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Journal of Functional Analysis 258 (2010) 893–912

JOURNAL OF Functional Analysis

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Asymptotics of Dirichlet eigenvalues and eigenfunctions of the Laplacian on thin domains in $\mathbb{R}^d$

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Received 23 May 2009; accepted 25 July 2009

Available online 5 August 2009

Communicated by J.-M. Coron

Abstract

We consider the Laplace operator with Dirichlet boundary conditions on a domain in $\mathbb{R}^d$ and study the effect that performing a scaling in one direction has on the eigenvalues and corresponding eigenfunctions as a function of the scaling parameter around zero. This generalizes our previous results in two dimensions and, as in that case, allows us to obtain an approximation for Dirichlet eigenvalues for a large class of domains, under very mild assumptions. As an application, we derive a three-term asymptotic expansion for the first eigenvalue of d-dimensional ellipsoids.

Keywords: Laplace spectrum; Thin domain asymptotics

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[1] The author was partially supported by RFBR and gratefully acknowledges the support from Deligne 2004 Balzan prize in mathematics and the grant of the President of Russia for young scientists and for Leading Scientific Schools (NSh-2215.2008.1).

[2] The author was partially supported by POCTI/POCI2010, Portugal, and by project LC06002 of the Czech Ministry of Education, Youth and Sports. Part of this work was done while the author was visiting the Doppler Institute in Prague, and he would like to thank the people there, and in particular P. Exner and D. Krejčiřík, for their hospitality.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2009.07.014

1. Introduction

In his 1967 paper [7] Joseph studied families of domains indexed by one parameter to obtain perturbation formulae approximating eigenvalues in a neighbourhood of a given domain. Within this context, he derived an elegant expression for the first eigenvalue of ellipses parametrized by their eccentricity e, namely,

$$\begin{aligned}\lambda_1(e) = \lambda_1 - \frac{\lambda_1}{2}e^2 - \frac{\lambda_1}{16}\left(3 - \frac{\lambda_1}{2}\right)e^4 \\ - \frac{\lambda_1}{32}\left(3 - \frac{\lambda_1}{2}\right)e^6 + \mathcal{O}\big(e^8\big), \quad \text{as } e \to 0, \end{aligned} \tag{1.1}$$

where $\lambda_1 = \lambda_1(0)$ is the first eigenvalue of the disk – to obtain the eigenvalue of ellipses of, say, area π, for instance, this should be divided by $\sqrt{1-e^2}$ and $\lambda_1(0)$ be the corresponding value for the disk. The coefficient of order e^6 in Joseph's paper is actually incorrect – we are indebted to M. Ashbaugh for pointing this out to us, and also for mentioning Henry's book [6] where this has been corrected. Although in principle quite general, the approach used by Joseph yields formulae which, in the case of domain perturbations, will allow us to obtain explicit asymptotic expansions only in very special cases such as that of ellipses above. The failure to obtain these expressions may be the case even when the eigenvalues and eigenfunctions of the original domain are known, as this does not necessarily mean that the coefficients appearing in the expansion may be computed in closed form. An example of this is the perturbation of a rectangle into a parallelogram, which Joseph considered as an example of what he called "pure shear."

With the purpose of obtaining approximations that can be computed explicitly, in a previous paper we considered instead the scaling of a given two-dimensional domain in one direction and studied the resulting singular perturbation as the domain approached a segment in the limit [1]. This approach may, of course, have the disadvantage that we might now be starting too far from the original domain. However, it allows for the explicit derivation of the coefficients in the expansion in terms of the functions defining the boundary of the domain. As was to be expected, and can be seen from the examples given in that paper, these four–term approximations are quite accurate close to the thin limit. A more interesting feature of this approach is that in some cases it also allows us to approximate eigenvalues quite well away from this limit, as may be seen from the following examples. The application of our formula to the ellipses considered above yields

$$\lambda_1(\varepsilon) = \frac{\pi^2}{4\varepsilon^2} + \frac{\pi}{2\varepsilon} + \frac{3}{4} + \left(\frac{11}{8\pi} + \frac{\pi}{12}\right)\varepsilon + \mathcal{O}\big(\varepsilon^2\big), \quad \text{as } \varepsilon \to +0, \tag{1.2}$$

where we now considered ellipses of radii 1 and ε, ε being the stretch factor. The error in the approximation is comparable to that in Joseph's formula, except that Eq. (1.2) is more accurate closer to the thin limit while (1.1) provides better approximations near the circle. This is also an advantage, since it is natural for numerical methods to perform better away from the thin limit, but to have more difficulties the closer they are to the singular case, suggesting that our formulae may also be useful for checking numerical methods close to the limit case.

As another application we mention the case of the lemniscate

$$\big(x_1^2 + x_2^2\big)^2 = x_1^2 - x_2^2.$$

for which we have

$$\lambda_1(\varepsilon)=\frac{2\pi^2}{\varepsilon^2}+\frac{2\sqrt{3}\pi}{\varepsilon}+\frac{97}{24}+\left(\frac{593}{64\sqrt{3}\pi}+\frac{\sqrt{3}\pi}{4}\right)\varepsilon+\mathcal{O}\left(\varepsilon^2\right),\quad \text{as } \varepsilon\to+0,$$

yielding an error at ε equal to one which is in fact smaller than in the case of the disk above. For details, see [1].

In the present paper we extend the results in [1] to general dimension, in the sense that we now consider domains in $\mathbb{R}^d$ which are being scaled in one direction and approach a $(d-1)$-dimensional set in the limit as the stretch parameter goes to zero. Due to the increase in complexity in the corresponding formulae as a consequence of the fact that we are now considering arbitrary dimensions, we only obtained the first three non-zero coefficients in the asymptotic expansion of the principal eigenvalue. However, because of smoothness assumption near the point of global maximum, these include the coefficients of the two unbounded terms plus the constant term in the expansion – we know from the two-dimensional case that lack of smoothness at the point of maximum will yield other intermediate powers of ε [3,4].

As an example, we obtain an expansion for the first eigenvalue of the general d-dimensional ellipsoid

$$\mathcal{E}=\left\{(x_1,\ldots,x_d)\in\mathbb{R}^d\colon \left(\frac{x_1}{a_1}\right)^2+\cdots+\left(\frac{x_d}{a_d}\right)^2\leqslant 1\right\},$$

where the a_i's are positive real numbers. If we choose as projecting hyperplane that which is orthogonal to the x_d axis we obtain

$$\lambda_1(\mathcal{E}_\varepsilon)=\frac{\pi^2}{4a_d^2\varepsilon^2}+\frac{\pi}{2a_d\varepsilon}\sum_{i=1}^{d-1}\frac{1}{a_i}+\frac{1}{4}\left(3\sum_{i=1}^{d-1}\frac{1}{a_i^2}+\frac{1}{2}\sum_{i=1}^{d-1}\sum_{j=i+1}^{d-1}\frac{1}{a_ia_j}\right)+\mathcal{O}\left(\varepsilon^{1/2}\right),\quad \text{as } \varepsilon\to+0. \tag{1.3}$$

Besides the added complexity of the formulae, there are now extra technical difficulties related to the fact that there may exist multiple eigenvalues requiring a more careful approach. As in the two-dimensional case, the asymptotic expansions obtained depend on what happens locally at the point of global maximum width. Also as in that case, we cannot exclude the existence of a tail term approaching zero faster than any power of ε. However, we conjecture that if the boundary of the domain is analytic, then the expansions will actually correspond to the series developments of the corresponding eigenvalues.

In the next section we establish the notation and state the main results of the paper, which are then proved in Sections 3 and 4. In the last section, and as an application, we derive the above expression for the first eigenvalue of the ellipsoid.

2. Statement of results

Let $x=(x',x_d)$, $x'=(x_1,\ldots,x_{d-1})$ be Cartesian coordinates in $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$, respectively, $d\geqslant 2$, and $\omega\subset\mathbb{R}^{d-1}$ be a bounded domain having C^1-boundary. By $h_\pm=h_\pm(x')\in C(\overline{\omega})$ we denote two arbitrary functions such that $H(x') := h_+(x') + h_-(x') \geqslant 0$ for $x' \in \omega$. We consider the thin domain defined by

$$\Omega_\varepsilon := \{x\colon -\varepsilon h_-(x') < x_d < \varepsilon h_+(x'),\ x' \in \omega\},$$

where ε is a small positive parameter. We assume that the function $H(x')$ attains its global maximum at a single point $\bar{x} \in \omega$ and that there exists a ball $B'_\delta(\bar{x}) := \{x'\colon |x' - \bar{x}| < \delta\}$ such that $h_\pm \in C^\infty(B'_\delta(\bar{x}))$. Let $H_0 := H(\bar{x})$ and the Taylor expansions for H and h_- at $\bar{x}$ read as follows

$$H(x') = H_0 + \sum_{i=2k}^{\infty} H_i(x' - \bar{x}), \qquad h_-(x') = h_0 + \sum_{i=1}^{\infty} h_i(x' - \bar{x}), \tag{2.1}$$

where H_i and h_i are homogeneous polynomials of order i, $H_{2k}(x' - \bar{x}) < 0$ for $x' \neq \bar{x}$, and $k \geqslant 1$.

Our purpose is to study the asymptotic behaviour of the eigenvalues and eigenfunctions of the Dirichlet Laplacian $-\Delta^D_{\Omega_\varepsilon}$ in Ω_ε. Let $\chi = \chi(x') \in C^\infty(\mathbb{R}^{d-1})$ be a non-negative cut-off function equalling one as $|x' - \bar{x}| < \delta/3$ and vanishing for $|x' - \bar{x}| > \delta/2$. Denote $\Omega^\delta_\varepsilon := \Omega_\varepsilon \cap \{x\colon |x' - \bar{x}| < \delta\}$.

Let

$$G_n := -\Delta_{\xi'} - \frac{2\pi^2 n^2 H_{2k}(\xi')}{H_0^3}$$

be an operator in $L_2(\mathbb{R}^{d-1})$. The spectrum of this operator consists of countably many isolated eigenvalues of finite multiplicity having only one accumulation point at infinity [5, Ch. IV, Sec. 46, Th. 1]. By

$$\Lambda_{n,1} < \Lambda_{n,2} \leqslant \Lambda_{n,3} \ldots$$

we denote the eigenvalues of this operator arranged in non-decreasing order and taking the multiplicities into account. Denote by $\Psi_{n,m}$ the associated eigenfunctions orthonormalized in $L_2(\mathbb{R}^{d-1})$. It follows from [5, Ch. V, Sec. 43, Th. 2] that the functions $\Psi_{n,m}$ decay exponentially at infinity.

Our main results are the following. First, we obtain a two-parameter description for the eigenvalues.

Theorem 1. *Let $\Lambda = \Lambda_{n,M} = \Lambda_{n,M+1} = \cdots = \Lambda_{n,M+N-1}$ be a N-multiple eigenvalue of G_n for a given $n \in \mathbb{N}$. Then there exist eigenvalues $\lambda_{n,m}(\varepsilon)$ of $-\Delta^D_{\Omega_\varepsilon}$, $m = M, \ldots, M+N-1$ taken counting multiplicities whose asymptotics as $\varepsilon \to +0$ read as follows*

$$\lambda_{n,m}(\varepsilon) = \varepsilon^{-2} c_0^{(n,m)} + \varepsilon^{-2} \sum_{j=2k}^{\infty} c_j^{(n,m)} \eta^j, \quad \eta := \varepsilon^\alpha,\ \alpha := \frac{1}{k+1}, \tag{2.2}$$

$$c_0^{(n,m)} = \frac{\pi^2 n^2}{H_0^2}, \qquad c_{2k}^{(n,m)} = \Lambda, \tag{2.3}$$

and $-c_{2k+1}^{(n,m)}$ *are the eigenvalues of the matrix with the entries*

$$2\pi^2 n^2 H_0^{-3} \big(H_{2k+1}\Psi_{n,m}, \Psi_{n,l}\big)_{L_2(\mathbb{R}^{d-1})}, \quad m, l = M, \ldots, M+N-1.$$

The remaining coefficients are determined by Lemmas 3.6 *and* 3.7.

As in [1], for sufficiently small ε this allows us to derive the asymptotics for specific eigenvalues, and we give the explicit expansion for the first eigenvalue in terms of the functions H and h_- in the case where H_2 is negative for $x' \neq \bar{x}$.

Theorem 2. *For any* $N \geqslant 1$ *there exists* $\varepsilon_0 = \varepsilon_0(N)$ *such that for* $\varepsilon \leqslant \varepsilon_0$ *the first* N *eigenvalues of* $-\Delta_{\Omega_\varepsilon}^D$ *are* $\lambda_{1,m}(\varepsilon)$, $m = 1, \ldots, N$. *If*

$$k = 1, \quad H_2(x') = -\frac{1}{2}\sum_{i=1}^{d-1} \alpha_i^2 x_i^2, \tag{2.4}$$

the lowest eigenvalue $\lambda_{1,1}(\varepsilon)$ *has the asymptotic expansion*

$$\lambda_{1,1}(\varepsilon) = \frac{c_0^{(1,1)}}{\varepsilon^2} + \frac{c_2^{(1,1)}}{\varepsilon} + c_4^{(1,1)} + \mathcal{O}\big(\varepsilon^{1/2}\big), \quad \varepsilon \to +0,$$

$$c_0^{(1,1)} = \frac{\pi^2}{H_0^2}, \qquad c_2^{(1,1)} = \sum_{j=1}^{d-1} \theta_j, \quad \theta_j := \frac{\pi \alpha_j}{H_0^{3/2}},$$

$$\begin{aligned} c_4^{(1,1)} = {} & \frac{\pi^2}{H_0^4}\big(\big(3H_2^2(\xi') - 2H_0 H_4(\xi')\big)\Psi_0, \Psi_0\big)_{L_2(\mathbb{R}^{d-1})} \\ & + \frac{\pi^2}{H_0^2}\sum_{i=1}^{d-1}\left(\frac{\partial h_1}{\partial x_i}\right)^2 - \frac{2\pi^2}{H_0^3}\big(H_3(\xi')\widetilde{\Psi}_1, \Psi_0\big)_{L_2(\mathbb{R}^{d-1})}, \end{aligned} \tag{2.5}$$

$$\Psi_0(\xi') := \prod_{j=1}^{d-1} \frac{\theta_j^{1/4}}{\pi^{1/4}} \mathrm{e}^{-\frac{\theta_j \xi_j^2}{2}}, \tag{2.6}$$

$$\widetilde{\Psi}_1(\xi') := \Psi_0(\xi')\left(\sum_{p,j=1}^{d-1} \frac{3\pi^2 \beta_{ppj}\xi_j}{2H_0^3 \theta_j (2\theta_p + \theta_j)} - \sum_{p,q,j=1}^{d-1} \frac{\pi^2 \beta_{pqj}\xi_p \xi_q \xi_j}{H_0^3(\theta_p + \theta_q + \theta_j)}\right), \tag{2.7}$$

where it is assumed that $H_3(x')$ *is written as*

$$H_3(x') = \sum_{p,q,j=1}^{d-1} \beta_{pqj}\xi_p\xi_q\xi_j,$$

and the constants β_{pqj} *are invariant under each permutation of the indices* p, q, j:

$$\beta_{pqj} = \beta_{pjq} = \beta_{qpj} = \beta_{qjp} = \beta_{jpq} = \beta_{jqp}. \tag{2.8}$$

Remark 2.1. The assumption (2.4) for H_2 is not a restriction, since we can always achieve such form for H_2 by an appropriate change of variables.

3. Proof of Theorem 1

In this section we construct the asymptotics for the eigenvalues and the eigenfunctions of the operator $-\Delta^D_{\Omega_\varepsilon}$. This is first done formally, and justified rigorously afterwards. In the formal construction we employ the same approach as was used in [1, Sec. 3].

We are going to construct formally the asymptotic expansions for the eigenvalues $\lambda_{n,m}(\varepsilon)$, $m = M, \dots, M+N-1$ which we relabel as $\lambda_\varepsilon^{(m)}$, $m = 1, \dots, N$. We denote the associated eigenfunctions by $\psi_\varepsilon^{(m)}$. We construct their asymptotic expansions as the series

$$
\begin{gathered}
\lambda_\varepsilon^{(m)} = \varepsilon^{-2}\mu_\varepsilon^{(m)}, \qquad \mu_\varepsilon^{(m)} = c_0^{(m)} + \sum_{i=2k}^{\infty} c_i^{(m)} \eta^i, \\
\psi_\varepsilon^{(m)}(x) = \sqrt{H(x')}\,\widetilde{\psi}_\varepsilon^{(m)}(x), \qquad \widetilde{\psi}_\varepsilon^{(m)}(x) = \sum_{i=0}^{\infty} \eta^i \psi_i^{(m)}(\xi), \\
\xi = (\xi', \xi_d), \qquad \xi' := \frac{x' - \bar{x}}{\eta}, \qquad \xi_d := \frac{x_d + \varepsilon h_-(x')}{\varepsilon H(x')}.
\end{gathered}
\tag{3.1}
$$

We postulate the functions $\psi_i^{(m)}(\xi)$ to be exponentially decaying as $\xi' \to +\infty$. It means that they are exponentially small outside $\Omega_\varepsilon^\delta$ (with respect to ε). In terms of the variables ξ the domain $\Omega_\varepsilon^\delta$ becomes $\{\xi\colon |\xi'| < \delta\eta^{-1},\ 0 < \xi_d < 1\}$. As $\eta \to 0$, it "tends" to the layer $\Pi := \{\xi\colon 0 < \xi_d < 1\}$ and this is why we shall construct the functions ψ_i as defined on Π.

We rewrite the eigenvalue equation for ψ_ε and λ_ε in the variables ξ,

$$
\begin{aligned}
&-\Bigg[\eta^{2k}\Delta_{\xi'} + K_d \frac{\partial^2}{\partial \xi_d^2} + \sum_{i=1}^{d-1} \eta^{2k+1}\left(\frac{\partial}{\partial \xi_i} K_i \frac{\partial}{\partial \xi_d} + \frac{\partial}{\partial \xi_d} K_i \frac{\partial}{\partial \xi_i}\right) \\
&\qquad + \eta^{2k+2}\sum_{i=1}^{d-1} \frac{\partial}{\partial \xi_d} K_i^2 \frac{\partial}{\partial \xi_d} + \eta^{2k+2} K_0\Bigg]\widetilde{\psi}_\varepsilon^{(m)} = \mu_\varepsilon^{(m)}\widetilde{\psi}_\varepsilon^{(m)} \quad \text{in } \Pi, \\
&\widetilde{\psi}_\varepsilon^{(m)} = 0 \quad \text{on } \partial\Pi,
\end{aligned}
\tag{3.2}
$$

where $K_i = K_i(\xi, \eta)$, $i = 0, \dots, d$,

$$
\begin{aligned}
K_d(\xi, \eta) &= \frac{1}{H^2(\bar{x} + \eta\xi')}, \\
K_i(\xi, \eta) &= \frac{1}{H(\bar{x} + \eta\xi')}\left[\frac{\partial h_-}{\partial x_i}(\bar{x} + \eta\xi') - \xi_d \frac{\partial H}{\partial x_i}(\bar{x} + \eta\xi')\right], \\
K_0(\xi, \eta) &= \frac{1}{2} H^{-1}(\bar{x} + \eta\xi')\Delta_{x'} H(\bar{x} + \eta\xi') - \frac{1}{4} H^{-2}(\bar{x} + \eta\xi')\left|\nabla_{x'} H(\bar{x} + \eta\xi')\right|^2.
\end{aligned}
$$

Remark 3.1. We have introduced the factor $\sqrt{H(x')}$ in the series (3.1) for $\psi_\varepsilon^{(m)}$ in order to have a symmetric differential operator in the equation (3.2).

We expand the functions K_i into the Taylor series w.r.t. η and employ (2.1) to obtain

$$
\begin{aligned}
K_d(\xi,\eta) &= H_0^{-2} + \sum_{j=2k}^{\infty} \eta^j P_j^{(d)}(\xi'),\\
K_i(\xi,\eta) &= \sum_{j=0}^{\infty} \eta^j K_j^{(i)}(\xi),\\
K_j^{(i)}(\xi) &:= P_j^{(i)}(\xi') + \xi_d Q_j^{(i)}(\xi'), \quad i=1,\dots,d-1,\\
K_0(\xi,\eta) &= \sum_{i=0}^{\infty} \eta^i P_i^{(0)}(\xi'),
\end{aligned}
\tag{3.3}
$$

where $P_j^{(i)}$, $Q_j^{(i)}$ are polynomials, and, in particular,

$$
\begin{aligned}
&P_{2k}^{(d)}(\xi') = -\frac{2H_{2k}(\xi')}{H_0^3}, \qquad P_{2k+1}^{(d)}(\xi') = -\frac{2H_{2k+1}(\xi')}{H_0^3},\\
&P_0^{(i)}(\xi') = \frac{1}{H_0}\frac{\partial h_1}{\partial x_i}(\bar{x}), \qquad Q_0^{(i)}(\xi') = 0.
\end{aligned}
\tag{3.4}
$$

We substitute (3.1), (3.3), (3.4) into (3.2) and equate the coefficients of like powers of η. This leads us to the following boundary value problems for $\psi_i^{(m)}$,

$$
\begin{aligned}
&\left(\frac{1}{H_0^2}\frac{\partial^2}{\partial\xi_d^2} + c_0^{(m)}\right)\psi_j^{(m)} = 0 \quad \text{in } \Pi,\\
&\psi_j^{(m)} = 0 \quad \text{on } \partial\Pi, \quad j=0,\dots,2k-1,
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
&-\left(\frac{1}{H_0^2}\frac{\partial^2}{\partial\xi_d^2} + c_0^{(m)}\right)\psi_{2k}^{(m)} = \left(\Delta_{\xi'} - \frac{2H_{2k}(\xi')}{H_0^3}\frac{\partial^2}{\partial\xi_d^2} + c_{2k}^{(m)}\right)\psi_0^{(m)} \quad \text{in } \Pi,\\
&\psi_{2k}^{(m)} = 0 \quad \text{on } \partial\Pi,
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
&-\left(\frac{1}{H_0^2}\frac{\partial^2}{\partial\xi_d^2} + c_0^{(m)}\right)\psi_j^{(m)} = \left(\Delta_{\xi'} - \frac{2H_{2k}(\xi')}{H_0^3}\frac{\partial^2}{\partial\xi_d^2} + c_{2k}^{(m)}\right)\psi_{j-2k}^{(m)}\\
&\qquad\qquad + c_j^{(m)}\psi_0^{(m)} + \sum_{q=1}^{j-2k-1} c_{j-q}^{(m)}\psi_q^{(m)} + F_j^{(m)} \quad \text{in } \Pi,\\
&\psi_j^{(m)} = 0 \quad \text{on } \partial\Pi, \quad j \geqslant 2k+1,
\end{aligned}
\tag{3.7}
$$

$$F_j^{(m)} := \sum_{q=0}^{j-2k-1} \mathcal{L}_{j-q-2k} \psi_q^{(m)},$$

$$\mathcal{L}_j := \sum_{i=1}^{d-1} \left(\frac{\partial}{\partial \xi_d} K_{j-1}^{(i)} \frac{\partial}{\partial \xi_i} + \frac{\partial}{\partial \xi_i} K_{j-1}^{(i)} \frac{\partial}{\partial \xi_d} \right)$$
$$+ \sum_{i=1}^{d-1} \sum_{s=0}^{j-2} \frac{\partial}{\partial \xi_d} K_s^{(i)} K_{j-s-2}^{(i)} \frac{\partial}{\partial \xi_d} + P_{j+2k}^{(d)} \frac{\partial^2}{\partial \xi_d^2} + P_{j-2}^{(0)}, \tag{3.8}$$

where $P_{-1}^{(0)} = 0$. Problems (3.5) can be solved explicitly with

$$\psi_j^{(m)}(\xi) = \Psi_j^{(m)}(\xi') \sin \pi n \xi_d, \qquad c_0 = \frac{\pi^2 n^2}{H_0^2}, \tag{3.9}$$

where $j = 0, \ldots, 2k-1$, and $\Psi_j^{(m)}$ are the functions to be determined. The last identity proves formula (2.3) for $c_0^{(n,m)}$.

We consider the problem (3.6) as posed on the interval $(0,1)$ and depending on ξ'. It is solvable, if and only if the right-hand side is orthogonal to $\sin \pi n \xi_d$ in $L_2(0,1)$. It implies the equation

$$-\left(\Delta_{\xi'} + \frac{2\pi^2 n^2 H_{2k}(\xi')}{H_0^3} \right) \Psi_0^{(m)} = c_{2k}^{(m)} \Psi_0^{(m)} \quad \text{in } \mathbb{R}^{d-1}. \tag{3.10}$$

Thus, $c_{2k}^{(m)}$ is an eigenvalue of the operator G_n, i.e., $c_{2k}^{(m)} = \Lambda$. Then $\Psi_0^{(m)}$ is one of the eigenfunctions associated with Λ. These eigenfunctions are assumed to be orthonormalized in $L_2(\mathbb{R}^{d-1})$. We substitute Eq. (3.10) into (3.6) and see that formula (3.9) is valid also for $j = 2k$.

The problems (3.7) are solvable, if and only if the right-hand sides are orthogonal to $\sin \pi n \xi_d$ in $L_2(0,1)$. It gives rise to the equations

$$(G_n - \Lambda) \Psi_{j-2k}^{(m)} = f_j^{(m)} + \sum_{q=1}^{j-2k-1} c_{j-q}^{(m)} \Psi_q^{(m)} + c_j^{(m)} \Psi_0^{(m)},$$

$$\Psi_j^{(m)} = \Psi_j^{(m)}(\xi') := 2 \int_0^1 \psi_j^{(m)}(\xi) \sin \pi n \xi_d \, \mathrm{d}\xi_d, \tag{3.11}$$

$$f_j^{(m)} = f_j^{(m)}(\xi') := 2 \int_0^1 F_j^{(m)}(\xi) \sin \pi n \xi_d \, \mathrm{d}\xi_d. \tag{3.12}$$

To solve problems (3.7), (3.11) we need some auxiliary lemmata. The first of these follows from standard results in the spectral theory of self-adjoint operators.

Lemma 3.2. *Let $f \in L_2(\mathbb{R}^{d-1})$. The equation*

$$(G_n - \Lambda)u = f \tag{3.13}$$

is solvable, if and only if

$$\big(f, \Psi_0^{(m)}\big)_{L_2(\mathbb{R}^{d-1})} = 0, \quad m = 1, \dots, N.$$

The solution is unique up to a linear combination of the functions $\Psi_0^{(m)}$.

By $\mathfrak{g}_n$ we denote the sesquilinear form associated with G_n,

$$\mathfrak{g}_n[u, v] = (\nabla u, \nabla v)_{L_2(\mathbb{R}^{d-1})} - (H_{2k}u, v)_{L_2(\mathbb{R}^{d-1})}.$$

The domain of this form is

$$\mathcal{D}(\mathfrak{g}_n) = W_2^1\big(\mathbb{R}^{d-1}\big) \cap \big\{u\colon \big(1 + |\xi'|^k\big)u \in L_2\big(\mathbb{R}^{d-1}\big)\big\}.$$

By $\mathcal{D}(G_n)$ we denote the domain of G_n. The set $C_0^\infty(\mathbb{R}^{d-1})$ is dense in $\mathcal{D}(\mathfrak{g}_n)$ in the topology induced by $\mathfrak{g}_n$ [2, Ths. 1.8.1, 1.8.2].

Lemma 3.3. *Let $f \in L_2(\mathbb{R}^{d-1})$, $u \in L_2(\mathbb{R}^{d-1}) \cap W_2^1(S)$ for each bounded domain $S \subset \mathbb{R}^{d-1}$ and for each $\phi \in \mathcal{D}(\mathfrak{g})$ the identity*

$$\int\limits_{\mathbb{R}^{d-1}} \nabla u \cdot \nabla \phi \, d\xi' - \int\limits_{\mathbb{R}^{d-1}} \big(H_{2k}(\xi') - \Lambda\big)u\overline{\phi} \, d\xi' = \int\limits_{\mathbb{R}^{d-1}} f\overline{\phi} \, d\xi' \tag{3.14}$$

holds true. Then $u \in \mathcal{D}(G_n)$ and Eq. (3.13) *is valid.*

Proof. Let $\chi_1 = \chi_1(t)$ be a non-negative infinitely differentiable cut-off function taking values in $[0, 1]$, equalling one as $t < 1$, and vanishing as $t > 2$. It is clear that for each $t > 0$ the function $u(\xi')\chi_1(|\xi'|t)$ belongs to $\mathcal{D}(\mathfrak{g}_n)$. We substitute $\phi(\xi') = u(\xi')\chi_1(|\xi'|t)$ into (3.14) and integrate by parts,

$$\begin{aligned} &\|\chi_1 \nabla u\|^2_{L_2(\mathbb{R}^{d-1})} - (H_{2k}\chi_1 u, \chi_1 u)_{L_2(\mathbb{R}^{d-1})} \\ &\quad = (\chi_1 f, \chi_1 u)_{L_2(\mathbb{R}^{d-1})} + \frac{1}{2}\big(u\Delta_{\xi'}\chi_1^2, u\big)_{L_2(\mathbb{R}^{d-1})} + \Lambda\|\chi_1 u\|^2_{L_2(\mathbb{R}^{d-1})}. \end{aligned} \tag{3.15}$$

Hence,

$$\begin{aligned} &\|\nabla u\|^2_{L_2(\mathcal{B}'_{t^{-1}}(0))} - (H_{2k}u, u)_{L_2(\mathcal{B}'_{t^{-1}}(0))} \\ &\quad \leqslant \|f\|_{L_2(\mathbb{R}^{d-1})}\|u\|_{L_2(\mathbb{R}^{d-1})} + Ct^2\|u\|_{L_2(\mathbb{R}^{d-1})} + \Lambda\|u\|^2_{L_2(\mathbb{R}^{d-1})}, \end{aligned} \tag{3.16}$$

where the constant C is independent of t, and $\mathcal{B}'_r(a) := \{\xi'\colon |\xi' - a| < r\}$. Passing to the limit as $t \to +0$, we conclude that $u \in \mathcal{D}(\mathfrak{g}_n)$ and in view of (3.14) this function belongs to $\mathcal{D}(\mathfrak{g}_n)$ and solves Eq. (3.13). $\square$

Let $\mathfrak{V}$ be the space of the functions $f \in C^\infty(\mathbb{R}^{d-1})$ such that

$$\big(1+|\xi'|^\gamma\big)\frac{\partial^\tau f}{\partial \xi'^\tau} \in L_2\big(\mathbb{R}^{d-1}\big)$$

for each $\tau \in \mathbb{Z}_+^d$, $\gamma \in \mathbb{Z}_+$.

Lemma 3.4. *Let $f \in \mathfrak{V}$, and u be a solution to* (3.13). *Then $u \in \mathfrak{V}$.*

Proof. Since $u \in \mathcal{D}(G_n)$, we have $\nabla u \in L_2(\mathbb{R}^{d-1})$, $(1+|\xi'|^k)u \in L_2(\mathbb{R}^{d-1})$, and due to standard smoothness improving theorems $u \in C^\infty(\mathbb{R}^{d-1})$. The identity (3.15) is also valid with χ_1 replaced by $\chi_1(|\xi'|t)|\xi'|^\beta$. Employing this identity and proceeding as in (3.16), we check that $(1+|\xi'|^\beta)\nabla u \in L_2(\mathbb{R}^{d-1})$, $(1+|\xi'|^{k+\beta})u \in L_2(\mathbb{R}^{d-1})$, if $(1+|\xi'|^\beta)u \in L_2(\mathbb{R}^{d-1})$ for some $\beta \in \mathbb{Z}_+$. Applying this fact by induction and using that $(1+|\xi'|^k)u \in L_2(\mathbb{R}^{d-1})$, we conclude that $(1+|\xi'|^\gamma)\nabla u \in L_2(\mathbb{R}^{d-1})$, $(1+|\xi'|^{k+\gamma})u \in L_2(\mathbb{R}^{d-1})$ for each $\gamma \in \mathbb{Z}_+$.

We differentiate Eq. (3.13) w.r.t. ξ_i,

$$(G_n - \Lambda)\frac{\partial u}{\partial \xi_i} = \frac{\partial f}{\partial \xi_i} + \frac{\partial H_{2k}}{\partial \xi_i}u.$$

The right-hand side belongs to $L_2(\mathbb{R}^{d-1})$ and the function $\frac{\partial u}{\partial \xi_i}$ satisfies the hypothesis of Lemma 3.3. Applying this lemma, we see that $\frac{\partial u}{\partial \xi_i} \in \mathcal{D}(G_n)$. Proceeding as above, one can make sure that

$$\big(1+|\xi'|^\gamma\big)\nabla\frac{\partial u}{\partial \xi_i} \in L_2\big(\mathbb{R}^{d-1}\big)$$

for each $\gamma \in \mathbb{Z}_+$. Repeating the described process, we complete the proof. □

As it follows from Lemma 3.2, the solvability condition of Eq. (3.11) is

$$\big(f_j^{(m)}, \Psi_0^{(l)}\big)_{L_2(\mathbb{R}^{d-1})} + \sum_{q=1}^{j-2k-1} c_{j-q}^{(m)}\big(\Psi_q^{(m)}, \Psi_0^{(l)}\big)_{L_2(\mathbb{R}^{d-1})} + c_j^{(m)}\delta_{ml} = 0,$$

where $m, l = 1, \dots, N$, and δ_{ml} is the Kronecker delta. Here we have supposed that the functions $\Psi_0^{(m)}$ are orthonormalized in $L_2(\mathbb{R}^{d-1})$. In view of (3.12) these identities can be rewritten as

$$2\big(F_j^{(m)}, \psi_0^{(l)}\big)_{L_2(\Pi)} + 2\sum_{q=1}^{j-2k-1} c_{j-q}^{(m)}\big(\psi_q^{(m)}, \psi_0^{(l)}\big)_{L_2(\Pi)} + c_j^{(m)}\delta_{ml} = 0, \tag{3.17}$$

where $m, l = 1, \dots, N$.

Consider the problem (3.7) for $j = 2k+1$. The solvability condition is Eq. (3.11) for the same j. Since $\Psi_0^{(m)} \in \mathfrak{V}$, the same is true for $f_{2k+1}^{(m)}$. By (3.17), this equation is solvable, if and only if

$$T_{ml}^{(2k+1)} + c_{2k+1}^{(m)}\delta_{ml} = 0, \quad m, l = 1, \dots, N,$$
$$T_{ml}^{(2k+1)} := 2\big(\mathcal{L}_1\psi_0^{(m)}, \psi_0^{(l)}\big)_{L_2(\Pi)}. \tag{3.18}$$

The definition of $\mathcal{L}_1$ and (3.4) yield

$$T_{ml}^{(2k+1)} = 2\pi^2 n^2 H_0^{-3}\big(H_{2k+1}\Psi_0^{(m)}, \Psi_0^{(l)}\big)_{L_2(\mathbb{R}^{d-1})}. \tag{3.19}$$

Hence, the matrix $T^{(2k+1)}$ with the entries $T_{ml}^{(2k+1)}$ is symmetric. This matrix describes a quadratic form on the space spanned over $\Psi_0^{(m)}$, $m = 1, \dots, N$. By the theorem on the simultaneous diagonalization of two quadratic forms we conclude that the eigenfunctions $\Psi_0^{(m)}$ can be chosen as orthonormalized in $L_2(\mathbb{R}^{d-1})$ and, in addition, so that the matrix $T^{(2k+1)}$ is diagonal. In what follows we assume that these functions are chosen in such a way. Then identities (3.18) imply

$$c_{2k+1}^{(m)} = -\tau_m^{(2k+1)}, \tag{3.20}$$

where $\tau_m^{(2k+1)}$ are the eigenvalues of $T^{(2k+1)}$.

By Lemma 3.2 the solution to (3.11) for $j = 2k + 1$ reads as follows

$$\Psi_1^{(m)}(\xi') = \Phi_1^{(m)}(\xi') + \sum_{p=1}^{N} b_{p,1}^{(m)}\Psi_0^{(p)}, \tag{3.21}$$

where $\Phi_1^{(m)}$ is orthogonal to all $\Psi_0^{(l)}$, $l = 1, \dots, N$, in $L_2(\mathbb{R}^{d-1})$ and $b_{p,1}^{(m)}$ are constants to be found. It follows from Lemma 3.3 that $\Phi_1^{(m)} \in \mathfrak{V}$. The definition (3.8) of $\mathcal{L}_1$ and Eq. (3.11) for $j = 2k + 1$ imply that the right-hand side of the equation in (3.7) for $j = 2k + 1$ is zero. Hence, the solution to the problem (3.7) for $j = 2k + 1$ is given by formula (3.9), where $\Psi_{2k+1}^{(m)}$ is to be found. We substitute (3.9), (3.21) into the equation (3.11) for $j = 2k + 2$. In view of (3.17) and (3.20) the solvability condition for this equation is as follows

$$b_{l,1}^{(m)}\big(\tau_l^{(2k+1)} - \tau_m^{(2k+1)}\big) + c_{2k+2}^{(m)}\delta_{ml} + 2\big(\mathcal{L}_2\psi_0^{(m)} + \mathcal{L}_1\Phi_1^{(m)} \sin\pi n\xi_d, \psi_0^{(l)}\big)_{L_2(\Pi)} = 0,$$
$$l = 1, \dots, N. \tag{3.22}$$

Assume that all the eigenvalues $\tau_m^{(2k+1)}$ are different. In this case the last identities imply

$$b_{l,1}^{(m)} = \frac{2(\mathcal{L}_2\psi_0^{(m)} + \mathcal{L}_1\Phi_1^{(m)} \sin\pi n\xi_d, \psi_0^{(l)})_{L_2(\Pi)}}{\tau_m^{(2k+1)} - \tau_l^{(2k+1)}}, \quad m \neq l,$$
$$c_{2k+2}^{(m)} = -2\big(\mathcal{L}_2\psi_0^{(m)} + \mathcal{L}_1\Phi_1^{(m)} \sin\pi n\xi_d, \psi_0^{(m)}\big)_{L_2(\Pi)}, \tag{3.23}$$

and we can also let $b_{m,1}^{(m)} = 0$.

Now suppose that all the eigenvalues $\tau_m^{(2k+1)}$ are equal. In this case the equations (3.22) do not allow us to determine the constants $b_{l,1}^{(m)}$ for $m \neq l$. Consider the matrix $T^{(2k+2)}$ with the entries

$$T_{ml}^{(2k+2)} := 2\big(\mathcal{L}_2\psi_0^{(m)} + \mathcal{L}_1\Phi_1^{(m)} \sin\pi n\xi_d, \psi_0^{(l)}\big)_{L_2(\Pi)}.$$

Lemma 3.5. *The matrix $T^{(2k+2)}$ is symmetric.*

Proof. Integrating by parts, we obtain

$$T_{ml}^{(2k+2)} = 2\big(\psi_0^{(m)}, \mathcal{L}_2\psi_0^{(l)}\big)_{L_2(\Pi)} + 2\big(\Phi_1^{(m)} \sin\pi n\xi_d, \mathcal{L}_1\psi_0^{(l)}\big)_{L_2(\Pi)}.$$

Since by (3.7)

$$\mathcal{L}_1\psi_0^{(l)} = -\left(\frac{1}{H_0^2}\frac{\partial^2}{\partial\xi_d^2} + c_0^{(m)}\right)\psi_{2k+1}^{(m)} - c_{2k+1}^{(m)}\psi_0^{(m)},$$

in view of (3.9), (3.11), (3.21) we have

$$\begin{aligned} 2\big(\Phi_1^{(m)} \sin\pi n\xi_d, \mathcal{L}_1\psi_0^{(l)}\big)_{L_2(\Pi)} &= \big(\Phi_1^{(m)}, (G_n - \Lambda)\Phi_1^{(l)}\big)_{L_2(\mathbb{R}^{d-1})} \\ &= \big((G_n - \Lambda)\Phi_1^{(m)}, \Phi_1^{(l)}\big)_{L_2(\mathbb{R}^{d-1})} \\ &= 2\big(\mathcal{L}_1\psi_0^{(m)}, \Phi_1^{(m)} \sin\pi n\xi_d\big)_{L_2(\Pi)}. \qquad \square \end{aligned}$$

Since we supposed that all the eigenvalues of $T^{(2k+1)}$ are equal, we can make orthogonal transformation in the space spanned over $\Psi_0^{(m)}$, $m = 1, \dots, N$, without destroying the orthonormality in $L_2(\Pi)$ and diagonalization of $T^{(2k+1)}$. We employ this freedom to diagonalize the matrix $T^{(2k+2)}$ which is possible due to Lemma 3.5. After such diagonalization we see that the coefficients $c_{2k+2}^{(m)}$ are determined by the eigenvalues of the matrix $T^{(2k+2)}$:

$$c_{2k+2}^{(m)} = -\tau_m^{(2k+2)}.$$

If all these eigenvalues are distinct, we can determine the numbers $b_{l,1}^{(m)}$ at the next step by formulae similar to (3.23). If all these eigenvalues are identical, at the next step we should consider the next matrix $T^{(2k+3)}$ and diagonalize it.

There exists one more possibility. Namely, the matrix $T^{(2k+1)}$ can have different multiple eigenvalues. We do not treat this case here. The reason is that the formal construction of the asymptotics is rather complicated from the technical point of view and at the same time it does not require any new ideas in comparison with the cases discussed above. Thus, from now on, we consider two cases only. More precisely, in the first case we assume that the matrix $T^{(2k+1)}$ has N different eigenvalues $\tau_m^{(2k+1)}$, $m = 1, \dots, N$. In the second case we suppose that the matrix $T^{(2k+1)}$ has only one eigenvalue $\tau^{(2k+1)}$ with multiplicity N, while the matrix $T^{(2k+2)}$ has N different eigenvalues $\tau_m^{(2k+2)}$, $m = 1, \dots, N$.

Lemma 3.6. *Assume that the matrix $T^{(2k+1)}$ has N different eigenvalues and choose $\Psi_0^{(m)}$ being orthonormalized in $L_2(\mathbb{R}^{d-1})$ and so that the matrix $T^{(2k+1)}$ is diagonal. Then problems* (3.5), (3.6), (3.7) *have solutions*

$$\psi_j^{(m)}(\xi) = \widetilde{\psi}_j^{(m)}(\xi) + \widetilde{\Psi}_j^{(m)}(\xi') \sin\pi n\xi_d + \sum_{p=1}^{N} b_{j,p}^{(m)}\psi_0^{(p)}(\xi).$$

Here the functions $\widetilde{\psi}_j^{(m)}$ *are zero for* $j \leqslant 2k+1$*, while for other* j *they solve the problems*

$$-\left(\frac{1}{H_0^2}\frac{\partial^2}{\partial\xi_d^2} + c_0^{(m)}\right)\widetilde{\psi}_j^{(m)} = \left(\Delta_{\xi'} - \frac{2H_{2k}(\xi')}{H_0^3}\frac{\partial^2}{\partial\xi_d^2} + \Lambda\right)\widetilde{\psi}_{j-2k}^{(m)} + \sum_{q=2k+2}^{j-2k-1} c_{j-q}^{(m)}\widetilde{\psi}_q^{(m)} + F_j^{(m)}$$

$$- 2\big(F_j^{(m)}, \sin\pi n\xi_d\big)_{L_2(0,1)} \sin\pi n\xi_d \quad \text{in } \Pi,$$

$$\widetilde{\psi}_j^{(m)} = 0 \quad \text{on } \partial\Pi,$$

and are represented as finite sums

$$\widetilde{\psi}_j^{(m)}(\xi) = \sum_{\varsigma} \psi_{j,\varsigma,1}^{(m)}(\xi')\psi_{j,\varsigma,2}^{(m)}(\xi_d),$$

where $\psi_{j,\varsigma,1}^{(m)} \in \mathfrak{V}$*,* $\psi_{j,\varsigma,2}^{(m)} \in C_0^\infty[0,1]$*,* $\psi_{j,\varsigma,2}^{(m)}(0) = \psi_{j,\varsigma,2}^{(m)}(1) = 0$*, and the functions* $\psi_{j,\varsigma,2}^{(m)}$ *are orthogonal to* $\sin\pi n\xi_d$ *in* $L_2(0,1)$*. The functions* $\widetilde{\Psi}_j^{(m)} \in \mathfrak{V}$ *are solutions to Eqs.* (3.11) *and are orthogonal to all* $\Psi_0^{(l)}$*,* $l = 1,\dots,N$*, in* $L_2(\mathbb{R}^{d-1})$*. The constants* $b_{j,p}^{(m)}$ *and* $c_j^{(m)}$ *are determined by the formulae*

$$b_{0,l}^{(m)} = \delta_{ml}, \qquad b_{j,m}^{(m)} = 0, \quad j \geqslant 1,$$

$$b_{j,l}^{(m)} = \frac{2(\widetilde{F}_{j+2k+1}^{(m)}, \psi_0^{(l)}) + \sum_{q=1}^{j-1} c_{j+2k-q+1}^{(m)} b_{q,l}^{(m)}}{\tau_{2k+1}^{(m)} - \tau_{2k+1}^{(l)}}, \quad m \neq l,\ j \geqslant 1,$$

$$c_{2k}^{(m)} = \Lambda, \qquad c_{2k+1}^{(m)} = -\tau_m^{(2k+1)},$$

$$c_j^{(m)} = -2\big(\widetilde{F}_j^{(m)}, \psi_0^{(m)}\big)_{L_2(\Pi)}, \quad j \geqslant 2k+2,$$

$$\widetilde{F}_j^{(m)} = \sum_{q=0}^{j-2k-1} \mathcal{L}_{j-q-2k}\big(\widetilde{\psi}_q^{(m)} + \widetilde{\Psi}_q^{(m)}\sin\pi n\xi_d\big) + \sum_{q=0}^{j-2k-2}\sum_{p=1}^{N} b_{q,p}^{(m)}\mathcal{L}_{j-q-2k}\psi_0^{(p)}.$$

Lemma 3.7. *Assume that all the eigenvalues of the matrix* $T^{(2k+1)}$ *are identical and that the matrix* $T^{(2k+2)}$ *has* N *different eigenvalues, and choose* $\Psi_0^{(m)}$ *being orthonormalized in* $L_2(\mathbb{R}^{d-1})$ *so that the matrices* $T^{(2k+1)}$ *and* $T^{(2k+2)}$ *are diagonal. Then problems* (3.5)*,* (3.6)*,* (3.7) *have solutions*

$$\psi_j^{(m)}(\xi) = \widetilde{\psi}_j^{(m)}(\xi) + \widetilde{\Psi}_j^{(m)}(\xi')\sin\pi n\xi_d$$

$$+ \sum_{p=1}^{N} b_{j-1,p}^{(m)}\Phi_1^{(p)}(\xi')\sin\pi n\xi_d + \sum_{p=1}^{N} b_{j,p}^{(m)}\psi_0^{(p)}(\xi).$$

Here the functions $\widetilde{\psi}_j^{(m)}$ *are zero for* $j \leqslant 2k+1$*, while for other* j *they solve the problems*

$$-\left(\frac{1}{H_0^2}\frac{\partial^2}{\partial\xi_d^2}+c_0^{(m)}\right)\widetilde{\psi}_j^{(m)}=\left(\Delta_{\xi'}-\frac{2H_{2k}(\xi')}{H_0^3}\frac{\partial^2}{\partial\xi_d^2}+\Lambda\right)\widetilde{\psi}_{j-2k}^{(m)}+\sum_{q=2k+2}^{j-2k-1}c_{j-q}^{(m)}\widetilde{\psi}_q^{(m)}+\widetilde{F}_j^{(m)}$$
$$-2\big(\widetilde{F}_j^{(m)},\sin\pi n\xi_d\big)_{L_2(\Pi)}\sin\pi n\xi_d \quad \text{in } \Pi,$$
$$\widetilde{\psi}_j^{(m)}=0 \quad \text{on } \partial\Pi,$$
$$\widetilde{F}_j^{(m)}:=\sum_{q=0}^{j-2k-1}\mathcal{L}_{j-q-2k}\big(\widetilde{\psi}_q^{(m)}+\widetilde{\Psi}_q^{(m)}\sin\pi n\xi_d\big)+\sum_{p=1}^{N}\sum_{q=1}^{j-2k-2}b_{q-1,p}^{(m)}\mathcal{L}_{j-q-2k}\Phi_1^{(p)}\sin\pi n\xi_d$$
$$+\sum_{p=1}^{N}\sum_{q=0}^{j-2k-3}b_{q,p}^{(m)}\mathcal{L}_{j-q-2k}\psi_0^{(p)},$$

and are represented as finite sums

$$\widetilde{\psi}_j^{(m)}(\xi)=\sum_{\varsigma}\psi_{j,\varsigma,1}^{(m)}(\xi')\psi_{j,\varsigma,2}^{(m)}(\xi_d),$$

where $\psi_{j,\varsigma,1}^{(m)}\in\mathfrak{V}$*,* $\psi_{j,\varsigma,2}^{(m)}\in C_0^\infty[0,1]$*,* $\psi_{j,\varsigma,2}^{(m)}(0)=\psi_{j,\varsigma,2}^{(m)}(1)=0$*, and the functions* $\psi_{j,\varsigma,2}^{(m)}$ *are orthogonal to* $\sin\pi n\xi_d$ *in* $L_2(0,1)$*. The functions* $\widetilde{\Psi}_j^{(m)}\in\mathfrak{V}$ *are solutions to the equations*

$$(G_n-\Lambda)\widetilde{\Psi}_j^{(m)}=\widetilde{f}_{j+2k}^{(m)}+\sum_{q=1}^{j-1}c_{j+2k-q}^{(m)}\widetilde{\Psi}_q^{(m)}+\sum_{q=1}^{j-3}\sum_{p=1}^{N}c_{j+2k-q}^{(m)}b_{q,p}^{(m)}\Psi_0^{(p)}$$
$$+\sum_{q=1}^{j-2}\sum_{p=1}^{N}c_{j+2k-q}^{(m)}b_{q-1,p}^{(m)}\Phi_1^{(p)}-\sum_{p=1}^{N}\big(\widetilde{f}_{j+2k}^{(m)},\Psi_0^{(p)}\big)_{L_2(\mathbb{R}^{d-1})}\Psi_0^{(p)},$$

and are orthogonal to all $\Psi_0^{(l)}$*,* $l=1,\dots,N$*, in* $L_2(\mathbb{R}^{d-1})$*. The constants* $b_{j,p}^{(m)}$ *and* $c_j^{(m)}$ *are determined by the formulae*

$$b_{l,-1}^{(m)}=0,\qquad b_{0,l}^{(m)}=\delta_{ml},\qquad b_{j,m}^{(m)}=0,\quad j\geqslant 1,$$
$$b_{j,l}^{(m)}=\frac{2(\widetilde{F}_{j+2k+2}^{(m)},\psi_0^{(l)})+\sum_{q=1}^{j-1}c_{j+2k-q+2}^{(m)}b_{q,l}^{(m)}}{\tau_{2k+1}^{(m)}-\tau_{2k+1}^{(l)}},\quad m\neq l,\ j\geqslant 1,$$
$$c_{2k}^{(m)}=\Lambda,\qquad c_{2k+1}^{(m)}=-\tau^{(2k+1)},\qquad c_{2k+2}^{(m)}=-\tau_m^{(2k+2)},$$
$$c_j^{(m)}=-2\big(\widetilde{F}_j^{(m)},\psi_0^{(m)}\big)_{L_2(\Pi)},\quad j\geqslant 2k+3.$$

These lemmata can be proven by induction.

Remark 3.8. We observe that if Λ is simple, then $N=1$ and the hypothesis of Lemma 3.6 is obviously true.

We denote

$$\psi_{\varepsilon,s}^{(m)}(x) := \chi(x')\sqrt{H(x')}\sum_{j=0}^{s}\eta^j\psi_j^{(m)}\left(\frac{x'-\bar{x}}{\eta}, \frac{x_d+\varepsilon h_-(x')}{\varepsilon H(x')}\right),$$

$$\lambda_{\varepsilon,s}^{(m)} := \varepsilon^{-2}c_0^{(m)} + \varepsilon^{-2}\sum_{j=2k}^{s}\eta^j c_j^{(m)}, \quad s \geqslant 2k.$$

The next lemma follows from the construction of the functions $\psi_j^{(m)}$ and the constants $c_j^{(m)}$.

Lemma 3.9. *The functions $\psi_{\varepsilon,s}^{(m)}$ solve the boundary value problems*

$$-\left(\Delta_{\Omega_\varepsilon}^D + \lambda_{\varepsilon,s}^{(m)}\right)\psi_{\varepsilon,s}^{(m)} = g_{\varepsilon,s}^{(m)}, \quad m = 1, \ldots, N, \tag{3.24}$$

where the right-hand sides satisfy the estimate

$$\left\|g_{\varepsilon,s}^{(m)}\right\|_{L_2(\Omega_\varepsilon)} = \mathcal{O}\left(\eta^{s-\frac{3k-d}{2}-2}\right), \quad m = 1, \ldots, N. \tag{3.25}$$

We rewrite problem (3.24) as

$$\psi_{\varepsilon,s}^{(m)} = A_\varepsilon\psi_{\varepsilon,s}^{(m)} + \frac{1}{1+\lambda_{\varepsilon,s}^{(m)}}A_\varepsilon g_{\varepsilon,s}^{(m)},$$

where $A_\varepsilon := (-\Delta_{\Omega_\varepsilon}^D + 1)^{-1}$. This operator is self-adjoint, compact and satisfies the estimate $\|A_\varepsilon\| \leqslant 1$. In view of this estimate and (3.25) we have

$$\left\|\frac{1}{1+\lambda_{\varepsilon,s}^{(m)}}A_\varepsilon g_{\varepsilon,s}^{(m)}\right\|_{L_2(\Omega_\varepsilon)} \leqslant C_{m,s}\eta^{s-\frac{3k-d}{2}+2k}, \quad m = 1, \ldots, N,$$

where $C_{m,s}$ are constants. We apply Lemma 1.1 to conclude that there exists an eigenvalue $\varrho_s^{(m)}(\varepsilon)$ of A_ε such that

$$\left|\varrho_s^{(m)}(\varepsilon) - \left(1+\lambda_{\varepsilon,s}^{(m)}\right)^{-1}\right| \leqslant C_{m,s}\eta^{s-\frac{3k-d}{2}+2k}, \quad m = 1, \ldots, N.$$

Hence, the number $\lambda_s^{(m)}(\varepsilon) := \left(\varrho_s^{(m)}(\varepsilon)\right)^{-1} - 1$ is an eigenvalue of the operator $-\Delta_{\Omega_\varepsilon}^D$, which satisfies the inequality

$$\left|\lambda_s^{(m)}(\varepsilon) - \lambda_{\varepsilon,s}^{(m)}\right| \leqslant \widetilde{C}_{m,s}\eta^{s-\frac{7k-d}{2}-4}, \quad m = 1, \ldots, N, \tag{3.26}$$

where $\widetilde{C}_{m,s}$ are constants.

Let $\varepsilon_s^{(m)}$ be a monotone sequence such that $\widetilde{C}_{m,s}\eta \leqslant \widetilde{C}_{m,s-1}$ as $\varepsilon \leqslant \varepsilon_s^{(m)}$. We choose the eigenvalue $\lambda_\varepsilon^{(m)} := \lambda_{\varepsilon,s}^{(m)}$ as $\varepsilon \in [\varepsilon_s^{(m)}, \varepsilon_{s+1}^{(m)})$. Inequality (3.26) implies that the eigenvalue $\lambda_\varepsilon^{(m)}$ has the asymptotics (2.2). We employ Lemma 1.1 in [8, Ch. III, Sec. 1.1] once again with

$\alpha = C_{m,s}\eta^{s-\frac{3k-d}{2}}$, $d = \sqrt{\alpha}$. It yields that there exists a linear combination $\psi_s^{(m)}(x,\varepsilon)$ of the eigenfunctions of $-\Delta_{\Omega_\varepsilon}^D$ associated with the eigenvalues lying in $[\lambda_\varepsilon^{(m)} - d, \lambda_\varepsilon^{(m)} + d]$ such that

$$\left\| \psi_s^{(m)}(\cdot,\varepsilon) - \psi_{\varepsilon,s}^{(m)} \right\|_{L_2(\Omega_\varepsilon)} = \mathcal{O}\left(\eta^{\frac{2s-3k+d}{4}}\right), \quad m = 1,\dots,N.$$

Since the functions $\psi_{\varepsilon,s}^{(m)}$ are linearly independent for different m, the same is true for $\psi_s^{(m)}(\cdot,\varepsilon)$, if s is large enough. Thus, the total multiplicity of the eigenvalues $\lambda_\varepsilon^{(m)}$ is at least N. The proof is complete.

4. Proof of Theorem 2

In order to prove Theorem 2 we need to ensure that, for sufficiently small ε, the asymptotic expansions for $\lambda_{1,m}$, $m = 1,\dots,N$, provided by Theorem 1 do correspond to the first N eigenvalues of $-\Delta_{\Omega_\varepsilon}^D$ (counting multiplicities). In [1] this was done by means of adapting the proof of Theorem 1.1 in [4] from the situation where $h_- = 0$ to our case. In the present context we need to show that, under the conditions for $h_\pm$, this result may be extended to d dimensions. There are two important points that should be stressed here. On the one hand, we are assuming C^∞ regularity in a neighbourhood of the point of global maximum, and thus do not have to deal with what could now be more complex regularity issues at this point. On the other hand, since the proof of eigenvalue convergence given in [4] is based on convergence in the norm, it is not affected by details related to the possible higher multiplicities as was the case in the derivation of the formulae in the previous section.

While still using the notation defined in Section 2, we also refer to the notation in [4]. In particular, the function h and the operator $\mathbf{H}$ defined there correspond to our width function H and operator G_n, respectively. We begin by assuming H to be strictly positive in $\overline{\omega}$. Let thus

$$\psi(x', x_d) = \psi_\chi(x', x_d) = \chi(x')\sqrt{\frac{2}{\varepsilon H(x')}} \sin\left[\frac{\pi(x_d + \varepsilon h_-(x'))}{\varepsilon H(x')}\right].$$

As in [4], we have

$$\left\| \psi_\chi(x', x_d) \right\|_{L_2(\Omega_\varepsilon)} = \int_\omega \chi^2(x')\,\mathrm{d}x',$$

while now

$$\int_\omega \int_{-\varepsilon h_-(x')}^{\varepsilon h_+(x')} \left|\nabla \psi_\chi(x', x_d)\right|^2 = \int_\omega \left|\nabla \chi(x')\right|^2 + \left(\frac{\pi^2}{\varepsilon^2 H^2(x')} + v(x')\right)\chi^2(x')\,\mathrm{d}x',$$

with

$$v(x') = \frac{\pi^2}{H^2(x')}\left[\left|\frac{1}{2}\nabla H(x') - \nabla h_-(x')\right|^2 + \frac{1}{4}\left(\frac{1}{3} + \frac{1}{\pi^2}\right)\left|\nabla H(x')\right|^2\right].$$

In the notation of [4], the potential W_ε appearing in the quadratic form $q_\varepsilon[\chi]$ (Eq. (1.3) on page 3 in that paper) is now defined by

$$W_\varepsilon(x') = \frac{\pi^2}{\varepsilon^2}\left[\frac{1}{H^2(x')} - \frac{1}{H^2(\bar{x})}\right] + v(x').$$

We consider the scaling $x' = e^\alpha t$ as before, which causes the domain ω to be scaled to $\omega_\varepsilon = e^\alpha \omega$. Then the proofs of Lemma 2.1 and Theorem 1.2 go through with minor changes (note that $m = 2k$, while I and I_ε should be changed by ω and ω_ε, respectively). Similar remarks apply to the proofs in Section 4 of [4] leading to the proof of Theorem 1.3, except that due to regularity we do not need to worry about separating the domain into different parts as was necessary there for the intervals I_ε.

Finally, we relax the condition on the strict positivity of H mentioned above. This again follows in a similar fashion to what was done in Section 6.1 of [4].

We are now in conditions to proceed to the proof of (2.5). In the case considered the lowest eigenvalue of G_1 is $\Lambda = \sum_{j=1}^d \theta_j$, while the associated eigenfunction is given by (2.6). This proves the formula for $c_2^{(1,1)}$. In view of Remark 3.8, we can employ Lemma 3.6 to calculate $c_3^{(1,1)}$, $c_4^{(1,1)}$. Since Ψ_0 is even w.r.t. each ξ_i, $i = 1, \dots, d-1$, and $H_3(-\xi') = -H_3(\xi')$, we conclude by (3.19) that $T_{11}^{(3)} = 0$. By Theorem 1 it yields that $c_3^{(1,1)} = 0$.

Eq. (3.11) for $\widetilde{\Psi}_1$ with $j = 3$ reads as follows

$$(G_1 - \Lambda)\widetilde{\Psi}_1 = \frac{2\pi^2}{H_0^3} H_3 \Psi_0. \tag{4.1}$$

We seek the solution as $\widetilde{\Psi}_1 = R\Psi_0$, where R is a polynomial of the form

$$R(\xi') := \sum_{p,q,j=1}^{d-1} C_{pqj}\xi_p\xi_q\xi_j + \sum_{j=1}^{d-1} C_j\xi_j, \tag{4.2}$$

where C_{pqj}, C_j are constants to be found, and C_{pqj} are invariant under each permutation of the indices p, q, j. We also note that such a choice of R ensures that $(\widetilde{\Psi}_1, \Psi_0)_{L_2(\mathbb{R}^{d-1})} = 0$. We substitute (4.2) and the formula for $\widetilde{\Psi}_1$ into (4.1) taking into account (2.8),

$$\begin{aligned} &2\sum_{p,q,j=1}^{d-1} (\theta_p + \theta_q + \theta_j) C_{pqj}\xi_p\xi_q\xi_j + 2\sum_{j=1}^{d-1}\theta_j C_j \xi_j + 6\sum_{p,j=1}^{d-1} C_{ppj}\xi_j \\ &\quad = -\frac{2\pi^2}{H_0^3}\sum_{p,q,j=1}^{d-1} \beta_{pqj}\xi_p\xi_q\xi_j. \end{aligned}$$

It yields the formulae

$$C_{pqj} = -\frac{\pi^2 \beta_{pqj}}{H_0^3(\theta_p + \theta_q + \theta_j)}, \qquad C_j = \frac{3}{2}\sum_{p=1}^{d-1} \frac{\pi^2 \beta_{ppj}}{H_0^3 \theta_j (2\theta_p + \theta_j)}. \tag{4.3}$$

It is easy to check that

$$Q_1^{(i)}(\xi') = -\frac{1}{H_0}\frac{\partial H_2}{\partial x_i}(\xi'), \qquad P_0^{(0)} = \frac{1}{2H_0}\Delta_{x'} H_2,$$
$$P_4^{(d)}(\xi') = H_0^{-4}\big(3H_2^2(\xi') - 2H_0 H_4(\xi')\big). \tag{4.4}$$

Employing these identities, we write the formula for $c_4^{(1,1)}$ from Lemma 3.6

$$\begin{aligned}
c_4^{(1,1)} &= -2(\widetilde{F}_4, \psi_0)_{L_2(\Pi)} = -2(\mathcal{L}_2\psi_0, \psi_0)_{L_2(\Pi)} - 2(\mathcal{L}_1\widetilde{\Psi}_1 \sin\pi\xi_d, \psi_0)_{L_2(\Pi)} \\
&= \pi^2\big(P_4^{(d)}\Psi_0, \Psi_0\big)_{L_2(\mathbb{R}^{d-1})} - \big(P_0^{(0)}\Psi_0, \Psi_0\big)_{L_2(\mathbb{R}^{d-1})} \\
&\quad + 4\pi\sum_{i=1}^{d-1}\left(Q_1^{(i)}\frac{\partial\Psi_0}{\partial\xi_i}\sin\pi\xi_d, \Psi_0\xi_d\cos\pi\xi_d\right)_{L_2(\Pi)} \\
&\quad - 2\sum_{i=1}^{d-1}\big(K_0^{(i)}\big)^2\left(\frac{\partial^2\psi_0}{\partial\xi_d^2}, \psi_0\right)_{L_2(\Pi)} + \pi^2\big(P_3^{(d)}\widetilde{\Psi}_1, \Psi_0\big)_{L_2(\mathbb{R}^{d-1})} \\
&= \pi^2\big(P_4^{(d)}\Psi_0, \Psi_0\big)_{L_2(\mathbb{R}^{d-1})} + \frac{1}{2H_0}\sum_{i=1}^{d-1}\alpha_i^2 - \sum_{i=1}^{d-1}\left(Q_1^{(i)}\frac{\partial\Psi_0}{\partial\xi_i}, \Psi_0\right)_{L_2(\mathbb{R}^{d-1})} \\
&\quad + \frac{\pi^2}{H_0^2}\sum_{i=1}^{d-1}\left(\frac{\partial h_1}{\partial x_i}(\bar{x})\right)^2 + \pi^2\big(P_3^{(d)}\widetilde{\Psi}_1, \Psi_0\big)_{L_2(\mathbb{R}^{d-1})} \\
&= \pi^2\big(\big(P_4^{(d)} + P_3^{(d)}\big)\Psi_0, \Psi_0\big)_{L_2(\mathbb{R}^{d-1})} + \frac{\pi^2}{H_0^2}\sum_{i=1}^{d-1}\left(\frac{\partial h_1}{\partial x_i}(\bar{x})\right)^2.
\end{aligned}$$

We substitute (4.2), (4.3), (4.4) into this identity and arrive at the desired formula for $c_4^{(1,1)}$.

5. The d-dimensional ellipsoid

As an application of our results, we will derive the expression (1.3) for the asymptotic expansion for the first eigenvalue for a general ellipsoid. From the equation defining the boundary of $\mathcal{E}$ and assuming that, as mentioned in the Introduction, we are doing the scaling along the x_d axis, we have

$$h_\pm(x') = a_d\left[1 - \left(\frac{x_1}{a_1}\right)^2 - \cdots - \left(\frac{x_{d-1}}{a_{d-1}}\right)^2\right]^{1/2},$$

while $H(x') = 2h_\pm(x')$. We thus have $\bar{x}$ located at the origin and $H_0 = 2a_d$. Expanding H around $\bar{x}$ we have

$$H(x')=2a_d-a_d\left[\left(\frac{x_1}{a_1}\right)^2+\cdots+\left(\frac{x_{d-1}}{a_{d-1}}\right)^2\right]-\frac{a_d}{4}\left[\left(\frac{x_1}{a_1}\right)^4+\cdots+\left(\frac{x_{d-1}}{a_{d-1}}\right)^4\right.$$
$$\left.+2\left(\frac{x_1x_2}{a_1a_2}\right)^2+2\left(\frac{x_1x_3}{a_1a_3}\right)^2+\cdots+2\left(\frac{x_{d-2}x_{d-1}}{a_{d-2}a_{d-1}}\right)^2\right]+\cdots,$$

yielding $H_k=h_k=0$ for odd k and

$$H_2(x')=-a_d\sum_{i=1}^{d-1}\left(\frac{x_i}{a_i}\right)^2$$
$$H_4(x')=-\frac{a_d}{4}\sum_{i=1}^{d-1}\sum_{j=1}^{d-1}\left(\frac{x_ix_j}{a_ia_j}\right)^2.$$

Hence

$$\alpha_i=\frac{\sqrt{2a_d}}{a_i},\qquad \theta_i=\frac{\pi}{2a_ia_d}$$

and

$$\psi_0(x')=\frac{2^{\frac{1-d}{4}}a_d^{\frac{1-d}{4}}}{(a_1\ldots a_{d-1})^{1/2}}e^{-\frac{\pi}{4a_d}(\frac{x_1^2}{a_1}+\cdots+\frac{x_{d-1}^2}{a_{d-1}})}.$$

Note that since H_3 is identically zero, there is no need to compute $\widetilde{\Psi}_1$. It is now straightforward to obtain

$$c_0^{(1,1)}=\frac{\pi^2}{4a_d^2}\quad\text{and}\quad c_2^{(1,1)}=\frac{\pi}{2a_d}\sum_{i=1}^{d-1}\frac{1}{a_i}.$$

It remains to compute

$$c_4^{(1,1)}=\frac{\pi^2}{16a_d^2}\left(\left[3\left(\sum_{i=1}^{d-1}\left(\frac{x_i}{a_i}\right)^2\right)^2+\sum_{i=1}^{d-1}\sum_{j=1}^{d-1}\left(\frac{x_ix_j}{a_ia_j}\right)^2\right]\psi_0(x'),\psi_0(x')\right)_{L_2(\mathbb{R}^{d-1})}$$
$$=\frac{\pi^2}{2^{\frac{d+3}{2}}a_d^{\frac{d+3}{2}}(a_1\ldots a_{d-1})^{1/2}}\sum_{i=1}^{d-1}\sum_{j=1}^{d-1}\int_{\mathbb{R}^{d-1}}\left(\frac{x_ix_j}{a_ia_j}\right)^2e^{-\frac{\pi}{2a_d}(\frac{x_1^2}{a_1}+\cdots+\frac{x_{d-1}^2}{a_{d-1}})}\,\mathrm{d}x'$$

which, after some further simplifications, yields the desired result.

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