Samoilenko's Method to 
Differential Algebraic Systems 
with Integral Boundary Conditions

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Abstract—The numerical analytic method combined with the comparison one is used to establish solvability of differential algebraic systems with integral boundary conditions. Existence results are formulated under assumptions that corresponding functions satisfy the Lipschitz conditions in matrix notation. A problem with deviated arguments is also discussed. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

A useful approach in the studying of existence of solutions is Samoilenko's numerical analytic method (for details, see [1,2]). In this paper, we apply this technique to differential algebraic systems of the form

\[ \begin{align*}
    x'(t) &= f(t, x(t), y(t)), \quad t \in J = [0,T], \\
    y(t) &= g(t, x(t), y(t)), \quad t \in J
\end{align*} \]

with the integral boundary condition

\[ A_0 x(0) + \int_0^\xi D(s)x(s) \, ds + A_1 x(T) = d, \]

where \( f \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^p) \), \( g \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q) \). The value \( \xi \) is a fixed constant and \( 0 < \xi \leq T \). In the above, \( (A_0)_{p \times p} \), \( (A_1)_{p \times p} \), \( D_{p \times p} \), and \( d_{p \times 1} \) are given matrices. The application of numerical analytic method to differential systems \( x'(t) = f(t, x(t)) \) with condition (2) can be found, for example, in papers [1–10] if \( D(t) = 0 \) on \([0, \xi]\), see also [11].

The numerical analytic method combined with the comparison one is used to formulate corresponding existence results for problems of type (1),(2) under the assumption that \( f \) and \( g \) satisfy the Lipschitz conditions (with respect to the last two variables) in matrix notation. The aim
of the present paper is to discuss the conditions under which the solution can be obtained by the method of successive approximations and Seidel's method too. A more general differential algebraic problem with deviated arguments is also considered and corresponding existence results are given in Section 5.

2. ASSUMPTIONS

Put

$$L_x(t, x, y) = \left(1 - \frac{t}{T}\right) \int_0^t f(s, x(s), y(s)) \, ds - \frac{t}{T} \int_t^T f(s, x(s), y(s)) \, ds,$$

$$B_0 = \int_0^\xi sD(s) \, ds, \quad B_1 = \int_0^\xi D(s) \, ds,$$

$$B_2 = (A_1T + B_0)^{-1}, \quad B_3(x_0) = B_2 [d - (A_0 + A_1 + B_1)x_0],$$

assuming that the matrix $B_2$ exists. Apply the numerical analytic method to problem (1),(2) to obtain the following auxiliary system

$$x(t) = x_0 + L_x(t, x, y) - B_2t \int_0^\xi D(s)L_x(s, x, y) \, ds + tB_3(x_0) \equiv F(t, x, y; x_0), \quad t \in J, \quad (3)$$

$$y(t) = g(t, x(t), y(t)), \quad t \in J.$$

Note that if $x$ satisfies the first equation of problem (3), then condition (2) is satisfied too. Moreover, $F(0, x, y; x_0) = x_0$, so $x(0) = x_0$.

Let us introduce the following.

ASSUMPTION $H_1$.

1° There are matrices $K_{p \times p}, L_{p \times q}$ with nonnegative entries such that

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq K |x - \bar{x}| + L |y - \bar{y}|,$$

for all $t \in J, x, \bar{x} \in \mathbb{R}^p, y, \bar{y} \in \mathbb{R}^q$.

2° There are matrices $M_{q \times p}, N_{q \times q}$ with nonnegative entries, $\rho(N) < 1$, and such that

$$|g(t, x, y) - g(t, \bar{x}, \bar{y})| \leq M |x - \bar{x}| + N |y - \bar{y}|,$$

for all $t \in J, x, \bar{x} \in \mathbb{R}^p, y, \bar{y} \in \mathbb{R}^q$. Here $|\cdot|$ denotes the absolute value of the vector, so $|x| = (|x_1|, \ldots, |x_p|)^T$ or $|y| = (|y_1|, \ldots, |y_q|)^T$. Moreover, $\rho(N)$ denotes the spectral radius of the matrix $N$.

ASSUMPTION $H_2$. For any nonnegative function $h \in C(\mathbb{J} \times \mathbb{R}^p, \mathbb{R}^p)$, there exists a unique solution $u \in C(\mathbb{J}, \mathbb{R}^p)$ of the comparison equation

$$\Omega(t, u) + |B_2| t \int_0^\xi |D(s)|\Omega(s, u) \, ds + h(t, x_0) = u(t), \quad t \in J, \quad (4)$$

where

$$\Omega(t, u) = \left(1 - \frac{t}{T}\right) \int_0^t Au(s) \, ds + \frac{t}{T} \int_t^T Au(s) \, ds, \quad \text{with } A = K + L(I - N)^{-1}M.$$

Put

$$\Omega_1(t, u, v) = \left(1 - \frac{t}{T}\right) \int_0^t [Ku(s) + Lv(s)] \, ds + \frac{t}{T} \int_t^T [Ku(s) + Lv(s)] \, ds.$$
Then, by Assumption $H_1(1^0)$, for $t \in J$, we have
\[
|\mathcal{L}f(t, x, y) - \mathcal{L}f(t, x, y)| \leq \Omega_1(t, |x - \bar{x}|, |y - \bar{y}|),
\]
\[
|F(t, x, y; \bar{x}) - F(t, \bar{x}, \bar{y}; \bar{x})| \leq |\mathcal{L}f(t, x, y) - \mathcal{L}f(t, \bar{x}, \bar{y})|
\]
\[
+ |B_2| t \int_0^\xi |D(s)| \Omega_1(s, |x - \bar{x}|, |y - \bar{y}|) | ds \leq \Omega_1(t, |x - \bar{x}|, |y - \bar{y}|)
\]
\[
+ |B_2| t \int_0^\xi |D(s)| \Omega_1(s, |x - \bar{x}|, |y - \bar{y}|) | ds.
\]

3. LEMMAS

For $t \in J$, $n = 0, 1, \ldots$, let us define the sequences $\{u_n, w_n\}$ by formulas
\[
u_{n+1}(t) = \Omega_1(t, u_n, w_n) + |B_2| t \int_0^\xi |D(s)| \Omega_1(s, u_n, w_n) | ds, \quad u_0(t) = u(t),
\]
\[
w_{n+1}(t) = M u_n(t) + N w_n(t), \quad w_0(t) = (I - N)^{-1} [M u_0(t) + |g(t, x_0, y_0) - y_0(t)|],
\]
where $u$ is defined as in Assumption $H_2$ with
\[
h(t, \bar{x}_0) = |F(t, x_0, y_0; \bar{x}_0) - x_0(t)| + \tilde{\Omega}(t, r) + |B_2| t \int_0^\xi |D(s)| \tilde{\Omega}(s, r) | ds
\]
for $r(t) = |g(t, x_0, y_0) - y_0(t)|$. Here $\tilde{\Omega}$ is defined as $\Omega$ with the matrix $B = L(I - N)^{-1}$ instead of $A$.

To obtain a solution of problem (3), we shall first establish some properties for sequences $\{u_n, w_n\}$. They are given in the next two lemmas.

LEMMA 1. Let Assumptions $H_1$ and $H_2$ be satisfied. Assume that the matrix $B_2$ exists. Then
\[
\forall n+1(t) - u_n(t) \leq u_0(t), \quad w_{n+1}(t) \leq w_n(t) \leq w_0(t), \quad t \in J, \quad n = 0, 1, \ldots,
\]
and the sequences $\{u_n, w_n\}$ converge uniformly to zero functions, so $u_n(t) \to 0$, $w_n(t) \to 0$, $t \in J$ if $n \to \infty$.

PROOF. Note that the matrix $(I - N)^{-1}$ exists and its entries are nonnegative because of the condition $\rho(N) < 1$. Indeed, $\Omega_1(t, u_0, w_0) = \tilde{\Omega}(t, r)$. Then
\[
u_1(t) = \Omega_1(t, u_0, w_0) + |B_2| t \int_0^\xi |D(s)| \Omega_1(s, u_0, w_0) | ds
\]
\[
\leq \Omega(t, u_0) + |B_2| t \int_0^\xi |D(s)| \tilde{\Omega}(s, r) | ds \leq u_0(t),
\]
\[
w_1(t) = M u_0(t) + N(I - N)^{-1} [M u_0(t) + r(t)] \leq u_0(t), \quad t \in J.
\]
By induction in $n$, we are able to prove that
\[
u_{n+1}(t) \leq u_n(t), \quad w_{n+1}(t) \leq w_n(t), \quad t \in J, \quad n = 0, 1, \ldots,
\]
Now, if $n \to \infty$, then $u_n \to u$, $w_n \to w$, where the pair $(u, w)$ is a solution of the system
\[
u(t) = \Omega_1(t, u, w) + |B_2| t \int_0^\xi |D(s)| \Omega_1(s, u, w) | ds, \quad t \in J,
\]
\[
w(t) = M u + N w, \quad t \in J.
\]
Hence, $w(t) = (I - N)^{-1} M u(t)$, so $\Omega_1(t, u, w) = \Omega(t, u)$ showing that $u$ is a solution of problem $u(t) = \Omega(t, u) + |B_2| t \int_0^\xi |D(s)| \tilde{\Omega}(s, u) | ds$, $t \in J$. By Assumption $H_2$, $u(t) = 0$ on $J$ and then $w(t) = 0$, $t \in J$. The proof is complete.
LEMMA 2. Assume that $\tilde{f} \in C(J \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^p)$, $\bar{g} \in C(J \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q)$, and $(A_0)_{p \times p}$, $(A_1)_{p \times p}$, $B_{n_{\times p}}$, and $d_{p \times 1}$ are given matrices. Assume that the matrix $B_2$ exists. Let Assumptions $H_1$ and $H_2$ be satisfied. Then we have the estimates

$$
\begin{align*}
|x_n(t) - x_0(t)| &\leq u_0(t), \quad t \in J, \\
|x_{n+k}(t) - x_k(t)| &\leq u_k(t), \quad t \in J,
\end{align*}
$$

and

$$
\begin{align*}
|y_n(t) - y_0(t)| &\leq w_0(t), \quad t \in J, \\
|y_{n+k}(t) - y_k(t)| &\leq w_k(t), \quad t \in J,
\end{align*}
$$

(6)

where $x_0 \in C^1(J,\mathbb{R}^p)$, $y_0 \in C(J,\mathbb{R}^q)$, and

$$
x_{n+1}(t) = F(t, x_n, y_n; \bar{x}_0), \quad y_{n+1}(t) = g(t, x_n, y_n), \quad t \in J.
$$

Moreover,

$$
A_0 x_{n+1}(0) + A_1 x_{n+1}(T) + \int_0^T D(s)x_{n+1}(s) \, ds = d, \quad n = 0, 1, \ldots
$$

PROOF. Put $R(t; \bar{x}_0) = |F(t, x_0, y_0; \bar{x}_0) - x_0(t)|$, $r(t) = |g(t, x_0, y_0) - y_0(t)|$. Indeed,

$$
\begin{align*}
|x_1(t) - x_0(t)| &= R(t; \bar{x}_0) \leq h(t, \bar{x}_0) \leq u_0(t), \quad t \in J, \\
|y_1(t) - y_0(t)| &= r(t) \leq [N(I - N)^{-1} + J] r(t) \leq w_0(t), \quad t \in J.
\end{align*}
$$

Assume that

$$
|x_k(t) - x_0(t)| \leq u_0(t), \quad |y_k(t) - y_0(t)| \leq w_0(t), \quad t \in J,
$$

for some $k \geq 0$. Then, by (5), we have

$$
\begin{align*}
|x_{k+1}(t) - x_0(t)| &\leq |F(t, x_k, y_k; \bar{x}_0) - F(t, x_0, y_0; \bar{x}_0)| + R(t, \bar{x}_0) \\
&\leq \Omega_1(t, u_0, w_0) + |B_2| \int_0^t |D(s)| \Omega_1(s, u_0, w_0) \, ds + R(t; \bar{x}_0) = u_0(t), \\
|y_{k+1}(t) - y_0(t)| &\leq |g(t, x_k, y_k) - g(t, x_0, y_0)| + r(t) \leq Mu_0(t) + Nw_0(t) + r(t) = w_0(t).
\end{align*}
$$

Hence, by mathematical induction, we have

$$
|x_n(t) - x_0(t)| \leq u_0(t), \quad |y_n(t) - y_0(t)| \leq w_0(t), \quad t \in J,
$$

for $n = 0, 1, \ldots$. Basing on the above, let us assume that

$$
|x_{n+k}(t) - x_k(t)| \leq u_k(t), \quad |y_{n+k}(t) - y_k(t)| \leq w_k(t), \quad t \in J,
$$

for all $n$ and some $k \geq 0$. Then, by (5), we see that

$$
\begin{align*}
|x_{n+k+1}(t) - x_{k+1}(t)| &= |F(t; x_{n+k}, y_{n+k}; \bar{x}_0) - F(t; x_k, y_k; \bar{x}_0)| \\
&\leq \Omega_1(t, u_k, w_k) + |B_2| \int_0^t |D(s)| \Omega_1(s, u_k, w_k) \, ds = u_{k+1}(t) \\
|y_{n+k+1}(t) - y_{k+1}(t)| &= |g(t, x_{n+k}, y_{n+k}) - g(t, x_k, y_k)| \leq Mu_k(t) + Nw_k(t) = w_{k+1}(t),
\end{align*}
$$

for $t \in J$. Hence, by mathematical induction, the estimates (6) hold. It is quite simple to verify that $x_{n+1}$ satisfies integral boundary condition (2) for any $n = 0, 1, \ldots$. It ends the proof.
4. EXISTENCE RESULTS

Put

\[ \Lambda (x_0, y_0) = \{ (x, y) \in C^1 (J, \mathbb{R}^p) \times C (J, \mathbb{R}^q) : |x_0(t) - x(t)| \leq u_0(t), \ |y_0(t) - y(t)| \leq v_0(t) \} . \]

Combining Lemmas 1 and 2, we have the following.

**Theorem 1.** Assume that \( f \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^p) \), \( g \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q) \), and \( (A_0)_{p \times p}, (A_1)_{p \times p}, D_{p \times p}, \) and \( d_{p \times 1} \) are given matrices. Assume that the matrix \( B_2 \) exists. Let Assumptions \( H_1 \) and \( H_2 \) be satisfied. Then, for every \( \tilde{x}_0 \in \mathbb{R}^p \), there exists a solution \((\tilde{x}, \tilde{y})\) of problem (3) where \( x_n(t) \to \tilde{x}(t), \ y_n(t) \to \tilde{y}(t), \ t \in J \) as \( n \to \infty \), and we have the estimates

\[ |x_n(t) - \tilde{x}(t)| \leq u_n(t), \ |y_n(t) - \tilde{y}(t)| \leq v_n(t), \ t \in J. \]

The pair \((\tilde{x}, \tilde{y})\) is a unique solution of problem (3) in the class \( \Lambda (x_0, y_0) \).

Moreover, \((\tilde{x}, \tilde{y})\) is the solution of problem (1), (2) iff

\[ \frac{1}{T} \int_0^T f (s, \tilde{x}(s), \tilde{y}(s)) \, ds + B_2 \int_0^T \int_0^t \Omega (s, u - \tilde{u}) \, ds = B_3 (\tilde{x}_0) . \]

**Proof.** By Lemmas 1 and 2, \( x_n(t) \to \tilde{x}(t), \ y_n(t) \to \tilde{y}(t), \ t \in J \). Indeed, \((\tilde{x}, \tilde{y})\) is a solution of problem (3). We need to show the uniqueness of \((\tilde{x}, \tilde{y})\). Assume that problem (3) has another solution \((X, Y)\) such that \(|X(t) - x_0(t)| \leq u_0(t), \ |Y(t) - y_0(t)| \leq v_0(t)\) on \( J \). Then, by (5), we have

\[ |\tilde{x}(t) - X(t)| \leq u_{n+1}(t) + \int_0^t \Omega (s, X(t) - x_0(t)) \, ds \]

and

\[ |\tilde{y}(t) - Y(t)| \leq v_{n+1}(t) + M |x_n(t) - X(t)| + N |y_n(t) - Y(t)| , \]

for \( t \in J \). Hence, by mathematical induction, we have

\[ |\tilde{x}(t) - X(t)| \leq 2u_{n+1}(t), \ |y_n(t) - Y(t)| \leq 2v_{n+1}(t), \ t \in J, \ n = 0, 1, \]

showing that \( \tilde{x} = X, \ \tilde{y} = Y \) on \( J \). It ends the proof.

**Remark 1.** Let the matrix \( B_2 \) exist. Assumption \( H_2 \) is satisfied if

\[ \rho (Z) < 1, \]

where \( Z = \left[ I + |B_2| T \int_0^T |D(s)| \, ds \right] \frac{T}{2} A . \]

To get condition (8), we need to apply the Banach fixed-point theorem to equation (4). Denote the left-hand side of problem (4) by \( \Lambda \). Let \( u, \tilde{u} \in C(J, \mathbb{R}^p) \). Then

\[ |\Lambda u - \Lambda \tilde{u}| = |\Omega (t, u) - \Omega (t, \tilde{u})| + |B_2| \int_0^T |D(s)| |\Omega (s, u) - \Omega (s, \tilde{u})| \, ds \leq Z \max_{t \in J} |u(t) - \tilde{u}(t)| , \]

because

\[ |\Omega (t, u) - \Omega (t, \tilde{u})| \leq A \left[ \left(1 - \frac{t}{T}\right) \int_0^t |u(s) - \tilde{u}(s)| \, ds + \frac{t}{T} \int_t^T |u(s) - \tilde{u}(s)| \, ds \right] \leq 2A \left(1 - \frac{t}{T}\right) \max_{t \in J} |u(t) - \tilde{u}(t)| \leq \frac{T}{2} A \max_{t \in J} |u(t) - \tilde{u}(t)| , \]

where

\[ \rho (Z) = \left(1 - \frac{t}{T}\right) , \]

and for \( u, \tilde{u} \) in \( C(J, \mathbb{R}^p) \).
Hence, operator $A$ is a contraction mapping so problem (4) has a unique solution, by the Banach fixed-point theorem.

REMARK 2. If $D(t) = 0_{p \times p}$, $t \in [0, \xi]$, then $Z = (T/2)A$.

REMARK 3. If $A_0 = A_1 = 0_{p \times p}$, and $D(t) = I_{p \times p}$, $t \in [0, \xi]$, then $Z = (1 + (2T/\xi))(T/2)A$.

REMARK 4. Indeed, condition $\rho(Z) < 1$ holds if

$$
T\|A\left[1 + \|B_2\|\frac{\int_0^\xi \|D(s)\| \, ds}{T}\right] < 2,
$$

where $\| \cdot \|$ denotes the Tchebysheff maximum norm.

In place of the above considered process of successive approximations (7), it is sometimes convenient to use Seidel's method described by

$$
\begin{align*}
\tilde{u}_{n+1}(t) &= F(t, \tilde{x}_n, \tilde{y}_n; \tilde{x}_0), \\
\tilde{y}_{n+1}(t) &= g(t, \tilde{x}_n, \tilde{y}_n), \\
\tilde{x}_{n+1}(t) &= F(t, \tilde{x}_n, \tilde{y}_{n+1}; \tilde{x}_0),
\end{align*}
$$

(9)

for $t \in J$ and $n = 0, 1, \ldots$.

Let us define the following sequences:

$$
\begin{align*}
\tilde{u}_0(t) &= u_0(t), \\
\tilde{w}_0(t) &= w_0(t), \\
\tilde{u}_{n+1}(t) &= \Omega_1(t, \tilde{u}_n, \tilde{w}_n) + |B_2| t \int_0^\xi |D(s)| \vee \Omega_1(s, \tilde{u}_n, \tilde{w}_n) \, ds, \\
\tilde{w}_{n+1}(t) &= M\tilde{u}_{n+1}(t) + N\tilde{w}_n(t), \\
\tilde{u}_0(t) &= u_0(t), \\
\tilde{w}_0(t) &= w_0(t), \\
\tilde{w}_{n+1}(t) &= \Omega_1(t, \tilde{u}_n, \tilde{w}_{n+1}) + |B_2| t \int_0^\xi |D(s)| \Omega_3(s, \tilde{u}_n, \tilde{w}_{n+1}) \, ds, \\
\tilde{u}_{n+1}(t) &= \Omega_1(t, \tilde{u}_n, \tilde{w}_{n+1}) + |B_2| t \int_0^\xi |D(s)| \vee \Omega_1(s, \tilde{u}_n, \tilde{w}_{n+1}) \, ds,
\end{align*}
$$

for $t \in J$, $n = 0, 1, \ldots$. Now, we are able to show the following result by mathematical induction.

**LEMMA 3.** Let Assumptions $H_1$ and $H_2$ hold. Assume that $B_2$ exists. Then

$$
\begin{align*}
\tilde{u}_n(t) &\leq u_n(t), & \tilde{w}_n(t) &\leq w_n(t), & t \in J, & n = 0, 1, \ldots, \\
\tilde{u}_n(t) &\leq u_n(t), & \tilde{w}_n(t) &\leq w_n(t), & t \in J, & n = 0, 1, \ldots,
\end{align*}
$$

and

$$
\tilde{u}_n(t) \to 0, & \tilde{w}_n(t) \to 0, & \tilde{u}_n(t) \to 0, & \tilde{w}_n(t) \to 0, & if \ n \to \infty.
$$

The simple consequence of Lemma 3 is the following.

**THEOREM 2.** Assume that all assumptions of Theorem 1 are satisfied. Then the assertion of Theorem 1 holds and $\tilde{x}_n(t) \to \tilde{x}(t)$, $\tilde{y}_n(t) \to \tilde{y}(t)$, $\tilde{x}_n(t) \to \tilde{x}(t)$, $\tilde{y}_n(t) \to \tilde{y}(t)$, $t \in J$ as $n \to \infty$, for $\tilde{x}_0(t) = \tilde{x}_0(t) = x_0(t)$, $\tilde{y}_0(t) = \tilde{y}_0(t) = y_0(t)$, $t \in J$. Moreover, we have the estimates

$$
\begin{align*}
|\tilde{x}_n(t) - \tilde{x}(t)| &\leq \tilde{u}_n(t), & |\tilde{y}_n(t) - \tilde{y}(t)| &\leq \tilde{w}_n(t), & t \in J, \\
|\tilde{x}_n(t) - \tilde{x}(t)| &\leq \tilde{u}_n(t), & |\tilde{y}_n(t) - \tilde{y}(t)| &\leq \tilde{w}_n(t), & t \in J,
\end{align*}
$$

for $n = 0, 1, \ldots$.

Note that iterations (7) and (9) converge to $(\tilde{x}, \tilde{y})$ under the same conditions but basing on Lemma 3 we see that the error estimates for (9) are better in comparing with the corresponding estimates for (7). This notice is important since $\{x_n, y_n\}$, $\{\tilde{x}_n, \tilde{y}_n\}$, and $\{\tilde{x}_n, \tilde{y}_n\}$ are approximated solutions of problem (3).
5. DIFFERENTIAL ALGEBRAIC SYSTEMS WITH DEVIATED ARGUMENTS

Let \( \alpha, \beta, \gamma \in C(J, J) \). Let us consider the following problem:

\[
x'(t) = f \left( t, x(\alpha(t)), y(\beta(t)) \right), \quad t \in J = [0, T],
\]

\[
y(t) = g \left( t, x(\gamma(t)), y(t) \right), \quad t \in J.
\]

with condition (2), where \( f \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^p) \), \( g \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q) \). According to the numerical analytic method find the vector \( \delta \) such that

\[
x(t) = \bar{x}_0 + \mathcal{P}x(t) + \delta t,
\]

satisfies condition (2). Then, by substituting \( z(t) = \bar{x}_0 + \int_0^t z(s) \, ds \), and introducing it to problem (10), we have the following auxiliary problem:

\[
z(t) = f \left( t, \bar{x}_0 + \mathcal{P}z(\alpha(t)) - B_2\alpha(t) \int_0^t D(s) \mathcal{P}z(s) \, ds + B_3(\bar{x}_0) \alpha(t), y(\beta(t)) \right)
\]

\[
= \mathcal{F} \left( t, z, y; \bar{x}_0 \right), \quad t \in J,
\]

\[
y(t) = g \left( t, \bar{x}_0 + \mathcal{P}z(\gamma(t)) - B_2\gamma(t) \int_0^t D(s) \mathcal{P}z(s) \, ds + B_3(\bar{x}_0) \gamma(t), y(t) \right)
\]

\[
= \mathcal{G} \left( t, z, y; \bar{x}_0 \right), \quad t \in J,
\]

where the matrices \( B_2 \) and \( B_3 \) are defined as in Section 2 assuming that \( B_2 \) exists.

Now, let us define the sequences \( \{z_n, y_n\} \) by formulas

\[
z_{n+1}(t) = \mathcal{F} \left( t, z_n, y_n; \bar{x}_0 \right), \quad t \in J, \quad z_0 \in C(J \times \mathbb{R}^p, \mathbb{R}^p),
\]

\[
y_{n+1}(t) = \mathcal{G} \left( t, z_n, y_n; \bar{x}_0 \right), \quad t \in J, \quad y_0 \in C(J \times \mathbb{R}^q, \mathbb{R}^q).
\]

ASSUMPTION H3. For any nonnegative function \( H \in C(J \times \mathbb{R}^p, \mathbb{R}^p) \) there exists a unique solution \( v \in C(J, \mathbb{R}^p) \) of the comparison equation

\[
v(t) = KO_0(\alpha(t), v) + \left[ K |B_2| \alpha(t) + L(I - N)^{-1} M |B_2| \alpha(t) \right] \int_0^t |D(s)| \Omega_0(s, v) \, ds
\]

\[
+ L(I - N)^{-1} M \Omega_0(\gamma(\beta(t)), v) + H(t, \bar{x}_0),
\]

with

\[
\Omega_0(t, u) = \left( 1 - \frac{t}{T} \right) \int_0^t u(s) \, ds + \frac{t}{T} \int_t^T u(s) \, ds.
\]

Note that, by Assumption H1, we have

\[
|\mathcal{F} \left( t, x, y; \bar{x}_0 \right) - \mathcal{F} \left( t, \tilde{x}, \tilde{y}; \bar{x}_0 \right)|
\]

\[
= \left| f \left( t, \bar{x}_0 + \mathcal{P}z(\alpha(t)) - B_2\alpha(t) \int_0^t D(s) \mathcal{P}z(s) \, ds + B_3(\bar{x}_0) \alpha(t), y(\beta(t)) \right) - f \left( t, \tilde{x}_0 + \mathcal{P}z(\alpha(t)) - B_2\alpha(t) \int_0^t D(s) \mathcal{P}z(s) \, ds + B_3(\bar{x}_0) \alpha(t), \tilde{y}(\beta(t)) \right) \right|
\]

\[
\leq K |\mathcal{P}z(\alpha(t)) - \mathcal{P}\tilde{z}(\alpha(t))| + K |B_2| \alpha(t) \int_0^t |D(s)| |\mathcal{P}z(s) - \mathcal{P}\tilde{z}(s)| \, ds
\]

\[
+ L |y(\beta(t)) - \tilde{y}(\beta(t))| + L |y(\beta(t)) - \tilde{y}(\beta(t))| \leq K \Omega_0(\alpha(t), |x - \tilde{x}|) + K |B_2| \alpha(t) \int_0^t |D(s)| \Omega_0(s, |x - \tilde{x}|) \, ds
\]

\[
+ L |y(\beta(t)) - \tilde{y}(\beta(t))| \leq K \Omega_0(\alpha(t), |x - \tilde{x}|) + K |B_2| \alpha(t) \int_0^t |D(s)| \Omega_0(s, |x - \tilde{x}|) \, ds
\]

\[
+ L |y(\beta(t)) - \tilde{y}(\beta(t))|,
\]
and

\[ |G(t, x, y; \bar{x}_0) - G(t, \bar{y}; \bar{x}_0)| = \left| g \left( t, \bar{x}_0 + \mathcal{P}x(\gamma(t)) - B_2 \gamma(t) \int_0^\xi D(s) \mathcal{P}x(s) \, ds + B_3 (\bar{x}_0) \gamma(t), y(t) \right) \right| \]

\[ \leq M \left| \mathcal{P}x(\gamma(t)) - \mathcal{P}\bar{x}(\gamma(t)) \right| + M |B_2| \gamma(t) \int_0^\xi |D(s)||\mathcal{P}x(s) - \mathcal{P}\bar{x}(s)| \, ds \]

\[ + N |y(t) - \bar{y}(t)| \]

For \( n = 0, 1, \ldots \), let us define the sequences \( \{u_n, w_n\} \) by relations

\[ u_0(t) = v(t), \]

\[ u_{n+1}(t) = K_0(\alpha(t), u_n) + K |B_2| \alpha(t) \int_0^\xi |D(s)||\Omega_0(s, s, u_n) \, ds + Lw_n(\beta(t)) \] \hspace{1cm} (16)

\[ w_{n+1}(t) = M_0(\gamma(t), u_n) + M |B_2| \gamma(t) \int_0^\xi |D(s)||\Omega_0(s, s, u_n) \, ds + Nw_n(t), \]

\[ w_0(t) = (I - N)^{-1} \left[ M_0(\gamma(t), u_0) + M |B_2| \gamma(t) \int_0^\xi |D(s)||\Omega_0(s, s, u_0) \, ds + a(t) \right], \] \hspace{1cm} (17)

where \( v \) is defined as in Assumption H3 with

\[ H(t, \bar{x}_0) = L(I - N)^{-1}a(\beta(t)) + |\mathcal{F}(t, z_0, y_0; \bar{x}_0) - z_0(t)|, \quad \alpha(t) = |G(t, z_0, y_0; \bar{x}_0) - y_0(t)|. \]

**Lemma 4.** Assume that \( f \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^p) \), \( g \in C(J \times \mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^q) \), \( \alpha, \beta, \gamma \in C(J, J) \), and \( (A_0)_{p \times p}, (A_1)_{p \times p}, D_{p \times p}, \) and \( d_{p \times 1} \) are given matrices. Assume that the matrix \( B_2 \) exists. Let Assumptions H1 and H3 be satisfied. Then the sequences \( \{u_n, w_n\} \) of form (16),(17) are nonincreasing and converge uniformly to zero functions if \( n \to \infty \). Moreover, we can show that

\[ |x(t) - z_0(t)| \leq u_0(t), \quad |y(t) - y_0(t)| \leq w_0(t), \]

\[ |x_{n+k}(t) - z_k(t)| \leq u_k(t), \quad |y_{n+k}(t) - y_k(t)| \leq w_k(t), \] \hspace{1cm} (18)

for \( t \in J \) and \( n = 0, 1, \ldots \), where \( x_n \) and \( y_n \) are defined by (12).

**Proof.** By mathematical induction, it is simple to show that

\[ u_{n+1}(t) \leq u_n(t) \leq u_0(t), \quad w_{n+1}(t) \leq w_n(t) \leq w_0(t), \quad t \in J, \quad n = 0, 1, \ldots \]

Hence \( u_n \to 0, w_n \to 0 \) on \( J \), by Assumption H3. Now we are going to show (18). Note that

\[ |x_1(t) - z_0(t)| = |\mathcal{F}(t, z_0, y_0; \bar{x}_0) - z_0(t)| \leq u_0(t), \]

\[ |y_1(t) - y_0(t)| = |G(t, z_0, y_0; \bar{x}_0) - y_0(t)| = a(t) \leq [(I - N)(I - N)^{-1} + N(I - N)^{-1}]a(t) = (I - N)^{-1}a(t) \leq w_0(t), \quad t \in J. \]
Assume that \(|z_k(t) - z_0(t)| \leq u_0(t)|, \ |y_k(t) - y_0(t)| \leq w_0(t)|, t \in J\) for some \(k \geq 1\). By (14) and (15), we see that
\[
|z_{k+1}(t) - z_0(t)| \leq |F(t, z_k, y_k; \tilde{x}_0) - F(t, z_0, y_0; \tilde{x}_0)| + R(t, \tilde{x}_0)
\]
\[
= K\Omega_0(\alpha(t), u_0) + K|B_2|\alpha(t) \int_0^\xi |D(s)|\Omega_0(s, u_0) \, ds
\]
\[
+ Lw_0(\beta(t)) + R(t, \tilde{x}_0) = u_0(t), \quad t \in J,
\]
\[
|y_{k+1}(t) - y_0(t)| \leq |G(t, z_k, y_k; \tilde{x}_0) - G(t, z_0, y_0; \tilde{x}_0)| + a(t)
\]
\[
\leq M\Omega_0(\gamma(t), u_0) + M|B_2|\gamma(t) \int_0^\xi |D(s)|\Omega_0(s, u_0) \, ds
\]
\[
+ Nw_0(t) + a(t) = w_0(t), \quad t \in J.
\]
Hence, by mathematical induction, we have
\[
|z_n(t) - z_0(t)| \leq h_0(t), \quad |y_n(t) - y_0(t)| \leq w_0(t), \quad t \in J, \quad n = 0, 1, \ldots
\]
The rest of estimates (18) can be proved by similar argument. It ends the proof.

Lemma 4 follows.

THEOREM 3. Assume that all assumptions of Lemma 4 are satisfied. Then, for every \(\tilde{x}_0 \in \mathbb{R}^p\), system (12) of sequences \(\{z_n, y_n\}\) converges to the unique solution \((\tilde{z}, \tilde{y})\) of problem (11) (uniqueness in the class \(A(z_0, y_0)\)), so \(z_n(t) \to \tilde{z}(t), y_n(t) \to \tilde{y}(t)\) for \(t \in J\) if \(n \to \infty\) and for \(t \in J\) we have the error estimates
\[
\left\{
\begin{array}{ll}
|\tilde{z}(t) - z_0(t)| \leq u_0(t), \\
|z_n(t) - \tilde{z}(t)| \leq u_n(t),
\end{array}
\right.
\quad n = 0, 1, \ldots
\]
\[
\left\{
\begin{array}{ll}
|\tilde{y}(t) - y_0(t)| \leq w_0(t), \\
|y_n(t) - \tilde{y}(t)| \leq w_n(t),
\end{array}
\right.
\quad n = 0, 1, \ldots
\]
Moreover, \((\tilde{x}, \tilde{y})\) with \(\tilde{x} = \tilde{x}_0 + \int_0^T \tilde{z}(s) \, ds\) is the solution of problem (10),(2) iff
\[
B_2 \int_0^\xi D(s)P \tilde{z}(s) \, ds + \frac{1}{T} \int_0^T \tilde{z}(s) \, ds = B_3(\tilde{x}_0)
\]

REMARK 5. Note that Assumption H3 holds if we assume that \(\rho(W) < 1\), where
\[
W = K \max_{t \in J} Q(\alpha(t)) + [K + L(I - N)^{-1}M] |B_2| \frac{T^2}{2} \int_0^\xi |D(s)| \, ds
\]
\[
+ 2L(I - N)^{-1}M \max_{t \in J} Q(\gamma(\beta(t))), \quad \text{with} \quad Q(t) = \frac{t}{T}(T - t).
\]
Similarly as before to find a solution \((\tilde{z}, \tilde{y})\) of problem (11), we can apply Seidel’s method. It means that we can formulate the following.

THEOREM 4. Let all assumptions of Lemma 4 be satisfied. Then the results of Theorem 3 hold and \(z_n(t) \to \tilde{z}(t), z_n(t) \to \tilde{z}(t), y_n(t) \to \tilde{y}(t), y_n(t) \to \tilde{y}(t),\) where \(\{\tilde{z}, \tilde{y}\}\) and \(\{z_n, y_n\}\) are defined by
\[
\left\{
\begin{array}{ll}
\tilde{z}_{n+1}(t) = F(t, \tilde{z}_n, \tilde{y}_n; \tilde{x}_0), \\
\tilde{y}_{n+1}(t) = G(t, \tilde{z}_n, \tilde{y}_n; \tilde{x}_0),
\end{array}
\right.
\]
for \(t \in J, \ n = 0, 1, \ldots\) with \(\tilde{z}_0(t) = z_0(t), \tilde{y}_0(t) = y_0(t), t \in J.\)
Moreover, we have the error estimates
\[
\left\{
\begin{array}{ll}
|\tilde{z}(t) - \tilde{z}(t)| \leq \tilde{u}_n(t), \\
|\tilde{z}_n(t) - \tilde{z}(t)| \leq \tilde{u}_n(t),
\end{array}
\right.
\]
\[
\left\{
\begin{array}{ll}
|\tilde{y}(t) - \tilde{y}(t)| \leq \tilde{w}_n(t), \\
|\tilde{y}_n(t) - \tilde{y}(t)| \leq \tilde{w}_n(t),
\end{array}
\right.
\]
for $t \in J$, $n = 0, 1, \ldots$, where

$$
\begin{align*}
\bar{u}_0(t) &= u_0(t), \\
\bar{w}_0(t) &= w_0(t), \\
\bar{u}_{n+1}(t) &= K\Omega_0(\alpha(t), \bar{u}_n) + K|B_2|\alpha(t) \int_0^\xi |D(s)|\Omega_0(s, \bar{u}_n) \, ds + L\bar{w}_n(\beta(t)), \\
\bar{w}_{n+1}(t) &= M\Omega_0(\gamma(t), \bar{u}_{n+1}) + M|B_2|\gamma(t) \int_0^\xi |D(s)|\Omega_0(s, \bar{u}_{n+1}) \, ds + N\bar{w}_n(t), \\
\bar{v}_0(t) &= v_0(t), \\
\bar{v}_0(t) &= v_0(t), \\
\bar{v}_{n+1}(t) &= M\Omega_0(\gamma(t), \bar{u}_n) + M|B_2|\gamma(t) \int_0^\xi |D(s)|\Omega_0(s, \bar{v}_n) \, ds + N\bar{v}_n(t), \\
\bar{v}_{n+1}(t) &= K\Omega_0(\alpha(t), \bar{v}_n) + K|B_2|\alpha(t) \int_0^\xi |D(s)|\Omega_0(s, \bar{u}_n) \, ds + L\bar{v}_{n+1}(\beta(t)),
\end{align*}
$$

with $u_0, v_0$ defined as in relations (16) and (17), respectively.

REFERENCES


