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Samoilenko's Method to Differential Algebraic Systems with Integral Boundary Conditions

T. JANKOWSKI
Technical University of Gdańsk
Department of Differential Equations
11/12 G.Narutowicz Str., 80–952 Gdańsk, Poland

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Abstract—The numerical analytic method combined with the comparison one is used to establish solvability of differential algebraic systems with integral boundary conditions. Existence results are formulated under assumptions that corresponding functions satisfy the Lipschitz conditions in matrix notation. A problem with deviated arguments is also discussed.

Keywords—Numerical analytic method, Integral boundary conditions, Differential algebraic systems, Comparison method.

1. INTRODUCTION

A useful approach in the studying of existence of solutions is Samoilenko's numerical analytic method (for details, see [1,2]). In this paper, we apply this technique to differential algebraic systems of the form

$$\begin{aligned} x'(t) &= f(t, x(t), y(t)), \qquad t \in J = [0, T], \\ y(t) &= g(t, x(t), y(t)), \qquad t \in J \end{aligned} \tag{1}$$

with the integral boundary condition

$$A_0 x(0) + \int_0^{\xi} D(s)x(s)\,ds + A_1 x(T) = d, \tag{2}$$

where $f \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^p)$, $g \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q)$. The value ξ is a fixed constant and $0 < \xi \le T$. In the above, $(A_0)_{p\times p}$, $(A_1)_{p\times p}$, $D_{p\times p}$, and $d_{p\times 1}$ are given matrices. The application of numerical analytic method to differential systems $x'(t) = f(t, x(t))$ with condition (2) can be found, for example, in papers [1–10] if $D(t) = 0$ on $[0, \xi]$, see also [11].

The numerical analytic method combined with the comparison one is used to formulate corresponding existence results for problems of type (1),(2) under the assumption that f and g satisfy the Lipschitz conditions (with respect to the last two variables) in matrix notation. The aim

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of the present paper is to discuss the conditions under which the solution can be obtained by the method of successive approximations and Seidel's method too. A more general differential algebraic problem with deviated arguments is also considered and corresponding existence results are given in Section 5.

2. ASSUMPTIONS

Put

$$\mathcal{L}f(t,x,y) = \left(1-\frac{t}{T}\right)\int_0^t f(s,x(s),y(s))\,ds - \frac{t}{T}\int_t^T f(s,x(s),y(s))\,ds,$$

$$B_0 = \int_0^\xi sD(s)\,ds, \qquad B_1 = \int_0^\xi D(s)\,ds,$$
$$B_2 = (A_1T + B_0)^{-1}, \qquad B_3(\bar{x}_0) = B_2\left[d - (A_0 + A_1 + B_1)\bar{x}_0\right],$$

assuming that the matrix B_2 exists. Apply the numerical analytic method to problem (1),(2) to obtain the following auxiliary system

$$\begin{aligned} x(t) &= \bar{x}_0 + \mathcal{L}f(t,x,y) - B_2t\int_0^\xi D(s)\mathcal{L}f(s,x,y)\,ds + tB_3(\bar{x}_0) \equiv F(t,x,y;\bar{x}_0), && t \in J, \\ y(t) &= g(t,x(t),y(t)), && t \in J. \end{aligned} \tag{3}$$

Note that if x satisfies the first equation of problem (3), then condition (2) is satisfied too. Moreover, $F(0,x,y;\bar{x}_0) = \bar{x}_0$, so $x(0) = \bar{x}_0$.

Let us introduce the following.

ASSUMPTION H_1.

1° *There are matrices $K_{p\times p}, L_{p\times q}$ with nonnegative entries such that*

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \le K|x-\bar{x}| + L|y-\bar{y}|,$$

for all $t \in J$, $x,\bar{x} \in \mathbb{R}^p$, $y,\bar{y} \in \mathbb{R}^q$.

2° *There are matrices $M_{q\times p}, N_{q\times q}$ with nonnegative entries, $\rho(N) < 1$, and such that*

$$|g(t,x,y) - g(t,\bar{x},\bar{y})| \le M|x-\bar{x}| + N|y-\bar{y}|,$$

for all $t \in J$, x, $\bar{x} \in \mathbb{R}^p$, y, $\bar{y} \in \mathbb{R}^q$. Here $|\cdot|$ denotes the absolute value of the vector, so $|x| = (|x_1|,\dots,|x_p|)^\top$ or $|y| = (|y_1|,\dots,|y_q|)^\top$. Moreover, $\rho(N)$ denotes the spectral radius of the matrix N.

ASSUMPTION H_2. *For any nonnegative function $h \in C(J\times\mathbb{R}^p,\mathbb{R}^p_+)$, there exists a unique solution $u \in C(J,\mathbb{R}^p_+)$ of the comparison equation*

$$\Omega(t,u) + |B_2|\,t\int_0^\xi |D(s)|\Omega(s,u)\,ds + h(t,\bar{x}_0) = u(t), \qquad t \in J, \tag{4}$$

where

$$\Omega(t,u) = \left(1-\frac{t}{T}\right)\int_0^t Au(s)\,ds + \frac{t}{T}\int_t^T Au(s)\,ds, \qquad \text{with } A = K + L(I-N)^{-1}M.$$

Put

$$\Omega_1(t,u,v) = \left(1-\frac{t}{T}\right)\int_0^t [Ku(s)+Lv(s)]\,ds + \frac{t}{T}\int_t^T [Ku(s)+Lv(s)]\,ds.$$

Then, by Assumption $H_1(1^\circ)$, for $t \in J$, we have

$$\begin{aligned} |\mathcal{L}f(t,x,y) - \mathcal{L}f(t,\bar{x},\bar{y})| &\le \Omega_1(t, |x-\bar{x}|, |y-\bar{y}|), \\ |F(t,x,y;\bar{x}_0) - F(t,\bar{x},\bar{y};\bar{x}_0)| &\le |\mathcal{L}f(t,x,y) - \mathcal{L}f(t,\bar{x},\bar{y})| \\ &\quad + |B_2|\, t \int_0^{\xi} |D(s)[\mathcal{L}f(s,x,y) - \mathcal{L}f(s,\bar{x},\bar{y})]|\, ds \\ &\le \Omega_1(t, |x-\bar{x}|, |y-\bar{y}|) \\ &\quad + |B_2|\, t \int_0^{\xi} |D(s)| \Omega_1(s, |x-\bar{x}|, |y-\bar{y}|)\, ds. \end{aligned} \tag{5}$$

3. LEMMAS

For $t \in J$, $n = 0, 1, \dots$, let us define the sequences $\{u_n, w_n\}$ by formulas

$$\begin{aligned} u_{n+1}(t) &= \Omega_1(t, u_n, w_n) + |B_2|\, t \int_0^{\xi} |D(s)| \Omega_1(s, u_n, w_n)\, ds, \qquad u_0(t) = u(t), \\ w_{n+1}(t) &= M u_n(t) + N w_n(t), \qquad w_0(t) = (I-N)^{-1}[M u_0(t) + |g(t,x_0,y_0) - y_0(t)|], \end{aligned}$$

where u is defined as in Assumption H_2 with

$$h(t,\bar{x}_0) = |F(t,x_0,y_0;\bar{x}_0) - x_0(t)| + \bar{\Omega}(t,r) + |B_2|\, t \int_0^{\xi} |D(s)| \bar{\Omega}(s,r)\, ds$$

for $r(t) = |g(t,x_0,y_0) - y_0(t)|$. Here $\bar{\Omega}$ is defined as Ω with the matrix $B = L(I-N)^{-1}$ instead of A.

To obtain a solution of problem (3), we shall first establish some properties for sequences $\{u_n, w_n\}$. They are given in the next two lemmas.

LEMMA 1. *Let Assumptions H_1 and H_2 be satisfied. Assume that the matrix B_2 exists. Then*

$$u_{n+1}(t) = u_n(t) \le u_0(t), \quad w_{n+1}(t) \le w_n(t) \le w_0(t), \qquad t \in J, \quad n = 0, 1, \dots,$$

and the sequences $\{u_n, w_n\}$ converge uniformly to zero functions, so $u_n(t) \to 0$, $w_n(t) \to 0$, $t \in J$ if $n \to \infty$.

PROOF. Note that the matrix $(I-N)^{-1}$ exists and its entries are nonnegative because of the condition $\rho(N) < 1$. Indeed, $\Omega_1(t,u_0,w_0) = \Omega(t,u_0) + \bar{\Omega}(t,r)$. Then

$$\begin{aligned} u_1(t) &= \Omega_1(t,u_0,w_0) + |B_2| t \int_0^{\xi} |D(s)| \Omega_1(s,u_0,w_0)\, ds \\ &= \Omega(t,u_0) + |B_2| t \int_0^{\xi} |D(s)| \Omega(s,u_0)\, ds + \bar{\Omega}(t,r) + |B_2| t \int_0^{\xi} |D(s)| \bar{\Omega}(s,r)\, ds \le u_0(t), \\ w_1(t) &= M u_0(t) + N(I-N)^{-1}[M u_0(t) + r(t)] \le w_0(t), \qquad t \in J. \end{aligned}$$

By induction in n, we are able to prove that

$$u_{n+1}(t) \le u_n(t), \quad w_{n+1}(t) \le w_n(t), \qquad t \in J, \quad n = 0, 1, \dots.$$

Now, if $n \to \infty$, then $u_n \to u$, $w_n \to w$, where the pair (u,w) is a solution of the system

$$\begin{aligned} u(t) &= \Omega_1(t,u,w) + |B_2| t \int_0^{\xi} |D(s)| \Omega_1(s,u,w)\, ds, \qquad && t \in J, \\ w(t) &= Mu + Nw, && t \in J. \end{aligned}$$

Hence, $w(t) = (I-N)^{-1} M u(t)$, so $\Omega_1(t,u,w) = \Omega(t,u)$ showing that u is a solution of problem $u(t) = \Omega(t,u) + |B_2| t \int_0^{\xi} |D(s)| \Omega(s,u)\, ds$, $t \in J$. By Assumption H_2, $u(t) = 0$ on J and then $w(t) = 0$, $t \in J$. The proof is complete.

LEMMA 2. *Assume that $f \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^p)$, $g \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q)$, and $(A_0)_{p\times p}$, $(A_1)_{p\times p}$, $D_{p\times p}$, and $d_{p\times 1}$ are given matrices. Assume that the matrix B_2 exists. Let Assumptions H_1 and H_2 be satisfied. Then we have the estimates*

$$\begin{cases} |x_n(t) - x_0(t)| \le u_0(t), & t \in J, \\ |x_{n+k}(t) - x_k(t)| \le u_k(t), & t \in J, \end{cases} \qquad \begin{cases} |y_n(t) - y_0(t)| \le w_0(t), & t \in J, \\ |y_{n+k}(t) - y_k(t)| \le w_k(t), & t \in J, \end{cases} \tag{6}$$

where $x_0 \in C^1(J, \mathbb{R}^p)$, $y_0 \in C(J, \mathbb{R}^q)$, and

$$x_{n+1}(t) = F(t, x_n, y_n; \bar{x}_0), \quad y_{n+1}(t) = g(t, x_n, y_n), \qquad t \in J. \tag{7}$$

Moreover,

$$A_0 x_{n+1}(0) + A_1 x_{n+1}(T) + \int_0^\xi D(s) x_{n+1}(s)\, ds = d, \qquad n = 0, 1, \dots.$$

PROOF. Put $R(t; \bar{x}_0) = |F(t, x_0, y_0; \bar{x}_0) - x_0(t)|$, $r(t) = |g(t, x_0, y_0) - y_0(t)|$. Indeed,

$$\begin{aligned} |x_1(t) - x_0(t)| &= R(t, \bar{x}_0) \le h(t, \bar{x}_0) \le u_0(t), && t \in J, \\ |y_1(t) - y_0(t)| &= r(t) \le \left[N(I-N)^{-1} + I\right] r(t) \le w_0(t), && t \in J. \end{aligned}$$

Assume that

$$|x_k(t) - x_0(t)| \le u_0(t), \quad |y_k(t) - y_0(t)| \le w_0(t), \qquad t \in J,$$

for some $k \ge 0$. Then, by (5), we have

$$\begin{aligned} |x_{k+1}(t) - x_0(t)| &\le |F(t, x_k, y_k; \bar{x}_0) - F(t, x_0, y_0; \bar{x}_0)| + R(t, \bar{x}_0) \\ &\le \Omega_1(t, u_0, w_0) + |B_2|\, t \int_0^\xi |D(s)| \Omega_1(s, u_0, w_0)\, ds + R(t; \bar{x}_0) = u_0(t), \\ |y_{k+1}(t) - y_0(t)| &\le |g(t, x_k, y_k) - g(t, x_0, y_0)| + r(t) \le M u_0(t) + N w_0(t) + r(t) = w_0(t). \end{aligned}$$

Hence, by mathematical induction, we have

$$|x_n(t) - x_0(t)| \le u_0(t), \quad |y_n(t) - y_0(t)| \le w_0(t), \qquad t \in J,$$

for $n = 0, 1, \dots$. Basing on the above, let us assume that

$$|x_{n+k}(t) - x_k(t)| \le u_k(t), \quad |y_{n+k}(t) - y_k(t)| \le w_k(t), \qquad t \in J,$$

for all n and some $k \ge 0$. Then, by (5), we see that

$$\begin{aligned} |x_{n+k+1}(t) - x_{k+1}(t)| &= |F(t, x_{n+k}, y_{n+k}; \bar{x}_0) - F(t, x_k, y_k; \bar{x}_0)| \\ &\le \Omega_1(t, u_k, w_k) + |B_2|\, t \int_0^\xi |D(s)| \Omega_1(s, u_k, w_k)\, ds = u_{k+1}(t) \\ |y_{n+k+1}(t) - y_{k+1}(t)| &= |g(t, x_{n+k}, y_{n+k}) - g(t, x_k, y_k)| \le M u_k(t) + N w_k(t) = w_{k+1}(t), \end{aligned}$$

for $t \in J$. Hence, by mathematical induction, the estimates (6) hold. It is quite simple to verify that x_{n+1} satisfies integral boundary condition (2) for any $n = 0, 1, \dots$. It ends the proof.

4. EXISTENCE RESULTS

Put

$$\Lambda(x_0, y_0) = \left\{(x,y) \in C^1(J,\mathbb{R}^p) \times C(J,\mathbb{R}^q) : |x_0(t) - x(t)| \le u_0(t),\ |y_0(t) - y(t)| \le w_0(t)\right\}.$$

Combining Lemmas 1 and 2, we have the following.

THEOREM 1. *Assume that* $f \in C(J\times\mathbb{R}^p\times\mathbb{R}^q,\mathbb{R}^p)$, $g \in C(J\times\mathbb{R}^p\times\mathbb{R}^q,\mathbb{R}^q)$, *and* $(A_0)_{p\times p}$, $(A_1)_{p\times p}$, $D_{p\times p}$, *and* $d_{p\times 1}$ *are given matrices. Assume that the matrix* B_2 *exists. Let Assumptions* H_1 *and* H_2 *be satisfied. Then, for every* $\bar{x}_0 \in \mathbb{R}^p$, *there exists a solution* $(\bar{x}, \bar{y})$ *of problem (3) where* $x_n(t) \to \bar{x}(t)$, $y_n(t) \to \bar{y}(t)$, $t \in J$ *as* $n \to \infty$, *and we have the estimates*

$$|x_n(t) - \bar{x}(t)| \le u_n(t), \quad |y_n(t) - \bar{y}(t)| \le w_n(t), \qquad t \in J.$$

The pair $(\bar{x}, \bar{y})$ *is a unique solution of problem (3) in the class* $\Lambda(x_0, y_0)$.

Moreover, $(\bar{x}, \bar{y})$ *is the solution of problem (1),(2) iff*

$$\frac{1}{T}\int_0^T f(s,\bar{x}(s),\bar{y}(s))\,ds + B_2 \int_0^\xi D(s)\mathcal{L}f(s,\bar{x},\bar{y})\,ds = B_3(\bar{x}_0).$$

PROOF. By Lemmas 1 and 2, $x_n(t) \to \bar{x}(t)$, $y_n(t) \to \bar{y}(t)$, $t \in J$. Indeed, $(\bar{x}, \bar{y})$ is a solution of problem (3). We need to show the uniqueness of $(\bar{x}, \bar{y})$. Assume that problem (3) has another solution (X, Y) such that $|X(t) - x_0(t)| \le u_0(t)$, $|Y(t) - y_0(t)| \le w_0(t)$ on J. Then, by (5), we have

$$\begin{aligned}|\bar{x}(t) - X(t)| &\le |\bar{x}(t) - x_{n+1}(t)| + |F(t, x_n, y_n; \bar{x}_0) - F(t, X, Y; \bar{x}_0)| \\ &\le u_{n+1}(t) + \Omega_1(t, |x_n - X|, |y_n - Y|) + |B_2|\, t \int_0^\xi \Omega_1(s, |x_n - X|, |y_n - Y|)\,ds\end{aligned}$$

and

$$|\bar{y}(t) - Y(t)| \le w_{n+1}(t) + M|x_n(t) - X(t)| + N|y_n(t) - Y(t)|,$$

for $t \in J$. Hence, by mathematical induction, we have

$$|\bar{x}(t) - X(t)| \le 2u_{n+1}(t), \quad |y_n(t) - Y(t)| \le 2w_{n+1}(t), \qquad t \in J, \quad n = 0, 1, \dots,$$

showing that $\bar{x} = X$, $\bar{y} = Y$ on J. It ends the proof.

REMARK 1. Let the matrix B_2 exist. Assumption H_2 is satisfied if

$$\rho(Z) < 1, \qquad \text{where } Z = \left[I + |B_2|\, T \int_0^\xi |D(s)|\,ds\right] \frac{T}{2} A. \tag{8}$$

To get condition (8), we need to apply the Banach fixed-point theorem to equation (4). Denote the left-hand side of problem (4) by Λ. Let $u, \bar{u} \in C(J, \mathbb{R}^p_+)$. Then

$$|\Lambda u - \Lambda\bar{u}| = \left|\Omega(t,u) - \Omega(t,\bar{u}) + |B_2|\, t \int_0^\xi |D(s)|\,[\Omega(s,u) - \Omega(s,\bar{u}]\,ds\right| \le Z \max_{t\in J}|u(t) - \bar{u}(t)|,$$

because

$$\begin{aligned}|\Omega(t,u) - \Omega(t,\bar{u})| &\le A\left[\left(1 - \frac{t}{T}\right)\int_0^t |u(s) - \bar{u}(s)|\,ds + \frac{t}{T}\int_t^T |u(s) - \bar{u}(s)|\,ds\right] \\ &\le 2A\left(1 - \frac{t}{T}\right) t \max_{t\in J}|u(t) - \bar{u}(t)| \le \frac{T}{2} A \max_{t\in J}|u(t) - \bar{u}(t)|.\end{aligned}$$

Hence, operator Λ is a contraction mapping so problem (4) has a unique solution, by the Banach fixed-point theorem.

REMARK 2. If $D(t) = 0_{p\times p}$, $t \in [0,\xi]$, then $Z = (T/2)A$.

REMARK 3. If $A_0 = A_1 = 0_{p\times p}$, and $D(t) = I_{p\times p}$, $t \in [0,\xi]$, then $Z = (1 + (2T/\xi))(T/2)A$.

REMARK 4. Indeed, condition $\rho(Z) < 1$ holds if

$$T\|A\| \left[1 + \|B_2\| T \int_0^\xi \|D(s)\|\, ds\right] < 2,$$

where $\|\cdot\|$ denotes the Tchebysheff maximum norm.

In place of the above considered process of successive approximations (7), it is sometimes convenient to use Seidel's method described by

$$\begin{cases} \bar{x}_{n+1}(t) = F(t, \bar{x}_n, \bar{y}_n; \bar{x}_0), \\ \bar{y}_{n+1}(t) = g(t, \bar{x}_{n+1}, \bar{y}_n), \end{cases} \quad \text{or} \quad \begin{cases} \tilde{y}_{n+1}(t) = g(t, \tilde{x}_n, \tilde{y}_n), \\ \tilde{x}_{n+1}(t) = F(t, \tilde{x}_n, \tilde{y}_{n+1}; \tilde{x}_0), \end{cases} \tag{9}$$

for $t \in J$ and $n = 0, 1, \dots$.

Let us define the following sequences:

$$\begin{gathered} \tilde{u}_0(t) = u_0(t), \qquad \tilde{w}_0(t) = w_0(t), \\ \tilde{u}_{n+1}(t) = \Omega_1(t, \tilde{u}_n, \tilde{w}_n) + |B_2|\, t \int_0^\xi |D(s)| \vee \Omega_1(s, \tilde{u}_n, \tilde{w}_n)\, ds, \\ \tilde{w}_{n+1}(t) = M\tilde{u}_{n+1}(t) + N\tilde{w}_n(t), \end{gathered}$$

$$\begin{gathered} \bar{u}_0(t) = u_0(t), \qquad \bar{w}_0(t) = w_0(t), \\ \bar{w}_{n+1}(t) = M\bar{u}_n(t) + N\bar{w}_n(t), \\ \bar{u}_{n+1}(t) = \Omega_1(t, \bar{u}_n, \bar{w}_{n+1}) + |B_2|\, t \int_0^\xi |D(s)| \Omega_1(s, \bar{u}_n, \bar{w}_{n+1})\, ds, \end{gathered}$$

for $t \in J$, $n = 0, 1, \dots$. Now, we are able to show the following result by mathematical induction.

LEMMA 3. *Let Assumptions H_1 and H_2 hold. Assume that B_2 exists. Then*

$$\begin{aligned} &\bar{u}_n(t) \le u_n(t), \quad \bar{w}_n(t) \le w_n(t), \qquad t \in J, \quad n = 0, 1, \dots, \\ &\tilde{u}_n(t) \le u_n(t), \quad \tilde{w}_n(t) \le w_n(t), \qquad t \in J, \quad n = 0, 1, \dots, \end{aligned}$$

and

$$\bar{u}_n(t) \to 0, \quad \bar{w}_n(t) \to 0, \quad \tilde{u}_n(t) \to 0, \quad \tilde{w}_n(t) \to 0, \qquad \text{if } n \to \infty.$$

The simple consequence of Lemma 3 is the following.

THEOREM 2. *Assume that all assumptions of Theorem 1 are satisfied. Then the assertion of Theorem 1 holds and $\bar{x}_n(t) \to \bar{x}(t)$, $\bar{y}_n(t) \to \bar{y}(t)$, $\tilde{x}_n(t) \to \bar{x}(t)$, $\tilde{y}_n(t) \to \bar{y}(t)$, $t \in J$ as $n \to \infty$, for $\bar{x}_0(t) = \tilde{x}_0(t) = x_0(t)$, $\bar{y}_0(t) = \tilde{y}_0(t) = y_0(t)$, $t \in J$. Moreover, we have the estimates*

$$\begin{aligned} &|\bar{x}_n(t) - \bar{x}(t)| \le \bar{u}_n(t), \quad |\bar{y}_n(t) - \bar{y}(t)| \le \bar{w}_n(t), \qquad t \in J, \\ &|\tilde{x}_n(t) - \bar{x}(t)| \le \tilde{u}_n(t), \quad |\tilde{y}_n(t) - \bar{y}(t)| \le \tilde{w}_n(t), \qquad t \in J, \end{aligned}$$

for $n = 0, 1, \dots$.

Note that iterations (7) and (9) converge to $(\bar{x}, \bar{y})$ under the same conditions but basing on Lemma 3 we see that the error estimates for (9) are better in comparing with the corresponding estimates for (7). This notice is important since $\{x_n, y_n\}$, $\{\bar{x}_n, \bar{y}_n\}$, and $\{\tilde{x}_n, \tilde{y}_n)$ are approximated solutions of problem (3).

5. DIFFERENTIAL ALGEBRAIC SYSTEMS WITH DEVIATED ARGUMENTS

Let $\alpha, \beta, \gamma \in C(J, J)$. Let us consider the following problem:

$$\begin{aligned} x'(t) &= f\left(t, x(\alpha(t)), y(\beta(t))\right), && t \in J = [0, T], \\ y(t) &= g\left(t, x(\gamma(t)), y(t)\right), && t \in J, \end{aligned} \tag{10}$$

with condition (2), where $f \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^p)$, $g \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q)$. According to the numerical analytic method find the vector δ such that

$$x(t) = \bar{x}_0 + \mathcal{P}z(t) + \delta t, \qquad \text{with } \mathcal{P}z(t) = \left(1 - \frac{t}{T}\right) \int_0^t z(s)\, ds - \frac{t}{T} \int_t^T z(s)\, ds,$$

satisfies condition (2). Then, by substituting $x(t) = \bar{x}_0 + \int_0^t z(s)\, ds$, and introducing it to problem (10), we have the following auxiliary problem:

$$\begin{aligned} z(t) &= f\left(t, \bar{x}_0 + \mathcal{P}z\left(\alpha(t)\right) - B_2 \alpha(t) \int_0^\xi D(s) \mathcal{P}z(s)\, ds + B_3\left(\bar{x}_0\right) \alpha(t), y(\beta(t))\right) \\ &\equiv \mathcal{F}\left(t, z, y; \bar{x}_0\right), \qquad t \in J, \\ y(t) &= g\left(t, \bar{x}_0 + \mathcal{P}z\left(\gamma(t)\right) - B_2 \gamma(t) \int_0^\xi D(s) \mathcal{P}z(s)\, ds + B_3\left(\bar{x}_0\right) \gamma(t), y(t)\right) \\ &\equiv \mathcal{G}\left(t, z, y; \bar{x}_0\right), \qquad t \in J, \end{aligned} \tag{11}$$

where the matrices B_2 and B_3 are defined as in Section 2 assuming that B_2 exists.

Now, let us define the sequences $\{z_n, y_n\}$ by formulas

$$\begin{aligned} z_{n+1}(t) &= \mathcal{F}\left(t, z_n, y_n; \bar{x}_0\right), && t \in J, \quad z_0 \in C\left(J, \mathbb{R}^p\right), \\ y_{n+1}(t) &= \mathcal{G}\left(t, z_n, y_n; \bar{x}_0\right), && t \in J, \quad y_0 \in C\left(J, \mathbb{R}^q\right). \end{aligned} \tag{12}$$

ASSUMPTION H_3. *For any nonnegative function $H \in C(J \times \mathbb{R}^p, \mathbb{R}^p_+)$ there exists a unique solution $v \in C(J, \mathbb{R}^p_+)$ of the comparison equation*

$$\begin{aligned} v(t) = K\Omega_0(\alpha(t), v) + \left[K|B_2|\alpha(t) + L(I-N)^{-1}M|B_2|\gamma(\beta(t))\right] \int_0^\xi |D(s)| \Omega_0(s, v)\, ds \\ + L(I-N)^{-1}M\Omega_0(\gamma(\beta(t)), v) + H(t, \bar{x}_0), \end{aligned}$$

with

$$\Omega_0(t, u) = \left(1 - \frac{t}{T}\right) \int_0^t u(s)\, ds + \frac{t}{T} \int_t^T u(s)\, ds. \tag{13}$$

Note that, by Assumption H_1, we have

$$\begin{aligned} &\left|\mathcal{F}\left(t, x, y; \bar{x}_0\right) - \mathcal{F}\left(t, \bar{x}, \bar{y}; \bar{x}_0\right)\right| \\ &\quad = \Bigg| f\left(t, \bar{x}_0 + \mathcal{P}x(\alpha(t)) - B_2 \alpha(t) \int_0^\xi D(s) \mathcal{P}x(s)\, ds + B_3\left(\bar{x}_0\right) \alpha(t), y(\beta(t))\right) \\ &\qquad - f\left(t, \bar{x}_0 + \mathcal{P}\bar{x}(\alpha(t)) - B_2 \alpha(t) \int_0^\xi D(s) \mathcal{P}\bar{x}(s)\, ds + B_3\left(\bar{x}_0\right) \alpha(t), \bar{y}(\beta(t))\right) \Bigg| \\ &\quad \le K\left|\mathcal{P}x(\alpha(t)) - \mathcal{P}\bar{x}(\alpha(t))\right| + K\left|B_2\right| \alpha(t) \int_0^\xi |D(s)| \left|\mathcal{P}x(s) - \mathcal{P}\bar{x}(s)\right|\, ds \\ &\qquad + L|y(\beta(t)) - \bar{y}(\beta(t))| \\ &\quad \le K\Omega_0\left(\alpha(t), |x - \bar{x}|\right) + K\left|B_2\right| \alpha(t) \int_0^\xi |D(s)| \Omega_0\left(s, |x - \bar{x}|\right) ds \\ &\qquad + L\left|y(\beta(t)) - \bar{y}(\beta(t))\right|, \end{aligned} \tag{14}$$

and

$$
\begin{aligned}
&|\mathcal{G}(t,x,y;\bar{x}_0)-\mathcal{G}(t,\bar{x},\bar{y};\bar{x}_0)| \\
&=\left|g\left(t,\bar{x}_0+\mathcal{P}x(\gamma(t))-B_2\gamma(t)\int_0^{\xi}D(s)\mathcal{P}x(s)\,ds+B_3(\bar{x}_0)\gamma(t),y(t)\right)\right. \\
&\left.-g\left(t,\bar{x}_0+\mathcal{P}\bar{x}(\gamma(t))-B_2\gamma(t)\int_0^{\xi}D(s)\mathcal{P}\bar{x}(s)\,ds+B_3(\bar{x}_0)\gamma(t),\bar{y}(t)\right)\right| \\
&\le M\,|\mathcal{P}x(\gamma(t))-\mathcal{P}\bar{x}(\gamma(t))|+M\,|B_2|\,\gamma(t)\int_0^{\xi}|D(s)|\,|\mathcal{P}x(s)-\mathcal{P}\bar{x}(s)|\,ds \\
&\quad+N\,|y(t)-\bar{y}(t)| \\
&\le M\Omega_0(\gamma(t),|x-\bar{x}|)+M\,|B_2|\,\gamma(t)\int_0^{\xi}|D(s)|\Omega_0(s,|x-\bar{x}|)\,ds \\
&\quad+N\,|y(t)-\bar{y}(t)|.
\end{aligned}
\tag{15}
$$

For $n=0,1,\dots$, let us define the sequences $\{u_n,w_n\}$ by relations

$$
\begin{aligned}
&u_0(t)=v(t), \\
&u_{n+1}(t)=K\Omega_0(\alpha(t),u_n)+K\,|B_2|\,\alpha(t)\int_0^{\xi}|D(s)|\Omega_0(s,u_n)\,ds+Lw_n(\beta(t)) \quad ,
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
w_{n+1}(t)&=M\Omega_0(\gamma(t),u_n)+M|B_2|\gamma(t)\int_0^{\xi}|D(s)|\Omega_0(s,u_n)\,ds+Nw_n(t), \\
w_0(t)&=(I-N)^{-1}\left[M\Omega_0(\gamma(t),u_0)+M\,|B_2|\,\gamma(t)\int_0^{\xi}|D(s)|\Omega_0(s,u_0)\,ds+a(t)\right],
\end{aligned}
\tag{17}
$$

where v is defined as in Assumption H_3 with

$$
H(t,\bar{x}_0)=L(I-N)^{-1}a(\beta(t))+|\mathcal{F}(t,z_0,y_0;\bar{x}_0)-z_0(t)|, \qquad a(t)=|\mathcal{G}(t,z_0,y_0;\bar{x}_0)-y_0(t)|.
$$

LEMMA 4. *Assume that $f\in C(J\times\mathbb{R}^p\times\mathbb{R}^q,\mathbb{R}^p)$, $g\in C(J\times\mathbb{R}^p\times\mathbb{R}^q,\mathbb{R}^q)$, $\alpha,\beta,\gamma\in C(J,J)$, and $(A_0)_{p\times p}$, $(A_1)_{p\times p}$, $D_{p\times p}$, and $d_{p\times 1}$ are given matrices. Assume that the matrix B_2 exists. Let Assumptions H_1 and H_3 be satisfied. Then the sequences $\{u_n,w_n\}$ of form (16),(17) are nonincreasing and converge uniformly to zero functions if $n\to\infty$. Moreover, we can show that*

$$
\begin{cases}|z_n(t)-z_0(t)|\le u_0(t),\\ |z_{n+k}(t)-z_k(t)|\le u_k(t),\end{cases}\qquad \begin{cases}|y_n(t)-y_0(t)|\le w_0(t),\\ |y_{n+k}(t)-y_k(t)|\le w_k(t),\end{cases}
\tag{18}
$$

for $t\in J$ and $n=0,1,\dots$, where z_n and y_n are defined by (12).

PROOF. By mathematical induction, it is simple to show that

$$
u_{n+1}(t)\le u_n(t)\le u_0(t),\quad w_{n+1}(t)\le w_n(t)\le w_0(t),\qquad t\in J,\quad n=0,1,\dots
$$

Hence $u_n\to 0$, $w_n\to 0$ on J, by Assumption H_3. Now we are going to show (18). Note that

$$
\begin{aligned}
|z_1(t)-z_0(t)|&=|\mathcal{F}(t,z_0,y_0;\bar{x}_0)-z_0(t)|\equiv R(t,\bar{x}_0)\le u_0(t), \\
|y_1(t)-y_0(t)|&=|\mathcal{G}(t,z_0,y_0;\bar{x}_0)-y_0(t)|=a(t)\le\left[(I-N)(I-N)^{-1}+N(I-N)^{-1}\right]a(t) \\
&=(I-N)^{-1}a(t)\le w_0(t),\qquad t\in J.
\end{aligned}
$$

Assume that $|z_k(t) - z_0(t)| \le u_0(t)$, $|y_k(t) - y_0(t)| \le w_0(t)$, $t \in J$ for some $k \ge 1$. By (14) and (15), we see that

$$\begin{aligned}
|z_{k+1}(t) - z_0(t)| &\le |\mathcal{F}(t, z_k, y_k; \bar{x}_0) - \mathcal{F}(t, z_0, y_0; \bar{x}_0)| + R(t, \bar{x}_0) \\
&\le K\Omega_0(\alpha(t), u_0) + K|B_2|\alpha(t) \int_0^\xi |D(s)|\Omega_0(s, u_0)\, ds \\
&\quad + Lw_0(\beta(t)) + R(t, \bar{x}_0) = u_0(t), \qquad t \in J, \\
|y_{k+1}(t) - y_0(t)| &\le |\mathcal{G}(t, z_k, y_k; \bar{x}_0) - \mathcal{G}(t, z_0, y_0; \bar{x}_0)| + a(t) \\
&\le M\Omega_0(\gamma(t), u_0) + M|B_2|\gamma(t) \int_0^\xi |D(s)|\Omega_0(s, u_0)\, ds \\
&\quad + Nw_0(t) + a(t) = w_0(t), \qquad t \in J.
\end{aligned}$$

Hence, by mathematical induction, we have

$$|z_n(t) - z_0(t)| \le u_0(t), \quad |y_n(t) - y_0(t)| \le w_0(t), \qquad t \in J, \quad n = 0, 1, \dots.$$

The rest of estimates (18) can be proved by similar argument. It ends the proof.

Lemma 4 follows.

THEOREM 3. *Assume that all assumptions of Lemma 4 are satisfied. Then, for every $\bar{x}_0 \in \mathbb{R}^p$, system (12) of sequences $\{z_n, y_n\}$ converges to the unique solution $(\bar{z}, \bar{y})$ of problem (11) (uniqueness in the class $\Lambda(z_0, y_0)$), so $z_n(t) \to \bar{z}(t)$, $y_n(t) \to \bar{y}(t)$ for $t \in J$ if $n \to \infty$ and for $t \in J$ we have the error estimates*

$$\begin{cases} |\bar{z}(t) - z_0(t)| \le u_0(t), \\ |z_n(t) - \bar{z}(t)| \le u_n(t), \end{cases} \quad n = 0, 1, \dots, \qquad \begin{cases} |\bar{y}(t) - y_0(t)| \le w_0(t), \\ |y_n(t) - \bar{y}(t)| \le w_n(t), \end{cases} \quad n = 0, 1, \dots.$$

Moreover, $(\bar{x}, \bar{y})$ with $\bar{x} = \bar{x}_0 + \int_0^t \bar{z}(s)\, ds$ is the solution of problem (10),(2) iff

$$B_2 \int_0^\xi D(s)\mathcal{P}\bar{z}(s)\, ds + \frac{1}{T}\int_0^T \bar{z}(s)\, ds = B_3(\bar{x}_0)$$

REMARK 5. Note that Assumption H_3 holds if we assume that $\rho(W) < 1$, where

$$\begin{aligned}
W = K \max_{t\in J} Q(\alpha(t)) + \left[K + L(I - N)^{-1}M\right] |B_2| \frac{T^2}{2} \int_0^\xi |D(s)|\, ds \\
+ 2L(I - N)^{-1}M \max_{t\in J} Q(\gamma(\beta(t))), \qquad \text{with } Q(t) = \frac{t}{T}(T - t).
\end{aligned}$$

Similarly as before to find a solution $(\bar{z}, \bar{y})$ of problem (11), we can apply Seidel's method. It means that we can formulate the following.

THEOREM 4. *Let all ssumptions of Lemma 4 be satisfied. Then the results of Theorem 3 hold and $\bar{z}_n(t) \to \bar{z}(t)$, $\tilde{z}_n(t) \to \bar{z}(t)$, $\bar{y}_n(t) \to \bar{y}(t)$, $\tilde{y}_n(t) \to \bar{y}(t)$, where $\{\bar{z}_n, \bar{y}_n\}$ and $\{\tilde{z}_n, \tilde{y}_n\}$ are defined by*

$$\begin{cases} \bar{z}_{n+1}(t) = \mathcal{F}(t, \bar{z}_n, \bar{y}_n; \bar{x}_0), \\ \bar{y}_{n+1}(t) = \mathcal{G}(t, \bar{z}_{n+1}, \bar{y}_n; \bar{x}_0), \end{cases} \qquad \begin{cases} \tilde{y}_{n+1}(t) = \mathcal{F}(t, \tilde{z}_n, \tilde{y}_n; \bar{x}_0), \\ \tilde{z}_{n+1}(t) = \mathcal{G}(t, \tilde{z}_n, \tilde{y}_{n+1}; \bar{x}_0), \end{cases}$$

for $t \in J$, $n = 0, 1, \dots$ with $\bar{z}_0(t) = \tilde{z}_0(t) = z_0(t)$, $\bar{y}_0(t) = \tilde{y}_0(t) = y_0(t)$, $t \in J$.

Moreover, we have the error estimates

$$\begin{cases} |\bar{z}_n(t) - \bar{z}(t)| \le \bar{u}_n(t), \\ |\tilde{z}_n(t) - \bar{z}(t)| \le \tilde{u}_n(t), \end{cases} \qquad \begin{cases} |\bar{y}_n(t) - \bar{y}(t)| \le \bar{w}_n(t), \\ |\tilde{y}_n(t) - \bar{y}(t)| \le \tilde{w}_n(t), \end{cases}$$

for $t \in J$, $n = 0, 1, \dots$, *where*

$$\bar{u}_0(t) = u_0(t), \qquad \bar{w}_0(t) = w_0(t),$$

$$\bar{u}_{n+1}(t) = K\Omega_0(\alpha(t), \bar{u}_n) + K|B_2|\alpha(t)\int_0^\xi |D(s)|\Omega_0(s, \bar{u}_n)\,ds + L\bar{w}_n(\beta(t)),$$

$$\bar{w}_{n+1}(t) = M\Omega_0(\gamma(t), \bar{u}_{n+1}) + M|B_2|\gamma(t)\int_0^\xi |D(s)|\Omega_0(s, \bar{u}_{n+1})\,ds + N\bar{w}_n(t),$$

$$\tilde{u}_0(t) = u_0(t), \qquad \tilde{w}_0(t) = w_0(t),$$

$$\tilde{w}_{n+1}(t) = M\Omega_0(\gamma(t), \tilde{u}_n) + M|B_2|\gamma(t)\int_0^\xi |D(s)|\Omega_0(s, \tilde{u}_n)\,ds + N\tilde{w}_n(t),$$

$$\tilde{u}_{n+1}(t) = K\Omega_0(\alpha(t), \tilde{u}_n) + K|B_2|\alpha(t)\int_0^\xi |D(s)|\Omega_0(s, \tilde{u}_n)\,ds + L\tilde{w}_{n+1}(\beta(t)),$$

with u_0, w_0 *defined as in relations (16) and (17), respectively.*

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