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The Nonexistence of Certain Type of Finite Simple Group

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In this paper we study certain case which is left open in the recent work of Timmesfeld [3]. This is necessary for the completion of the "extraspecial problem." More precisely, we prove the following

THEOREM. *Let G be a finite group which possesses an involution z such that the centralizer M of z in G satisfies the following conditions:*

(i) *The subgroup $Q = O_2(M)$ is an extraspecial group of order 2^9 and* $C_M(Q) \subseteq Q$.

(ii) $M/Q \simeq \Sigma_3 \times A_6$ *or* $\Sigma_3 \times \Sigma_6$.

Then G is not simple.

According to [3] we set $\tilde{M} = M/\langle z\rangle$ and $\bar{M} = M/Q$ and we use the "bar convention." Further we write $C(X) = C_G(X)$ and $N(X) = N_G(X)$ for any subset X of G. The other notation is fairly standard.

1. SOME PROPERTIES OF M

From now on we assume that G is a finite simple group. We first prove some properties of M which shall be used in the proof of the theorem.

By [1] we can assume that z is conjugate in G to some involution a in $Q - \langle z\rangle$. Let $L = Q(Q_a \cap M)$ where $Q_a = O_2(C(a))$. Then by (3.11) [3], $\bar{L}$ is of order 2^3 and by (7.6) [3], $\langle \bar{L}_{\bar{M}}\rangle \simeq \Sigma_3 \times A_6$. Set $\bar{M}_0 = \langle \bar{L}_{\bar{M}}\rangle$. Let M_0 be the inverse image of $\bar{M}_0$ in M, then $|M : M_0| \leqslant 2$. Put $\bar{M}_0 = \bar{F}_0 \times \bar{F}_1$ where $\bar{F}_0 \simeq \Sigma_3$ and $\bar{F}_1 \simeq A_6$. Let x_0 be an element of order 3 of M_0 such that $\bar{x}_0 \in \bar{F}_0$, then $N_M(\langle x_0\rangle)$ covers $\bar{M}$. By [2], $\bar{M}$ acts irreducibly on $\tilde{Q}$, thus $N_M(\langle x_0\rangle) \cap Q = \langle z\rangle$. Set $X = N_M(\langle x_0\rangle) \cap M_0$, then $\tilde{X} = X/\langle z\rangle \simeq \Sigma_3 \times A_6$. Put $\tilde{X} = \tilde{X}_0 \times \tilde{X}_1$

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with $\tilde{X}_0 \simeq \Sigma_3$ and $\tilde{X}_1 \simeq A_6$. Let X_0 and X_1 (resp.) be the inverse image of $\tilde{X}_0$ and $\tilde{X}_1$ in X. We have $X_0 \cap X_1 = \langle z \rangle$. Let $P_0 = \langle \rho_0 \rangle$ be the S_3-subgroup of X_0 and P_1 be an S_3-subgroup of X_1. Then P_1 is elementary abelian of order 9. Let τ_0 be a 2-element of $X_0 - \langle z \rangle$. Then $\rho_0^{\tau_0} = \rho_0^{-1}$ and $\rho_0^{\tau_0} = \rho$ for all $\rho \in P_1$. We now prove that if $C_Q(\rho) \neq \langle z \rangle$ for some $\rho \in P_1^{\#}$, then $C_Q(\rho) \simeq Q_8 \times Q_8$. Suppose that $C_Q(\rho) \neq Q_8 \times Q_8$, then either $B = [Q, \langle \rho \rangle] \simeq Q_8 \times Q_8 \times Q_8$ or $B \simeq Q_8$. If $B \simeq Q_8 \times Q_8 \times Q_8$, then $B = C_B(\rho_0\rho)$. $C_B(\rho_0^{-1}\rho)$ as $(\rho_0\rho)^{\tau_0} = \rho_0^{-1}\rho$ and ρ_0, ρ act fixed-point-free on B, thus $C_B(\rho_0\rho) \cap C_B(\rho_0^{-1}\rho) \neq \langle z \rangle$, contradicting $C_{\tilde{Q}}(\tilde{\rho}_0) = 1$. If $B \simeq Q_8$, then $\tilde{\tau}_0$ centralizes $\tilde{B}$ as $[\tau_0, \rho] = 1$. From the structure of X_1 there exists an element ρ^* of P_1 of order 3 such that $P_1 = \langle \rho, \rho^* \rangle$ and $\rho \sim_{X_1} \rho^*$, then $B^* = [Q, \langle \rho^* \rangle] \simeq Q_8$ and $B \cap B^* = \langle z \rangle$. Further $C_Q(\rho) \cap C_Q(\rho^*) = 2^5$ and $\tilde{Q} = \widetilde{C_Q(\rho) \cap C_Q(\rho^*)} \times \tilde{B} \times \tilde{B}^*$. Now $\tilde{\tau}_0$ centralizes $\tilde{B}$ and $\tilde{B}^*$ and $\widetilde{C_Q(\rho) \cap C_Q(\rho^*)}$ is $\tilde{\tau}_0$-invariant, thus $|C_{\tilde{Q}}(\tilde{\tau}_0)| \geqslant 2^5$ which contradicts $|C_{\tilde{Q}}(\tilde{\tau}_0)| \leqslant 2^4$ as $\tilde{\tau}_0$ inverts $\tilde{\rho}_0$ and $\tilde{\rho}_0$ acts fixed-point-free on $\tilde{Q}$. Hence we have shown $C_Q(\rho) \simeq Q_8 \times Q_8$ if $C_Q(\rho) \neq \langle z \rangle$ for $\rho \in P_1^{\#}$. Suppose that ρ^* does not centralize $B = [Q, \langle \rho \rangle] \simeq Q_8 \times Q_8$, then $C_B(\rho^*) \simeq Q_8$ as $C_Q(\rho^*) \simeq Q_8 \times Q_8$, thus P_1 acts faithfully on B. On the other hand τ_0 centralizes P_1, so $\tilde{\tau}_0$ centralizes $\widetilde{C_B(\rho^*)}$ and $\widetilde{[B, \langle \rho^* \rangle]}$, hence $\tilde{\tau}_0$ centralizes $\tilde{B}$, thus $|C_{\tilde{Q}}(\tilde{\tau}_0)| \geqslant 2^5$, a contradiction. Therefore ρ^* centralizes B and $C_Q(\rho) = [Q, \langle \rho^* \rangle] \simeq Q_8 \times Q_8$ and $[Q, \langle \rho \rangle] = C_Q(\rho^*) \sim Q_8 \times Q_8$ and $\tilde{\rho}\tilde{\rho}^*$ acts fixed-point-free on $\tilde{Q}$. Acting with $\langle \rho_0, \rho\rho^* \rangle$ on Q we see that $Q = C_Q(\rho_0\rho\rho^*)$. $C_Q(\rho_0^{-1}\rho\rho^*)$ as $(\rho_0\rho\rho^*)^{\tau_0} = (\rho_0^{-1}\rho\rho^*)$. Moreover $C_Q(\rho_0\rho\rho^*) \simeq C_Q(\rho_0^{-1}\rho\rho^*) \simeq Q_8 \times Q_8$. This implies that $\widetilde{C_Q(\tau_0)}$ is elementary abelian of order 2^5 and $\widetilde{C_Q(\tau_0)} = C_{\tilde{Q}}(\tilde{\tau}_0)$. Now $C_{Q_a}(z) = \langle z \rangle \times K$ where $K \simeq D_8 \times D_8 \times D_8$ and $Q_a \cap Q = E$ is elementary abelian of order 2^5, so $K \cap Q = K \cap E$ is of order 2^4. By the structure of K there exists an elementary abelian subgroup D of order 2^3 such that $K = (K \cap E) \cdot D$, so $Q \cap D = 1$ and $L = Q \cdot D$. This implies that the coset $Q\tau_0$ contains involutions, hence τ_0 is an involution as $C_Q(\tau_0) \simeq E_{2^5}$. Let $\tilde{\tau}_1$ be an involution in $\tilde{X}_1$ and τ_1 be an inverse image of $\tilde{\tau}_1$ in X_1. From the structure of X_1 we can assume that τ_1 inverts $\rho\rho^*$. So $(\rho_0\rho\rho^*)^{\tau_1} = \rho_0\rho^{-1}\rho^{*-1}$. By the same argument as before we conclude that $C_Q(\tau_1) \simeq E_{2^5}$ and $\widetilde{C_Q(\tau_1)} = C_{\tilde{Q}}(\tilde{\tau}_1)$. Hence τ_1 is an involution. Therefore we have prove that $X_0 = \langle z \rangle \times F_0$ and $X_1 = \langle z \rangle \times F_1$ with $F_0 \simeq \Sigma_3$ and $F_1 \simeq A_6$. In other words the group M_0 splits over Q. We have proved the following

PROPOSITION 1. *$M_0 = Q \cdot (F_0 \times F_1)$ with $F_0 \simeq \Sigma_3$ and $F_1 \simeq A_6$. Let $P_0 = \langle \rho_0 \rangle$ be the S_3-subgroup of F_0. Then $\tilde{P}_0$ acts fixed-point-free on $\tilde{Q}$. There exists two elements ρ_1 and ρ_2 such that $P_1 = \langle \rho_1, \rho_2 \rangle$ is an S_3-subgroup of F_1 and $\tilde{\rho}_1$ acts fixed-point-free on $\tilde{Q}$ and $C_Q(\rho_2) \simeq Q_8 * Q_8$. Let τ_0 be an involution in F_0. Then $\widetilde{C_Q(\tau_0)} = C_{\tilde{Q}}(\tilde{\tau}_0)$ and $C_Q(\tau_0)$ is elementary abelian of order 2^5. Let τ_1 be an* *involution in* F_1. *Then* $\widetilde{C_Q(\tau_1)} = C_{\tilde{Q}}(\tilde{\tau}_1)$ *and* $C_Q(\tau_1)$ *is elementary abelian of order* 2^5.

Note that there exists an involution τ_1 in F_1 which acts invertingly on P_1. Set $P = P_0 \times P_1 = \langle \rho_0, \rho_1, \rho_2 \rangle$. Then $P/\langle \rho_2 \rangle$ acts faithfully on $C_Q(\rho_2) \simeq Q_8 * Q_8$ and $Q \cdot P$ acts transitively on 18 non-central involutions of $C_Q(\rho_2)$. Acting with $\langle \rho_0, \rho_1 \rangle$ on Q, we have $Q = C_Q(\rho_0\rho_1) \cdot C_Q(\rho_0\rho_1^{-1})$ as $(\rho_0\rho_1)^{\tau_1} = \rho_0\rho_1^{-1}$ and $\tilde{\rho}_0, \tilde{\rho}_1$ act fixed-point-free on $\tilde{Q}$. Further $C_Q(\rho_0\rho_1) = [Q, \langle \rho_0\rho_1^{-1} \rangle] \simeq Q_8 * Q_8$ and $C_Q(\rho\rho_1^{-1}) = [Q, \langle \rho_0\rho_1 \rangle] \simeq Q_8 * Q_8$ and $P/\langle \rho_0\rho_1 \rangle$ acts faithfully on $C_Q(\rho_0\rho_1)$ and $Q \cdot P$ acts transitively on 18 non-central involutions of $C_Q(\rho_0\rho_1)$. We now consider the action of $\langle \rho_0, \rho_2 \rangle$ on Q. Since $(\rho_0\rho_2)^{\tau_1} = \rho_0\rho_2^{-1}$ and $\tilde{\rho}_0$ acts fixed-point-free on $\tilde{Q}$, we see that $\widetilde{\rho_0\rho_2}$ and $\widetilde{\rho_0\rho_2^{-1}}$ act fixed-point-free on $\widetilde{C_Q(\rho_2)}$. As $\tilde{\rho}_0$ and $\tilde{\rho}_2$ act fixed-point-free on $\widetilde{[Q, \langle \rho_2 \rangle]}$ we have $C_{[Q,\langle \delta_2 \rangle]}(\rho_0\rho_2) \simeq C_{[Q,\langle \delta_2 \rangle]}(\rho_0\rho_2^{-1}) \simeq Q_8$. So $C_Q(\rho_0\rho_2) \simeq C_Q(\rho_0\rho_2^{-1}) \simeq Q_8$. Therefore we have proved that $9 \nmid |C_M(t)|$ for all involution $t \in Q - \langle z \rangle$. Let R be an S_5-subgroup of F_1. Then $\tilde{R}$ acts fixed-point-free on $\tilde{Q}$ as $[P_0, R] = 1$. This implies that $3 \mid |C_M(t)|$. Now the group M contains two conjugate classes of subgroups of order 3 which centralize some non-central involution of Q with the representatives $\langle \rho_2 \rangle$ and $\langle \rho_0\rho_1 \rangle$. Hence M has 2 conjugate classes of involutions which are contained in $Q - \langle z \rangle$ with the representatives t_1 and t_2 where t_1 is centralized by $\langle \rho_2 \rangle$ and t_2 by $\langle \rho_0\rho_1 \rangle$. We have $|M : C_M(t_i)| = 2^{a_i} \cdot 3^2 \cdot 5$, $a_i = 1$ or 2, $i = 1, 2$. The both cases $a_i = 1$ and $a_i = 2$ occur. It is easy to see that $|C_M(t_1)/C_Q(t_1)| = 2^4 \cdot 3$ and $|C_M(t_2)/C_Q(t_2)| = 2^3 \cdot 3$. Hence $|M : C_M(t_1)| = 2 \cdot 3^2 \cdot 5 = 90$ and $|M : C_M(t_2)| = 2^2 \cdot 3^2 \cdot 5 = 180$. It follows from (3.11) [3] that $t_1 \sim_G z$. By (3.13) [3] $t_2 \not\sim_G z$. We have proved the following

PROPOSITION 2. *The group* M *possesses precisely two conjugate classes of involutions which are contained in* $Q - \langle z \rangle$ *with the representatives* t_1 *and* t_2. *We have* $|t_1^M| = 2 \cdot 3^2 \cdot 5 = 90$ *and* $|t_2^M| = 2^2 \cdot 3^2 \cdot 5 = 180$. Moreover $r_1 \sim_G z$ *and* $t_2 \not\sim_G z$.

Let T be an S_2-subgroup of M containing τ_0. By proposition 1 $C_Q(\tau_0)$ is an elementary abelian group of order 2^5. We set $A = C_Q(\tau_0) \times \langle \tau_0 \rangle$. Then A is a maximal elementary abelian subgroup of order 2^6 of T and A is self-centralizing in M. It is easy to see that $N_M(A)/A$ is isomorphic to an elementary abelian group of order 2^4 extended by the group W where $W \simeq A_6$ or Σ_6 according as $M = M_0$ or $M \neq M_0$. Suppose that $A \cap Q = C_Q(\tau_0)$ contains an involution v which is conjugate in M to t_2. Then $|C_{N_M(A)}(v)/C_Q(v)| = 2^3 \cdot 3$ and hence $|N_M(A) : C_{N_M(A)}(v)| = 2^2 \cdot 3 \cdot 5 = 60$. This means that $A \cap Q$ contains 60 conjugates of v, but this is not the case. Hence $(A \cap Q) - \langle z \rangle$ contains only the involutions which are conjugate in M to t_1. Let $t \in (A \cap Q) - \langle z \rangle$ then $|N_M(A) : C_{N_M(A)}(t)| = 2 \cdot 3 \cdot 5 = 30$. Thus the group $N_M(A)$ contains exactly two conjugate classes of involutions which are contained in $A \cap Q$ with the representatives z (1 conjugate) and t (30 conjugates). Note that the coset $\tau_0 Q$ contains precisely 2^5 involutions which are of form $\tau_0 u$ with $u \in C_Q(\tau_0)$ and τ_0 is not conjugate to $\tau_0 z$ under the action of Q. Hence $\tau_0 Q$ contains under the action of Q precisely 2^4 conjugates of τ_0 and 2^4 conjugates of $\tau_0 z$. Let $x \in Q_{ax} - M$, then $z^x = a$. Since $A \subseteq L = Q \cdot (Q_a \cap M)$, so $A^x \subseteq L^x = Q_a \cdot (Q \cap C(a))$.

Further as $A^x \subseteq M$, so $A^x \subseteq L$. Thus $|A^x \cap Q| = 2^5$ and so $A^x \cap (L \cap M_0) = \langle \tau \rangle$, hence we can choose $y \in M_0$ such that $\tau^y \in Z(T \cap M_0)$, it follows that $A^{xy} \leq T$. Since T is self-normalizing in G, so $A = A^{xy}$. Thus $N(A) \not\subseteq M$. We have proved.

PROPOSITION 3. *Let T be an S_2-subgroup of M containing τ_0. Then $A = C_Q(\tau_0) \times \langle \tau_0 \rangle$ is a maximal elementary abelian subgroup of T, A is self-centralizing in G and $A \leq T$. The group $N_M(A)/A$ is isomorphic to an elementary abelian group of order 2^4 extended by the group W where $W \simeq A_6$ or Σ_6, according as $M = M_0$ or $M \neq M_0$. $A \cap Q$ contains under the action of $N_M(A)/A$ two conjugate classes of involutions with the representatives z (1 conjugate) and t (30 conjugates) and $t \sim_M t_1$. $A - (A \cap Q)$ has under the action of Q two conjugate classes of involutions with the representatives τ_0 (16 conjugates) and $\tau_0 z$ (16 conjugates). Furthermore $N(A) \not\subseteq M$.*

2. PROOF OF THEOREM

By proposition 3 we have $N(A) \not\subseteq M$. Since $N(A)/A$ is isomorphic to a subgroup of $GL(6, 2)$, so $|N(A) : N_M(A)| = 31$ or 63, the length of the orbit of z in $N(A)$.

Suppose that $|N(A) : N_M(A)| = 31$. Then $|N(A)/A| = 2^7 \cdot 3^2 \cdot 5 \cdot 31$ or $2^8 \cdot 3^2 \cdot 5 \cdot 31$ according as $M = M_0$ or $M \neq M_0$. Let X_{31} be an S_{31}-subgroup of $N(A)$. Then by a Sylow's theorem we have $9 \mid |N(X_{31}) \cap N(A)|$. Hence there is a subgroup R of order 3 in $N(A)$ which centralizes X_{31}. But $[A, X_{31}]$ is of order 2^5 and $[A, X_{31}]$ is R-invariant, a contradiction. We have proved that $|N(A) : N_M(A)| = 63 = 3^2 \cdot 7$. Thus z is fused in $N(A)$ to other 62 involutions of A. Now $|N(A)/A| = 2^7 \cdot 3^4 \cdot 5 \cdot 7$ or $2^8 \cdot 3^4 \cdot 5 \cdot 7$ according as $M = M_0$ or $M \neq M_0$. Let X_7 be an S_7-subgroup of $N(A)$. Then X_7 acts fixed-point-free on A. In order to simplify the notation we set $N = N(A)$, $\bar{N} = N(A)/A$ and we use the "bar convention." First of all we see that an S_3-subgroup W_3 of $C_N(X_7)$ acts fixed-point-free on A, hence W_3 is cyclic of order 9. Note that $5 \nmid |N_N(X_7)|$. Let n_7 be the index in $\bar{N}$ of a 7-Sylow normalizer in $\bar{N}$. By a Sylow's theorem we have the following possibilities for n_7:

$$\begin{array}{ll} 5 \cdot 3 \cdot 2^6 & 5 \cdot 3^3 \cdot 2^8 \\ 5 \cdot 3 \cdot 2^3 & 5 \cdot 3^3 \cdot 2^5 \\ 5 \cdot 3 & 5 \cdot 3^3 \cdot 2^2 \end{array}$$

First we show that $n_7 \neq 5 \cdot 3$. Suppose that $n_7 = 5 \cdot 3$.

Let W_2 be an S_2-subgroup of $N_N(X_7)$ and $W_2^* = W_2 \cap C_N(X_7)$. Then $|W_2 : W_2^*| \leqslant 2$. If $W_2 = W_2^*$, then $Z(AW_2) \subseteq A$ is of order 2 and is X_7-invariant which is not the case. So $|W_2 : W_2^*| = 2$. Since $Z(AW_2^*) \subseteq A$ is X_7-invariant, so $|Z(AW_2^*)| = 2^3$, hence $|Z(AW_2)| \geqslant 2^2$ which contradicts the fact that AW_2 is an S_2-subgroup of G. Hence we have $n_7 \neq 5 \cdot 3$. Now we have $O_2(\bar{N}) = 1$, as $\bar{N}$ acts transitively on A. Further $7 \nmid |O(\bar{N})|$, otherwise an S_2-subgroup of $\bar{N}$ would normalize an S_7-subgroup of $\bar{N}$ which is not the case. Since $5 \nmid |N_{\bar{N}}(\bar{X}_7)|$, it follows that $5 \nmid |O(\bar{N})|$. Thus $O(\bar{N})$ is a 3-group if $O(\bar{N}) \neq 1$. As $|C_{\bar{N}}(\bar{X}_7)|_3 \leqslant 9$ and $7 \nmid |GL_3(3)|$, it follows that $\bar{X}_7$ centralizes $O(\bar{N})$ (here note that an S_3-subgroup $\bar{X}_3$ of $\bar{N}$ is an S_3-subgroup of $GL_6(2)$ and so $\bar{X}_3$ is isomorphic to an S_3-subgroup of $PSp_4(3)$ and $\bar{X}_3$ contains a unique elementary abelian subgroup of order 27). So $|O(\bar{N})| \leqslant 3^2$ and $O(\bar{N})$ is cyclic. Since an S_5-subgroup $\bar{X}_5$ of $\bar{N}$ centralizes $O(\bar{N})$ and $[A, \bar{X}_5] \simeq E_{2^4}$ and $O(\bar{N})$ acts fixed-point-free on A, it follows that $|O(\bar{N})| \leqslant 3$.

Suppose that $|O(\bar{N})| = 3$. Let $\bar{C} = C_{\bar{N}}(O(\bar{N}))$. Then $|\bar{N} : \bar{C}| = 2$, as $O(\bar{N})$ acts fixed-point-free on A. Let $\bar{\bar{C}} = \bar{C}/O(\bar{N})$ and $\bar{\bar{H}}$ be a minimal normal subgroup of $\bar{\bar{C}}$. Then $\bar{\bar{H}}$ is not solvable, otherwise $\bar{\bar{H}}$ is an elementary abelian 2-group and $C_A(\bar{\bar{H}}) \simeq E_{2^3}$, as $C_A(\bar{\bar{H}})$ is X_7-invariant. On the other hand $C_A(\bar{\bar{H}})$ is $O(\bar{N})$-invariant and $O(\bar{N})$ acts fixed-point-free on A which is impossible. So $\bar{\bar{H}}$ is a direct product of isomorphic nonabelian simple groups. Since $[\bar{\bar{X}}_7, \bar{\bar{X}}_5] \neq 1$, for any subgroup $\bar{\bar{X}}_5$ of order 5 of $\bar{\bar{C}}$, it follows that $5, 7 \mid |\bar{\bar{H}}|$. From the order of $\bar{\bar{C}}$ we conclude that $\bar{\bar{H}}$ is simple. Hence we have the following possibilities for $\bar{\bar{H}}$: $\bar{\bar{H}} \simeq A_7$, A_8 or $L_3(4)$. If $\bar{\bar{H}} \simeq A_7$ or A_8, then there exists an element $\bar{\bar{x}}$ of order 3 of $\bar{\bar{C}} - \bar{\bar{H}}$ which centralizes $\bar{\bar{H}}$, in particular $\bar{\bar{x}}$ centralizes $\bar{\bar{X}}_7$ and so an inverse image $\bar{x}$ of $\bar{\bar{x}}$ in $\bar{C}$ is of order 9 and $\langle \bar{x}^3 \rangle = O(\bar{N})$ and $\bar{x}$ acts fixed-point-free on A. Further $\bar{x}$ centralizes an S_5-subgroup $\bar{X}_5$ of $\bar{C}$ and $C_A(\bar{X}_5) \simeq E_{2^2}$ is $\bar{x}$-invariant, a contradiction. So we have $\bar{\bar{H}} \simeq L_3(4)$. Since $|N_{\bar{\bar{H}}}(\bar{\bar{X}}_7)|_3 = 3$, then $n_7 = 5 \cdot 3 \cdot 2^6$ or $5 \cdot 3 \cdot 2^3$. Hence it follows from the structure of $\mathrm{Aut}(L_3(4))$ that there exists an element $\bar{\bar{x}}$ in $\bar{\bar{C}} - \bar{\bar{H}}$ of order 3 which centralizes $\bar{\bar{H}}$. Again we have a contradiction in the same way as before. So we have $O(\bar{N}) = 1$. Let $\bar{H}$ be a minimal normal subgroup of $\bar{N}$. Then $\bar{H}$ is simple and we get the following possibilities for $\bar{H}$: $\bar{H} \simeq A_7$, A_8, A_9 or $L_3(4)$. If $\bar{H} \simeq A_9$, then $n_7 = 5 \cdot 3^3 \cdot 2^5$ and $N_{\bar{H}}(\bar{X}_7)$ contains a diherdral group $\bar{D}$ of order 14. On the other hand there exists an involution $\bar{x} \in \bar{N} - \bar{H}$ which centralizes $\bar{D}$, thus $|C_A(\bar{x})| = 8$ and $C_A(\bar{x})$ is $\bar{D}$-invariant, contradicting $C_A(\bar{X}_7) = 1$.

If $\bar{H} \simeq L_3(4)$, A_7, A_8, then in any case $C_{\bar{N}}(\bar{H})$ is a nontrivial solvable normal subgroup of $\bar{N}$, contradicting $O_2(\bar{N}) = O(\bar{N}) = 1$. The proof is complete.

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