White noise driven SPDEs with reflection: Existence, uniqueness and large deviation principles

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Abstract

In the first part of this paper, we prove the uniqueness of the solutions of SPDEs with reflection, which was left open in the paper [C. Donati-Martin, E. Pardoux, White noise driven SPDEs with reflection, Probab. Theory Related Fields 95 (1993) 1–24]. We also obtain the existence of the solution for more general coefficients depending on the past with a much shorter proof. In the second part of the paper, we establish a large deviation principle for SPDEs with reflection. The weak convergence approach is proven to be very efficient on this occasion.

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1. Introduction and framework

Consider the following stochastic partial differential equation (SPDE) with reflection:

\[
\begin{aligned}
\frac{\partial u(x, t)}{\partial t} - \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t; u) &= \sigma(x, t; u)\dot{W}(x, t) + \eta, & x \in [0, 1], t \geq 0; \\
u(x, 0) &= u_0; \\
u(0, t) &= u(1, t) = 0,
\end{aligned}
\]

(1.1)

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where $\hat{W}$ denotes the space-time white noise defined on a complete probability space $(\Omega, \mathcal{F}, [\mathcal{F}_t]_{t\geq0}, \mathbb{P})$, where $\mathcal{F}_t = \sigma(W(x, s) : x \in [0, 1], 0 \leq s \leq t)$; $u_0$ is a non-negative continuous function on $[0, 1]$, which vanishes at 0 and 1; $\eta(x, t)$ is a random measure which is a part of the solution pair $(u, \eta)$. The coefficients $f$ and $\sigma$ are measurable mappings from $[0, 1] \times \mathbb{R}_+ \times C([0, 1] \times \mathbb{R}_+)$ into $\mathbb{R}$. The following definition is taken from [5,10].

**Definition 1.1.** A pair $(u, \eta)$ is said to be a solution of Eq. (1.1) if

(i) $u$ is a continuous process on $[0, 1] \times \mathbb{R}_+$; $u(x, t)$ is $\mathcal{F}_t$ measurable and $u(x, t) \geq 0$ a.s.

(ii) $\eta$ is a random measure on $(0, 1) \times \mathbb{R}_+$ such that

(a) $\eta((0, 1) \times \{t\}) = 0, \forall t \geq 0$.

(b) $\int_0^t \int_0^1 x(1-x)\eta(dx, ds) < \infty, t \geq 0$.

(c) $\eta$ is adapted in the sense that for any measurable mapping $\psi : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$

$$
\int_0^t \int_0^1 \psi(x, s)\eta(dx, ds) \text{ is } \mathcal{F}_t \text{ measurable.}
$$

(iii) $(u, \eta)$ solves the parabolic SPDE in the following sense: $(\cdot, \cdot)$ denotes the scalar product in $L^2([0, 1])$: $\forall \phi \in C^2([0, 1])$ with $\phi(0) = \phi(1) = 0$,

$$(u(t), \phi) - \int_0^t (u(s), \phi^\prime \prime)ds + \int_0^t \phi(\cdot, s; u)ds$$

$$= (u_0, \phi) + \int_0^t \int_0^1 \phi(x)\sigma(x, s; u)W(dx, ds) + \int_0^t \int_0^1 \phi(x)\eta(dx, ds) \quad \text{a.s.,}
$$

where $u(t) := u(\cdot, t)$.

(iv) $\int_Q ud\eta = 0$, where $Q = (0, 1) \times \mathbb{R}_+$.

This equation was first studied by Nualart and Pardoux in [10] when $\sigma(\cdot) = 1$, and by Donati-Martin and Pardoux in [5] for general diffusion coefficient $\sigma$. Various properties of the solution of Eq. (1.1) were studied later in [4,6,9,13,14]. SPDEs with reflection can also be used to model the evolution of random interfaces near a hard wall. It was proved by T. Funaki and S. Olla in [8] that the fluctuations of a $\nabla \phi$ interface model near a hard wall converge in law to the stationary solution of a SPDE with reflection.

We now introduce the precise assumptions on the coefficients. Let $f, \sigma$ be two measurable mappings

$$f, \sigma : [0, 1] \times \mathbb{R}_+ \times C([0, 1] \times \mathbb{R}_+) \to \mathbb{R}$$

satisfying:

**I** For any $u, v \in C([0, 1] \times \mathbb{R}_+)$, $(x, t) \in [0, 1] \times \mathbb{R}_+$ such that $u^t = v^t$,

$$f(x, t; u) = f(x, t; v)$$

$$\sigma(x, t; u) = \sigma(x, t; v),$$

where $u^t$, $v^t$ denote the restriction of $u$, $v$ to $[0, 1] \times [0, t]$ respectively.

**II** For any $T, M > 0$, there exists a constant $c(T, M)$ depending on $T, M$ such that for any $x \in [0, 1]$, $t \in [0, T]$, $u, v \in C([0, 1] \times \mathbb{R}_+)$ satisfying $\sup_{x \in [0, 1], t \in [0, T]} |u(x, t)| \leq M, \sup_{x \in [0, 1], t \in [0, T]} |v(x, t)| \leq M$,
\[ |f(x, t; u) - f(x, t; v)| + |\sigma(x, t; u) - \sigma(x, t; v)| \leq c(T, M) \sup_{y \in [0,1], s \in [0, t]} |u(y, s) - v(y, s)|. \]  

(1.2)

**Remark 1.1.** \( f(x, t; u), \sigma(x, t; u) \) are functionals that depend on the past of \( u \). If \( F(z), G(z) \) are real-valued local Lipschitz functions, then \( \sigma(x, t; u) \coloneqq G(u(x, t)) \) and \( f(x, t; u) \coloneqq F(u(x, t)) \) are particular functionals that satisfy assumptions (I), (II) and (III).

The purpose of this paper is twofold. In the first part, we prove the uniqueness of the solution of (1.1) which was left open in [5]. In the paper [5], the authors obtained the existence of a minimal solution of SPDEs with reflection through approximations of penalized SPDEs. Our approach also allows us to prove the existence of the solution by iteration. This provides a much shorter proof of the existence of the solution and for more general equations with coefficients depending on the past. In the second part of the paper, we consider the small noise perturbation of (1.1), that is

\[
\begin{align*}
\frac{\partial u^\varepsilon(x, t)}{\partial t} - \frac{\partial^2 u^\varepsilon(x, t)}{\partial x^2} + f(x, t; u^\varepsilon) &= \varepsilon \sigma(x, t; u^\varepsilon) \dot{W}(x, t) + \eta^\varepsilon; \\
u^\varepsilon(\cdot, 0) &= u_0; \\
u^\varepsilon(0, t) &= u^\varepsilon(1, t) = 0.
\end{align*}
\]

(1.4)

We will establish a large deviation principle for \( \{u^\varepsilon(\cdot, \cdot), \varepsilon > 0\} \) in \( C_+([0, 1] \times [0, T]) \), the space of non-negative continuous functions in \([0, 1] \times [0, T]\) endowed with the uniform topology. Small noise large deviations for stochastic partial differential equations driven by space-time white noise was first obtained by R. Sowers in [11] and later by Cerrai and Röckner in [3]. In these papers, the method is to establish some exponential estimates for solutions of the corresponding SPDEs, which are difficult to get for the current Eq. (1.1) because of the reflection. Instead of proving exponential estimates, we will use the weak convergence approach in the theory of large deviations, which is proven to be very efficient on this occasion. The reader is referred to [1,2,7] for details of this method. One of the key elements in this approach is to prove the weak convergence of the perturbation of Eq. (1.4) in the random Cameron–Martin directions. We remark that using the weak convergence approach one can also provide a simpler proof even for large deviations of SPDEs without reflection.

Denote by \( |\cdot|_{\infty} \) the uniform norm in \( C([0, 1] \times [0, T]) \). \( f, \sigma \) are said to satisfy the globally Lipschitz assumption, if for any \( T > 0 \),

\[ |f(x, t; u) - f(x, t; v)| + |\sigma(x, t; u) - \sigma(x, t; v)| \leq K(T)|u - v|_{\infty}. \]

(1.5)

The organization of this paper is as follows: In Section 2, we study the deterministic obstacle problem and prove the existence and uniqueness of the solution of Eq. (1.1). In Section 3, we study the skeleton equations and establish a large deviation principle for solutions of SPDEs with reflection.
2. The existence and uniqueness

2.1. Deterministic obstacle problem

Consider the following deterministic parabolic obstacle problem:

\[
\begin{align*}
\frac{\partial z(x, t)}{\partial t} - \frac{\partial^2 z(x, t)}{\partial x^2} &\geq 0; \\
z(x, t) &\geq -v(x, t); \\
\left( \frac{\partial z(x, t)}{\partial t} - \frac{\partial^2 z(x, t)}{\partial x^2}, z(x, t) + v(x, t) \right) &= 0,
\end{align*}
\]

(2.1)

where \( v \in C([0, 1] \times [0, T]) \) with \( v(x, 0) = u_0(x) \). If a pair \((z, \eta)\) satisfies

1. \( z \) is a continuous function on \([0, 1] \times [0, T]\) and
\[
z(x, 0) = 0, z(0, t) = z(1, t) = 0, \quad z \geq -v.
\]

2. \( \eta \) is a measure on \((0, 1) \times \mathbb{R}_+\) such that for all \( \varepsilon > 0, T > 0 \)
\[
\eta((\varepsilon, (1 - \varepsilon) \times [0, T])) < \infty.
\]

Furthermore, \( \eta \) also satisfies (a), (b) of (ii) in Definition 1.1.

3. For all \( t \geq 0, \phi \in C^2(0, 1), \)
\[
(z(t), \phi) - \int_0^t (z(s), \phi'') ds = \int_0^t \int_0^1 \phi(x) \eta(dx, ds).
\]

4. \( \int_0^t \int_0^1 (z(x, s) + v(x, s)) \eta(dx, ds) = 0 \), then \((z, \eta)\) is called a solution to problem (2.1). The following result was proved in [10] (Theorem 1.4).

Proposition 2.1 ([10]). If \( v(x, 0) = u_0(x) \) for all \( x \in [0, 1] \), \( v(0, t) = v(1, t) = 0 \) for all \( t \geq 0 \), Eq. (2.1) admits a unique solution. Moreover, \( |z|_\infty^T \leq |v|_\infty^T \).

Remark 2.1. The conclusion \( |z|_\infty^T \leq |v|_\infty^T \) was not stated in Theorem 1.4 in [10]. However, it follows from the estimate (13) in the proof of Theorem 1.4 in [10] by choosing \( \hat{v} = 0 \) and noticing \( \hat{z}^\varepsilon = 0 \) because \( f = 0 \).

2.2. SPDEs with reflection

Theorem 2.1. Under the assumptions (I), (II), (III), Eq. (1.1) admits a unique solution \( u \). Moreover \( E(|u|_T^\infty)^p < \infty, \forall p \geq 1 \).

Proof (Existence). Firstly, we may assume that \( f, \sigma \) satisfy the globally Lipschitz condition (II’).

We will use successive iteration. Let
\[
\begin{align*}
v_1(x, t) &= \int_0^1 G_t(x, y) u_0(y) dy - \int_0^t \int_0^1 G_{t-s}(x, y) f(y, s; u_0) dy ds \\
&\quad + \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; u_0) W(dy, ds),
\end{align*}
\]

where \( u_0 \) is the solution of the deterministic obstacle problem.
where $G(\cdot, \cdot)$ is Green’s function associated to the operator $\frac{\partial^2}{\partial x^2}$ with Dirichlet boundary conditions. As in [12], it is seen that $v_1(x, t)$ satisfies the following SPDE:

\[
\begin{aligned}
\frac{\partial v_1(x, t)}{\partial t} - \frac{\partial^2 v_1(x, t)}{\partial x^2} &= f(x, t; u_0) + \sigma(x, t; u_0) \dot{W}(x, t); \\
v_1(x, 0) &= u_0(x); \\
v_1(0, t) &= v_1(1, t) = 0,
\end{aligned}
\]

and $v_1(\cdot, \cdot) \in C([0, 1] \times [0, T])$. Denote by $(z_1, \eta_1)$ the unique random solution of (2.1) with $v = v_1$. Set $u_1 = z_1 + v_1$, then we can easily verify that $(u_1, \eta_1)$ is the unique solution of the following reflected SPDE:

\[
\begin{aligned}
\frac{\partial u_1(x, t)}{\partial t} - \frac{\partial^2 u_1(x, t)}{\partial x^2} + f(x, t; u_0) &= \sigma(x, t; u_0) \dot{W}(x, t) + \eta_1; \\
u_1(\cdot, 0) &= u_0; \\
u_1(0, t) &= u_1(1, t) = 0.
\end{aligned}
\]

(2.2)

Iterating this procedure, suppose $u_{n-1}$ has been defined. Let

\[
v_n(x, t) = \int_0^1 G_t(x, y)u_0(y)dy - \int_0^t \int_0^1 G_{t-s}(x, y)f(y, s; u_{n-1})dy ds + \int_0^t \int_0^1 G_{t-s}(x, y)\sigma(y, s; u_{n-1}) \dot{W}(dy, ds),
\]

and $(z_n, \eta_n)$ be the unique random solution of Eq. (2.1) with $v(x, t) = v_n(x, t)$. Set $u_n = z_n + v_n$, then $(u_n, \eta_n)$ is the unique solution of the following reflected SPDE:

\[
\begin{aligned}
\frac{\partial u_n(x, t)}{\partial t} - \frac{\partial^2 u_n(x, t)}{\partial x^2} + f(x, t; u_{n-1}) &= \sigma(x, t; u_{n-1}) \dot{W}(x, t) + \eta_n; \\
u_n(\cdot, 0) &= u_0; \\
u_n(0, t) &= u_n(1, t) = 0.
\end{aligned}
\]

(2.4)

From the proof of Theorem 1.4 in [10], we have

\[
|z_n - z_{n-1}|_T^T \leq |v_n - v_{n-1}|_T^T,
\]

hence,

\[
|u_n - u_{n-1}|_T^T \leq 2|v_n - v_{n-1}|_T^T.
\]

(2.5)

By Proposition 2.1,

\[
|z_n|_T^T \leq |v_n|_T^T.
\]

Therefore,

\[
|u_n|_T^T \leq 2|v_n|_T^T.
\]

(2.6)

From the proof of Theorem 3.1 in [5], we have

\[
E(|u_1|_T^T)^p \leq 2^p E(|v_1|_T^T)^p \leq c(T, p, |u_0|_T^T) < +\infty,
\]

where $c(T, p, |u_0|_T^T)$ is a constant depending on $T, p, |u_0|_T^T$. If $E(|u_{n-1}|_T^T)^p < \infty$, by the same reason, we have

\[
E(|u_n|_T^T)^p \leq 2^p E(|v_n|_T^T)^p \leq c(T, p, |u_0|_T^T, E(|u_{n-1}|_T^T)^p) < +\infty.
\]
In view of (2.5), we have
\[
(\|u_n - u_{n-1}\|_\infty)^p \leq 2^p (\|v_n - v_{n-1}\|_\infty)^p
\]
\[
\leq c(p) \left( \sup_{x \in [0,1], t \in [0,T]} \left| \int_0^t \int_0^1 G_{t-s}(x,y) \left[ f(y,s; u_{n-1}) - f(y,s; u_{n-2}) \right] dy ds \right| \right)^p
\]
\[
+ c(p) \left( \sup_{x \in [0,1], t \in [0,T]} \left| \int_0^t \int_0^1 G_{t-s}(x,y) \left[ \sigma(y,s; u_{n-1}) - \sigma(y,s; u_{n-2}) \right] W(dy, ds) \right| \right)^p. \tag{2.7}
\]

Set
\[
I_1(x, t) := \int_0^t \int_0^1 G_{t-s}(x,y) \left[ f(y,s; u_{n-1}) - f(y,s; u_{n-2}) \right] dy ds,
\]
\[
I_2(x, t) := \int_0^t \int_0^1 G_{t-s}(x,y) \left[ \sigma(y,s; u_{n-1}) - \sigma(y,s; u_{n-2}) \right] W(dy, ds).
\]

By (1.5) and Hölder’s inequality, for \( p > 6 \),
\[
E(I_{1,T}^p) \leq K(T)^p \left( \sup_{x \in [0,1], t \in [0,T]} \left| \int_0^t \int_0^1 G_{t-s}^q(x,y) dy ds \right| \right)^{\frac{p}{q}}
\]
\[
\times E \int_0^T (\|u_{n-1} - u_{n-2}\|_\infty)^p dr, \tag{2.8}
\]

where \( \frac{1}{q} + \frac{1}{p} = 1 \). Using (1.5) and following the same calculation as in the proof of Corollary 3.4 in [12] or Theorem 3.1 in [5], we deduce that for \( p > 6 \),
\[
E|I_2(x, t) - I_2(y, s)|^p \leq c_T \left( E \int_0^{(t \vee s)} (\|u_{n-1} - u_{n-2}\|_\infty)^p dr \right)^p
\]
\[
\times (|x, t) - (y, s)|^\frac{p}{4} - 3. \tag{2.9}
\]

Choosing \( p > 20 \) and applying Lemma 3.1 in [5], we obtain that
\[
|I_2(x, t) - I_2(y, s)|^p \leq N(\omega)^p (|x, t) - (y, s)|^\frac{p}{4} - 5 \left( \log \left( \frac{y}{|x, t) - (y, s)|} \right) \right)^2, \tag{2.10}
\]

where \( N(\omega) \) is a random variable satisfying
\[
E[N^p] \leq ac_T \left( E \int_0^{(t \vee s)} (\|u_{n-1} - u_{n-2}\|_\infty)^p dr \right)^p, \tag{2.11}
\]

where \( a, c_T \) are constants depending on \( p \) and \( T \). Choosing \( s = 0 \) in (2.10), we see that there exists a constant \( c_T \) such that
\[
E \left( \sup_{x \in [0,1], t \in [0,T]} \left| \int_0^t \int_0^1 G_{t-s}(x,y) \left[ \sigma(y,s; u_{n-1}) - \sigma(y,s; u_{n-2}) \right] W(dy, ds) \right|^p \right)^p
\]
\[
\leq c_T E \int_0^T (\|u_{n-1} - u_{n-2}\|_\infty)^p dr. \tag{2.12}
\]
Putting (2.7), (2.8), (2.12) together and using the fact that

\[ \sup_{x \in [0,1], t \in [0, T]} \int_0^t \int_0^1 G_s^r(x, y) dy ds < \infty, \quad (0 < r < 3), \]  

we get that

\[
E( |u_n - u_{n-1}|_T^\infty)^p \leq c(p, K, T) E \int_0^T (|u_{n-1} - u_{n-2}|_\infty^t)^p dt \\
\leq c^2(p, K, T) E \int_0^T (|u_{n-2} - u_{n-3}|_\infty^t)^p (T - t) dt \\
\leq \cdots \leq c^{n-1}(p, K, T) E \int_0^T (|u_1 - u_0|_\infty^t)^p (T - t)^{n-2} (n - 2)! dt \\
\leq c^{n-1}(p, K, T) E (|u_1 - u_0|_\infty^T)^p \frac{T^{n-1}}{(n - 1)!},
\]

where \( c(p, K, T) \) is a constant depending on \( p, K, T \). Hence, for any \( m \geq n \geq 1 \),

\[
[E( |u_m - u_n|_T^\infty)^p]^\frac{1}{p} \leq \sum_{n=1}^{m-1} \left[ \frac{(cT)^k}{k!} \right]^\frac{1}{p} [E( |u_1 - u_0|_\infty^T)^p]^\frac{1}{p} \\
\rightarrow 0 \quad \text{as} \quad n, m \rightarrow \infty.
\]

Hence, there exists a random field \( u(x, t) \in C([0, 1] \times [0, T]) \) such that

\[ E( |u|_T^\infty)^p < \infty, \]

and

\[ \lim_{n \rightarrow \infty} E( |u_n - u|_T^\infty)^p = 0. \]

Next, we will show that \( u \) is a solution of Eq. (1.1). Since \( u_n(x, t) \geq 0 \) a.s., it follows that \( u(x, t) \geq 0 \) a.s. For any \( \phi \in C_0^\infty((0, 1) \times \mathbb{R}_+) \), let \( \phi(s) \) denote the function \( \phi(\cdot, s) \) and \( f(s, u) \) denote the function \( f(\cdot, s; u) \). Since \( (u_n, \eta_n) \) is the solution of Eq. (2.4), we have

\[
(u_n(t), \phi(t)) - \int_0^t \left( u_n(s), \frac{\partial}{\partial s} \phi(s) \right) ds - \int_0^t (u_n(s), \phi''(s)) ds + \int_0^t (f(s, u_{n-1}), \phi(s)) ds \\
= (u_0, \phi(0)) + \int_0^t \int_0^1 \sigma(x, s; u_{n-1}) \phi(x, s) W(dx, ds) + \int_0^t \int_0^1 \phi(x, s) \eta_n(dx, ds).
\]

All the terms on the left-hand side of the above identity converge a.s to the corresponding terms with \( u_n \) replaced by \( u \), as \( n \rightarrow \infty \) (taking a subsequence, if necessary). On the other hand, it follows from (1.5) that

\[
\int_0^t \int_0^1 \sigma(x, s; u_{n-1}) \phi(x, s) W(dx, ds) \xrightarrow{L^2} \int_0^t \int_0^1 \sigma(x, s; u) \phi(x, s) W(dx, ds).
\]

Thus, \( \eta_n \) converges in the distributional sense to a positive distribution \( \eta \) on \((0, 1) \times \mathbb{R}_+ \) a.s and hence a measure on \((0, 1) \times \mathbb{R}_+ \). Letting \( n \rightarrow \infty \) in (2.17), we have

\[
(u(t), \phi(t)) - \int_0^t \left( u(s), \frac{\partial}{\partial s} \phi(s) \right) ds - \int_0^t (u(s), \phi''(s)) ds + \int_0^t (f(s, u), \phi(s)) ds
\]
\[ (u_0, \phi(0)) + \int_0^t \int_0^1 \sigma(x, s; u) \phi(x, s) W(dx, ds) + \int_0^t \int_0^1 \phi(x, s) \eta(dx, ds), \]  
(2.18)

so \((u, \eta)\) satisfies condition (iii) of Definition 1.1. To prove that \((u, \eta)\) satisfies condition (iv) of Definition 1.1, it is sufficient to prove
\[ (\text{supp } \eta)^c \supset \{u > 0\} = \{(x, t) \in [0, 1] \times [0, T] : u(x, t) > 0\}. \]
(2.19)

Now, for any \((x_0, t_0) \in \{u > 0\}\), there exists an open ball \(B((x_0, t_0), r)\), which is centered at \((x_0, t_0)\) with radius \(r\), such that
\[ u(x, t) \geq 2\alpha > 0, \quad (x, t) \in B((x_0, t_0), r), \]  
for some positive number \(\alpha\).

Since \(\lim_{n \to \infty} |u_n - u|^T = 0\), there exists a \(N_0\) such that \(\forall n \geq N_0,
\[ u_n(x, t) \geq \alpha > 0, \quad (x, t) \in B((x_0, t_0), r). \]

Choose a function \(\phi \in C_0^\infty([0, 1] \times [0, T])\), which satisfies \(0 \leq \phi \leq 1\) and
\[ \phi(x, t) = \begin{cases} 
1, & (x, t) \in B \left( (x_0, t_0), \frac{r}{2} \right) \\
0, & (x, t) \in (B((x_0, t_0), r))^c 
\end{cases}. \]

Then
\[ 0 \leq \int_0^T \int_0^1 \phi(x, t) \eta^n(dx, dt) \leq \int_{B((x_0, t_0), r)} 1 \eta^n(dx, dt) \]
\[ \leq \frac{1}{\alpha} \int_0^T \int_0^1 u_n(x, t) \eta^n(dx, dt) = 0, \]
where the last step follows from the fact that \((u^n, \eta^n)\) is the solution of Eq. (2.4). Hence,
\[ 0 \leq \eta \left( B \left( (x_0, t_0), \frac{r}{2} \right) \right) \leq \int_0^T \int_0^1 \phi(x, t) \eta(dx, dt) \]
\[ = \lim_{n \to \infty} \int_0^T \int_0^1 \phi(x, t) \eta^n(dx, dt) = 0, \]
which yields (2.19). Following the proof of Theorem 1.4 in [10], we can show condition (ii) of Definition 1.1 is also satisfied by \((u, \eta)\). Thus, we have shown that \((u, \eta)\) is a solution to Eq. (1.1).

Now, we assume that \(f, \sigma\) satisfy the locally Lipschitz condition (II). Note that an inequality analogous to (2.6) also holds for the solution \((u, \eta)\). Following the standard localization argument and applying (2.6) and linear growth condition (III), we can extend the existence to the locally Lipschitz case.

**Uniqueness.** Let \((u_1, \eta_1)\) and \((u_2, \eta_2)\) be two solutions of Eq. (1.1). Set
\[ v_i(x, t) = \int_0^1 G_t(x, y) u_0(y) dy - \int_0^t \int_0^1 G_{t-s}(x, y) f(y, s; u_i) dy ds \]
\[ + \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; u_i) W(dy, ds). \]
(2.20)

Then \((z_i = u_i - v_i, \eta_i)\) is the unique solution of Eq. (2.1) with \(v = v_i, i = 1, 2\). Similar to (2.5), we have
\[ |u_1 - u_2|^T \leq 2|v_1 - v_2|^T. \]
Define a stopping time by
\[ \tau_N := \inf\{t \geq 0, \sup_{x \in [0,1]} |u_1(x, t)| \vee \sup_{x \in [0,1]} |u_2(x, t)| > N \}. \]

By the similar proof as that for (2.14), we have
\[ E(|u_1 - u_2|_{T \wedge \tau_N}^T)^p \leq 2^p E(|v_1 - v_2|_{T \wedge \tau_N}^T)^p \leq c(p, K, T, N) E \int_0^T (|u_1 - u_2|_{\infty}^T)^p \, dt. \]

This implies that \( E|u_1 - u_2|_{\infty}^T = 0 \). Letting \( N \to \infty \), we get \( E|u_1 - u_2|_{\infty}^T = 0 \). Hence, \( u_1 = u_2 \), a.s, and consequently, \( \eta_1 = \eta_2 \), a.s. \( \square \)

3. Large deviation principles

From this section, we assume that \( f, \sigma \) satisfy the conditions (I), (II') and (III).

3.1. Skeleton equations and the rate function

The Cameron–Martin space associated with the Brownian sheet \( \{W(x, t), x \in [0, 1], t \in \mathbb{R}_+\} \) is given by
\[ \mathcal{H} = \left\{ \hat{h} = \int_0^\infty \int_0^s \hat{h}(x, s) \, dx \, ds; \int_0^T \int_0^1 \hat{h}_x(x, s) \, dx \, ds < \infty \right\}. \]

For \( h = \int_0^\infty \int_0^s \hat{h}(x, s) \, dx \, ds \in \mathcal{H} \), consider the following reflected deterministic PDE (the skeleton equation):
\[ \begin{aligned}
\frac{\partial h(x, t)}{\partial t} - \frac{\partial^2 h(x, t)}{\partial x^2} + f(x, t; s^h) = \sigma(x, t; s^h) \hat{h}(x, t) + \eta^h; \\
\hat{h}(-, 0) = u_0; \\
\hat{h}(0, t) = s^h(1, t) = 0.
\end{aligned} \quad (3.1) \]

Analogously, if there exists a pair of \((s^h, \eta^h)\) satisfying
(i) \( s^h \) is a continuous process on \([0, 1] \times \mathbb{R}_+ \) and \( s^h(x, t) \geq 0 \).
(ii) \( \eta \) is a measure on \((0, 1) \times \mathbb{R}_+ \) such that
(a) \( \eta((0, 1) \times \{t\}) = 0, \forall t \geq 0 \).
(b) \( \int_0^t \int_0^x (1 - x) \eta(dx, ds) < \infty, t \geq 0 \).
(iii) \((s^h, \eta^h)\) solves the parabolic PDE in the following sense \((, , \cdot)\) denotes the scalar product in \( L^2([0, 1]) \); \( \forall t \in \mathbb{R}_+, \phi \in C^2([0, 1]) \) with \( \phi(0) = \phi(1) = 0 \),
\[ (s^h(t), \phi) - \int_0^t (s^h(s), \phi''(s)) \, ds + \int_0^t (f(s, s^h), \phi) \, ds \\
= (u_0, \phi) + \int_0^t \int_0^1 \phi(x) \sigma(x, s; s^h) \hat{h}(x, s) \, dx \, ds + \int_0^t \int_0^1 \phi(x) \eta(dx, ds) \]
(iv) \( \int_0^t \int_0^1 s^h(x, s) \eta(dx, ds) = 0 \), then \((s^h, \eta^h)\) is called a solution of Eq. (3.1).

Theorem 3.1. For any \( h \in \mathcal{H} \), Eq. (3.1) admits a unique solution.
Proof. The proof is similar to that of Theorem 2.1, we sketch it here.

Existence. Let

\[ v^h_1(x, t) = \int_0^1 G_t(x, y)u_0(y)dy - \int_0^t \int_0^1 G_{t-s}(x, y)f(y, s; u_0)dyds \]

\[ + \int_0^t \int_0^1 G_{t-s}(x, y)\sigma(y, s; u_0)\hat{h}(y, s)dyds, \]

(3.2)

and \((z^h_1, \eta^h_1)\) be the solution of Eq. (2.1) with \(v = v^h_1\). Set \(s^h_1 = z^h_1 + v^h_1\), then \((s^h_1, \eta^h_1)\) is a solution of the following reflected PDE:

\[
\begin{cases}
\frac{\partial s^h_1(x, t)}{\partial t} - \frac{\partial^2 s^h_1(x, t)}{\partial x^2} + f(x, t; u_0) = \sigma(x, t; u_0)\hat{h}(x, t) + \eta^h_1; \\
s^h_1(t, 0) = u_0; \\
s^h_1(0, t) = s^h_1(1, t) = 0.
\end{cases}
\]

(3.3)

Generally, put

\[ v^h_n(x, t) = \int_0^1 G_t(x, y)u_0(y)dy - \int_0^t \int_0^1 G_{t-s}(x, y)f(y, s; s^h_{n-1})dyds \]

\[ + \int_0^t \int_0^1 G_{t-s}(x, y)\sigma(y, s; s^h_{n-1})\hat{h}(y, s)dyds, \]

(3.4)

and define \((z^h_n, \eta^h_n)\) to be the solution of (2.1) with \(v = v^h_n\).

Then \((s^h_n = z^h_n + v^h_n, \eta^h_n)\) is the solution of the following reflected PDE:

\[
\begin{cases}
\frac{\partial s^h_n(x, t)}{\partial t} - \frac{\partial^2 s^h_n(x, t)}{\partial x^2} + f(x, t; s^h_{n-1}) = \sigma(x, t; s^h_{n-1})\hat{h}(x, t) + \eta^h_n; \\
s^h_n(t, 0) = u_0; \\
s^h_n(0, t) = s^h_n(1, t) = 0.
\end{cases}
\]

(3.5)

Applying (2.5), one obtains that

\[
|s^h_n - s^h_{n-1}|^T_\infty \leq 2|v^h_n - v^h_{n-1}|^T_\infty
\]

\[
\leq 2 \sup_{x \in [0,1], t \in [0,T]} \left| \int_0^t \int_0^1 G_{t-s}(x, y)(f(y, s; s^h_{n-1}) - f(y, s; s^h_{n-2}))dyds \right|
\]

\[ + 2 \sup_{x \in [0,1], t \in [0,T]} \left| \int_0^t \int_0^1 G_{t-s}(x, y)(\sigma(y, s; s^h_{n-1}) - \sigma(y, s; s^h_{n-2}))\hat{h}(y, s)dyds \right|. \]

By (1.5) and Hölder’s inequality we get

\[
|s^h_n - s^h_{n-1}|^T_\infty \leq 2K \left( \sup_{x \in [0,1], t \in [0,T]} \int_0^t \int_0^1 G^2_t(x, y)dyds \right)^{\frac{1}{2}} \left( \int_0^T (|s^h_{n-1} - s^h_{n-2}|^2 \eta_\infty)dr \right)^{\frac{1}{2}}
\]

\[ + 2K \left( \sup_{x \in [0,1], t \in [0,T]} \int_0^t \int_0^1 G^2_t(x, y)dyds \right)^{\frac{1}{2}}
\]
\[
\times \left( \int_0^T \left[ (|s_{n-1}^h - s_{n-2}^h|^T_x)^2 \int_0^1 \dot{h}^2(y, t) \, dy \right] \right)^\frac{1}{2},
\]

therefore,
\[
(|s_n^h - s_{n-1}^h|^T_x)^2 \leq 4K^2 \left( \sup_{x \in [0,1], t \in [0, T]} \int_0^t \int_0^1 G_s^2(x, y) \, dy \right) \\
\times \left| \int_0^T (|s_{n-1}^h - s_{n-2}^h|^T_x)^2 \left( 1 + \int_0^1 \dot{h}^2(y, t) \, dy \right) \, dt \right|
\]

Set
\[
L := \sup_{x \in [0,1], t \in [0, T]} \int_0^t \int_0^1 G_s^2(x, y) \, dy, \\
C(t) := \left( 1 + \int_0^1 \dot{h}^2(y, t) \, dy \right), \\
\tilde{C}(t) := \int_0^t C(s) \, ds.
\]

Then,
\[
(|s_n^h - s_{n-1}^h|^T_x)^2 \leq 4K^2 L \int_0^T (|s_{n-1}^h - s_{n-2}^h|^T_x)^2 \, d\tilde{C}(t) \\
\leq \cdots \leq (4K^2 L)^{n-1} \int_0^T (|s_1^h - u_0|^T_x)^2 \frac{(\tilde{C}(T) - \tilde{C}(t))^{n-2}}{(n-2)!} \, d\tilde{C}(t) \\
\leq (|s_1^h - u_0|^T_x)^2 \frac{(4K^2 L)^{n-1} \tilde{C}(T)^{n-1}}{(n-1)!}.
\]

(3.6)

Therefore, for any \( m \geq n \geq 1, \)
\[
|s_m^h - s_n^h|^T_x \leq |s_1^h - u_0|^T_x \cdot \sum_{n} \left( \frac{(4K^2 L \tilde{C}(T))^k}{k!} \right)^\frac{1}{2} \\
\rightarrow 0, \quad \text{as } n, m \rightarrow +\infty.
\]

(3.7)

Let
\[
s^h(t, x) = \lim_{n \rightarrow +\infty} s_n^h(t, x).
\]

Since \( s_n^h(t, x) \geq 0, \) we have \( s^h(t, x) \geq 0. \) For any \( \phi \in C^\infty_0((0, 1) \times \mathbb{R}_+), \) we have
\[
(s_n^h(t), \phi(t)) - \int_0^t \left( s_n^h(s), \frac{\partial}{\partial s} \phi(s) \right) \, ds - \int_0^t (s_n^h(s), \phi''(s)) \, ds + \int_0^t (f(s, s_n^h), \phi(s)) \, ds \\
- \int_0^t \int_0^1 \sigma(x, s; s_n^h) \dot{h}(x, s) \phi(x, s) \, dx \, ds \\
= (u_0, \phi(0)) + \int_0^t \int_0^1 \phi(x, s) \sigma_n(dx, ds).
\]

(3.8)
From the left-hand side of the above identity, it is easy to see that $\eta^h_n$ converges in the distribution sense to a positive distribution $\eta^h$ on $(0, 1) \times \mathbb{R}_+$ and thus a measure on $(0, 1) \times \mathbb{R}_+$. As in the proof of Theorem 2.1, we can show that $(s^h, \eta^h)$ is the unique solution of (3.1). □

Let $B_N := \{h \in \mathcal{H}, |h|_\mathcal{H} \leq N\}$, where $|h|_\mathcal{H} = \left(\int_0^T \int_0^1 \hat{h}^2(x, s) \, ds \, dx\right)^{\frac{1}{2}}$. Then $B_N$ is a compact Polish space endowed with the weak topology of $\mathcal{H}$. Denote the convergence in $B_N$ by $\rightarrow_w$.

**Theorem 3.2.** $s^h(\cdot, \cdot)$ is a continuous mapping from $h \in B_N$ into $C([0, 1] \times [0, T])$.

**Proof.** Let $\{h_n\}_{n=1}^\infty$ be an arbitrary convergence sequence in $B_N$ and $h$ be the limit. We need to prove

$$
\lim_{n \to \infty} |s^{h_n} - s^h|^T_\infty = 0,
$$

where $s^{h_n}$, $s^h$ are the solutions of Eq. (3.1) associated with $h_n$, $h$ respectively. Let

$$
v^{h_n}(x, t) = \int_0^1 G_t(x, y)u_0(y) \, dy - \int_0^t \int_0^1 G_{t-s}(x, y) f(y, s; s^{h_n}) \, dy \, ds
$$

and

$$
v^h(x, t) = \int_0^1 G_t(x, y)u_0(y) \, dy - \int_0^t \int_0^1 G_{t-s}(x, y) f(y, s; s^h) \, dy \, ds
$$

In view of (2.5), we have

$$
|s^{h_n} - s^h|^T_\infty \leq 2|v^{h_n} - v^h|^T_\infty
\leq 2 \sup_{x \in [0, 1], t \in [0, T]} \left| \int_0^t \int_0^1 G_{t-s}(x, y) (f(y, s; s^{h_n}) - f(y, s; s^h)) \, dy \, ds \right|
+ 2 \sup_{x \in [0, 1], t \in [0, T]} \left| \int_0^t \int_0^1 G_{t-s}(x, y) (\sigma(y, s; s^{h_n}) - \sigma(y, s; s^h)) \hat{h}_n(y, s) \, dy \, ds \right|
+ 2 \sup_{x \in [0, 1], t \in [0, T]} \left| \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; s^h) (\hat{h}_n(y, s) - \hat{h}(y, s)) \, dy \, ds \right|

=: I_1 + I_2 + I_3.
$$

By (1.5),

$$
I_1 + I_2 \leq 2K \left( \sup_{x \in [0, 1], t \in [0, T]} \int_0^t \int_0^1 G^2_s(x, y) \, dy \, ds \right)^{\frac{1}{2}} \left( \int_0^T (|s^{h_n} - s^h|^T_\infty)^2 \, dr \right)^{\frac{1}{2}}
+ 2K \left( \sup_{x \in [0, 1], t \in [0, T]} \int_0^t \int_0^1 G^2_s(x, y) \, dy \, ds \right)^{\frac{1}{2}}
$$
Putting \( g_n \) compact in \( C \) continuous. According to Arzelà–Ascoli Theorem, we deduce that \( g_n \) is relatively compact in \( C([0, 1] \times [0, T]) \). Indeed,

\[
\sup_n |g_n|^T \leq M(1 + |s^h|_\infty^T) \left( \sup_{x \in [0, 1], t \in [0, T]} \left| \int_0^t \int_0^1 G_s^2(x, y)dy \right| \right)^{\frac{1}{2}} \\
\times \left( \int_0^T \int_0^1 |\dot{h}_n(y, s) - \dot{h}(y, s)|^2dy \right)^{\frac{1}{2}} \\
\leq 2N M(1 + |s^h|_\infty^T) \left( \sup_{x \in [0, 1], t \in [0, T]} \left| \int_0^t \int_0^1 G_s^2(x, y)dy \right| \right)^{\frac{1}{2}} < +\infty. \tag{3.14}
\]

Following the proof of Corollary 3.4 in [12], we can show that \( \{g_n(x, t), n \geq 1\} \) is also equi-continuous. According to Arzelà–Ascoli Theorem, we deduce that \( \{g_n(x, t), n \geq 1\} \) is relatively compact in \( C([0, 1] \times [0, T]) \). Combining with (3.13), one obtains

\[
\lim_{n \to \infty} |g_n|^T = 0. \tag{3.15}
\]

Putting (3.11), (3.12) together, and using Gronwall inequality, we get

\[
(|s^h|_\infty^T)^2 \leq 12(|g_n|^T)^2 \cdot \exp \left\{ 12K^2 \cdot \left( \sup_{x \in [0, 1], t \in [0, T]} \int_0^t \int_0^1 G_s^2(x, y)dy \right) \right\} \\
\times \left\{ \int_0^T \int_0^1 1 + \dot{h}_n^2(x, s)ds \right\} \\
\leq 12(|g_n|^T)^2 \cdot \exp \left\{ 12K^2(T + N^2) \right\} \\
\times \left( \sup_{x \in [0, 1], t \in [0, T]} \int_0^t \int_0^1 G_s^2(x, y)dy \right) \left\} \right. \\
\to 0, \quad \text{as } n \to \infty, \tag{3.16}
\]

which finishes the proof. \( \square \)
Define a function by

$$I(f) := \frac{1}{2} \inf_{\{h \in \mathcal{H}, s^h(\cdot, \cdot) = f\}} |h|^2_{\mathcal{H}}, \quad f \in C_+([0, 1] \times [0, T])$$

with the convention $\inf\{\emptyset\} = \infty$.

**Corollary 3.1.** The function $I(\cdot)$ defined above is a good rate function on $C_+([0, 1] \times [0, T])$, that is, $\{f : I(f) \leq a\}$ is compact for any $a \geq 0$.

**Proof.** For any $0 \leq a < \infty$,

$$\{f : I(f) \leq a\} = \{s^h(\cdot, \cdot) : |h|^2_{\mathcal{H}} \leq 2a\}.$$

Since $s^h(\cdot, \cdot)$ is continuous in $h$ by Theorem 3.2 and $\{h : |h|^2_{\mathcal{H}} \leq 2a\}$ is compact, we deduce that $\{f : I(f) \leq a\}$ is a compact set in $C_+([0, 1] \times [0, T])$. \(\square\)

### 3.2. Large deviation principles

Define $\mathcal{A}$ to be the class of random field $h(x, t)$ satisfying

$$h \in \mathcal{H}, \quad \text{P-a.s.,} \quad h(x, t) \text{ is } \mathcal{F}_t \text{ measurable.}$$

For any $N > 0$, define

$$\mathcal{A}_N := \{h \in \mathcal{A}, |h|_{\mathcal{H}} \leq N, \text{P-a.s.}\}.$$

For $h \in \mathcal{A}_N$, consider the following reflected SPDE:

$$\begin{cases}
\frac{\partial u^{\varepsilon, h}(x, t)}{\partial t} - \frac{\partial^2 u^{\varepsilon, h}(x, t)}{\partial x^2} + f(x, t; u^{\varepsilon, h}) \\
= \varepsilon \sigma(x, t; u^{\varepsilon, h}) \dot{W}(x, t) + \sigma(x, t; u^{\varepsilon, h}) \dot{h}(x, t) + \eta^{\varepsilon, h}; \\
u^{\varepsilon, h}(\cdot, 0) = u_0; \\
u^{\varepsilon, h}(0, t) = u^{\varepsilon, h}(1, t) = 0.
\end{cases} \tag{3.17}$$

The above equation admits a unique solution. This can be shown in the same way as Theorem 2.1. Recall the following from Theorem 4.4 in [1].

**Theorem 3.3** ([1]). Let $\Pi$ be a Polish space and for $\varepsilon > 0$, $\Gamma^\varepsilon$ be a measurable mapping from $C([0, 1] \times [0, T])$ into $\Pi$. Let $X^\varepsilon := \Gamma^\varepsilon(\mathcal{W}(\cdot, \cdot))$. Suppose that $\{\Gamma^\varepsilon\}_{\varepsilon > 0}$ satisfies the following assumptions: there exists a measurable map $\Gamma^0 : C([0, 1] \times [0, T]) \to \Pi$ such that

(I) For any family $\{h^\varepsilon\} \subset \mathcal{A}_N (N < +\infty)$ satisfying that $h^\varepsilon$ converges in distribution (as $B_N$-valued random elements) to $h$, $\Gamma^\varepsilon(\mathcal{W}(\cdot, \cdot) + \frac{1}{\varepsilon} \int_0^T \int_0^1 h^\varepsilon(x, s)dxds)$ converges in distribution to $\Gamma^0(\int_0^T \int_0^1 \tilde{h}(x, s)dxds)$.

(II) For every $N < +\infty$, the set

$$\Sigma_N := \left\{ \Gamma^0 \left( \int_0^T \int_0^1 \tilde{h}(x, s)dxds \right) : h \in B_N \right\}$$

is a compact subset of $\Pi$.

Then the family $\{X^\varepsilon\}_{\varepsilon > 0}$ satisfies a large deviation principle with the rate function $\bar{I}$ given by

$$\bar{I}(f) := \inf_{\{h \in \mathcal{H}, f = \Gamma^0(\int_0^T \int_0^1 \tilde{h}(x, s)dxds)\}} \left\{ \frac{1}{2} \int_0^T \int_0^1 \tilde{h}^2(x, s)dxds \right\}, \quad f \in \Pi$$

with the convention $\inf\{\emptyset\} = \infty$. 
Recall that \( u^\varepsilon \) denotes the solution of Eq. (1.4). The aim of this section is to prove the following result.

**Theorem 3.4.** The laws \( \{ \mu^\varepsilon \}_{\varepsilon > 0} \) of \( \{ u^\varepsilon (\cdot, \cdot) \}_{\varepsilon > 0} \) satisfy a large deviation principle on \( C_+([0, 1] \times [0, T]) \) with the rate function \( I(\cdot) \), i.e.,

(i) For any closed subset \( C \subseteq C_+([0, 1] \times [0, T]) \),

\[
\limsup_{\varepsilon \to 0} \varepsilon^2 \log \mu^\varepsilon (C) \leq - \inf_{f \in C} I(f).
\]

(ii) For any open set \( G \subseteq C_+([0, 1] \times [0, T]) \),

\[
\liminf_{\varepsilon \to 0} \varepsilon^2 \log \mu^\varepsilon (G) \geq - \inf_{f \in G} I(f).
\]

**Proof.** By Theorems 2.1 and 3.1, there exist measurable mappings \( I^\varepsilon \) and \( I^0 \) from \( C([0, 1] \times [0, T]) \) to \( C_+([0, 1] \times [0, T]) \) such that

\[
u^\varepsilon = I^\varepsilon (W(\cdot, \cdot)), \quad u^h = I^0 \left( \int_0^T \int_0^T \hat{h}(x, s)dxds \right).
\]

It is easy to see that \( u^{\varepsilon, h^\varepsilon} = I^\varepsilon (W(\cdot, \cdot) + \frac{1}{\varepsilon} \int_0^T \int_0^T \hat{h}^\varepsilon (x, s)dxds) \) is the solution of Eq. (3.17) with \( h = h^\varepsilon \). By Theorems 3.2 and 3.3, it is sufficient to prove that, for a family \( \{ h^\varepsilon \}_{\varepsilon > 0} \subseteq \mathcal{A}_N \) \((N < +\infty)\) that converges in distribution (as \( B_N \)-valued random elements) to \( h \), \( u^{\varepsilon, h^\varepsilon} \) converges in distribution to \( u^h \), where \( u^h \) is the solution of Eq. (3.1) associated with the random element \( h \in \mathcal{A} \).

By Skorohod’s embedding theorem, there exists a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) carrying a sequence \((\tilde{h}^\varepsilon, \tilde{W}^\varepsilon (\cdot, \cdot))\) of random fields such that the distributions of

\[(\tilde{h}^\varepsilon, \tilde{W}^\varepsilon) \quad \text{and} \quad (h^\varepsilon, W)\]

coincide, and for \( \tilde{P} \) almost every \( \tilde{\omega} \in \tilde{\Omega} \),

\[
\tilde{h}^\varepsilon \overset{w}{\to} h, \quad \sup_{x \in [0, 1], t \in [0, T]} |\tilde{W}^\varepsilon (x, t) - W(x, t)| \to 0,
\]

as \( \varepsilon \to 0 \). Clearly, \((\tilde{h}, \tilde{W})\) has the same law as \((h, W)\). Let \( \tilde{u}^{\varepsilon, h^\varepsilon} \) be the solution of the following reflected SPDE

\[
\begin{aligned}
\frac{\partial \tilde{u}^{\varepsilon, h^\varepsilon} (x, t)}{\partial t} - \frac{\partial^2 \tilde{u}^{\varepsilon, h^\varepsilon} (x, t)}{\partial x^2} + f(x, t; \tilde{u}^{\varepsilon, h^\varepsilon}) \\
= \varepsilon \sigma(x, t; \tilde{u}^{\varepsilon, h^\varepsilon}) \tilde{W} (x, t) + \sigma(x, t; \tilde{u}^{\varepsilon, h^\varepsilon}) \tilde{h}^\varepsilon (x, t) + \tilde{\eta}^\varepsilon; \quad \tilde{u}^{\varepsilon, h^\varepsilon} (x, 0) = u_0; \quad \tilde{u}^{\varepsilon, h^\varepsilon} (0, t) = \tilde{u}^{\varepsilon, h^\varepsilon} (1, t) = 0.
\end{aligned}
\]

Then, \( \tilde{u}^{\varepsilon, h^\varepsilon} \) has the same distribution as \( u^{\varepsilon, h^\varepsilon} \). Similarly, the distributions of \( \tilde{u}^h \) and \( u^h \) coincide, too, where \((\tilde{u}^h, \tilde{\eta})\) is the solution of Eq. (3.1) associated with \( \tilde{h} \). To show that \( u^{\varepsilon, h^\varepsilon} \) converge to \( u^h \) in distribution, it is sufficient to prove that \( \tilde{u}^{\varepsilon, h^\varepsilon} \) converge to \( \tilde{u}^h \) in probability w.r.t. \( \tilde{P} \). Write \( \tilde{u}^\varepsilon \) for \( \tilde{u}^{\varepsilon, h^\varepsilon} \) for simplicity. Let

\[
v^{\varepsilon, h^\varepsilon} (x, t) = \int_0^1 G_t(x, y)u_0(y)dy - \int_0^t \int_0^1 G_{t-s}(x, y)f(y, s; \tilde{u}^\varepsilon)dyds
\]
\[ + \varepsilon \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; \bar{u}^\varepsilon) \dot{W}^\varepsilon(dy, ds) \]
\[ + \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; \bar{u}^\varepsilon) \dot{h}(y, s)dyds, \]
and
\[ v^\bar{h}(x, t) = \int_0^1 G_t(x, y) u_0(y)dy - \int_0^t \int_0^1 G_{t-s}(x, y) f(y, s; \bar{u}^\bar{h})dyds \]
\[ + \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; \bar{u}^\bar{h}) \dot{\bar{h}}(y, s)dyds. \]

Then \((\bar{z}^\varepsilon = \bar{u}^\varepsilon - v^\bar{h}, \bar{\eta})\) and \((z^\bar{h} = \bar{u}^\bar{h} - v^\bar{h}, \eta)\) are the solutions of Eq. (2.1) with \(v = v^\bar{h}\) and \(v = v^\bar{h}\) respectively. By (2.5) again, we have
\[ |\bar{u}^\varepsilon - \bar{u}^\bar{h}|_T \leq 2|v^\bar{h} - v^\bar{h}|_T \]
\[ \leq 2 \sup_{x \in [0,1], t \in [0, T]} \left| \int_0^1 \int_0^1 G_{t-s}(x, y) (f(y, s; \bar{u}^\varepsilon) - f(y, s; \bar{u}^\bar{h}))dyds \right| \]
\[ + 2 \varepsilon \sup_{x \in [0,1], t \in [0, T]} \left| \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; \bar{u}^\varepsilon) \dot{W}^\varepsilon(dy, ds) \right| \]
\[ + 2 \sup_{x \in [0,1], t \in [0, T]} \left| \int_0^t \int_0^1 G_{t-s}(x, y) (\sigma(y, s; \bar{u}^\varepsilon) - \sigma(y, s; \bar{u}^\bar{h})) \dot{\bar{h}}(y, s)dyds \right| \]
\[ + 2 \sup_{x \in [0,1], t \in [0, T]} \left| \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; \bar{u}^\bar{h}) (\dot{\bar{h}}(y, s) - \dot{\bar{h}}(y, s))dyds \right| \]
\[ =: I_1 + I_2 + I_3 + I_4. \]

Using (1.5), one obtains
\[ I_1 + I_3 \leq 2K \left( \sup_{x \in [0,1], t \in [0, T]} \int_0^t \int_0^1 G_s^2(x, y)dyds \right)^{\frac{1}{2}} \left( \int_0^T (|\bar{u}^\varepsilon - \bar{u}^\bar{h}|_T^2)^{\frac{1}{2}} dt \right) \]
\[ + 2K \left( \sup_{x \in [0,1], t \in [0, T]} \int_0^t \int_0^1 G_s^2(x, y)dyds \right)^{\frac{1}{2}} \]
\[ \times \left( \int_0^T (|\bar{u}^\varepsilon - \bar{u}^\bar{h}|_T^2)^2 \int_0^1 (\dot{\bar{h}}(y, t))^2dydt \right)^{\frac{1}{2}}. \]

Using Gronwall’s inequality, we get from (3.20) that
\[ (|\bar{u}^\varepsilon - \bar{u}^\bar{h}|_T^2)^2 \leq c(I_2^2 + I_4^2) e^{8K^2 \left( \sup_{x \in [0,1], t \in [0, T]} \int_0^1 G_s^2(x, y)dyds \right)(1+N)}. \]

For \(I_2\), by (1.3) and the proof of Corollary 3.4 of [12] or Theorem 3.1 in [5], we deduce that
\[ \bar{E} \left| \int_0^1 \int_0^1 G_{t-r}(x, z) \sigma(z, r; \bar{u}^\varepsilon) \dot{W}^\varepsilon(dz, dr) \right| \]
\begin{equation}
- \int_0^s \int_0^1 G_{s-r}(y, z) \sigma(z, r; \bar{\mu}^\varepsilon) \bar{W}^\varepsilon (dz, dr) \right|_p^p \leq c_T \left( 1 + \bar{E} \int_0^{(t+\varepsilon)} (|\bar{\mu}^\varepsilon|_\infty^p) dr \right) \times |(x, t) - (y, s)|^{p-4},
\end{equation}

where \( p > 6 \) and \( \bar{E} \) denotes the expectation with respective to \( \tilde{P} \). Choosing \( p > 20 \) and applying Lemma 3.1 in [5], we get that there exists a random variable \( K(\omega)_{\varepsilon, p} \) such that

\begin{equation}
\left| \int_0^t \int_0^1 G_{t-r}(x, z) \sigma(z, r; \bar{\mu}^\varepsilon) \bar{W}^\varepsilon (dz, dr) - \int_0^s \int_0^1 G_{s-r}(y, z) \sigma(z, r; \bar{\mu}^\varepsilon) \bar{W}^\varepsilon (dz, dr) \right|_p^p \leq K(\omega)_{\varepsilon, p} |(x, t) - (y, s)|^{p-4} \log \left( \frac{y}{|x, t - (y, s)|} \right)^2,
\end{equation}

and

\begin{equation}
\bar{E}[K_{\varepsilon, p}^p] \leq ac_T \left( 1 + \bar{E} \int_0^{(t+\varepsilon)} (|\bar{\mu}^\varepsilon|_\infty^p) dr \right),
\end{equation}

where \( a, c_T \) are constants depending on \( p \) and \( T \). Choosing \( s = 0 \) in (3.24) and using (2.16), one obtains that there exists a constant \( c_T \) such that

\begin{equation}
\bar{E} \left( \sup_{x \in [0, 1], r \in [0, T]} \left| \int_0^t \int_0^1 G_{t-s}(x, y) \sigma(y, s; \bar{\mu}^\varepsilon) \bar{W}^\varepsilon (dy, dx) \right|_p^p \right) \leq c_T \left( 1 + \bar{E} \int_0^T (|\bar{\mu}^\varepsilon|_\infty^p) dr \right)
< +\infty,
\end{equation}

thus, \( I_2 \xrightarrow{\tilde{P}} 0 \) as \( \varepsilon \to 0 \). Following the proof for (3.15), we also have \( I_4 \to 0, \tilde{P}-a.s \) (\( \varepsilon \to 0 \)). In view of (3.22), we finally have

\begin{equation}
|\bar{u}^\varepsilon^\tilde{P}|_\infty^T \xrightarrow{\tilde{P}} 0, \quad \text{as } \varepsilon \to 0. \quad \square
\end{equation}

References