. Univ. Padova 62 (1980) 207–219.
[3] N. Dunford, J.T. Schwartz, Linear Operators. Part I, Wiley, New York, 1958.
[4] M. De la Sen, N. Luo, Discretization and FIR filtering of continuous linear systems with internal and external point delays, Internat. J. Control 60 (1994) 1223–1246.
[5] C.F. Alastruey, M. De la Sen, J.R. Gonzalez de Mendibil, The stabilizability of integro-differential systems with two distributed delays, Math. Comput. Modelling 21 (1995) 85–94.
[6] N.U. Ahmed, Optimal control of infinite dimensional systems governed by integro-differential equations, in: Differential Equations, Dynamical Systems and Control Science, in: Lecture Notes in Pure and Applied Mathematics, Vol. 152, Dekker, New York, 1994, pp. 383–402.
[7] T.A. Burton, Stability and Periodic Solutions of Ordinary and Differential Equations, Academic Press, New York, 1985.
[8] M.R.S. Kulenovic, G. Ladas, A. Meimaridou, Stability of solutions of linear delay differential equations, Proc. Amer. Math. Soc. 100 (1987) 433–441.
[9] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[10] J. Yu, J. Yan, Positive solutions and asymptotic behavior of delay differential equations with nonlinear impulses, J. Math. Anal. Appl. 207 (1997) 388–396.
[11] T. Sengadir, Asymptotic stability of nonlinear functional differential equations, Nonlinear Anal. 28 (1997) 1997–2003.
[12] S. Barnett, Polynomials and Linear Control Systems, Dekker, New York, 1983.
[13] M. De la Sen, An algebraic method for pole placement in multivariable systems with internal and external point delays by using single rate or multirate sampling, Dynam. Control 10 (2000) 5–31.