Stability of Functional Differential Equations of Neutral Type

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1. INTRODUCTION

The definition in Section 2 of a functional differential equation of neutral type is general enough to include differential difference equations of the form

$$\frac{d}{dt} [x(t) + g(t, x(t - 1))] = f(t, x(t), x(t - 1)), \quad (1.1)$$

where \(f, g\) are continuous functions of their arguments. In contrast to retarded equations \((g = 0)\), the theory of stability of \((1.1)\) is still in its infancy with some partial results based on a variation of constants formula contained in [1], [3].

For ordinary and retarded functional differential equations, one powerful tool in stability theory is the second method of Liapunov. In this paper, we define a class of equations of neutral type for which it is possible to develop a theory of stability using Liapunov functions which is as comprehensive as the known theory of Liapunov functions for retarded equations and includes the principle of invariance of LaSalle [5] and Hale [4].

This class of equations is general enough to permit \(g\) and \(f\) to depend on all values of \(x\) on the interval \([t - 1, t]\), but the function \(g\) must be linear in \(x\) and the operator on \(x\) which is differentiated in \((1.1)\) must be stable (see Section 3). The characterization of stable operators in terms of specific properties of \(g\) seems to be extremely difficult. On the other hand, if \(g = 0\), the operator is stable so that the theory includes the retarded functional

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differential equations. Also, if \( g = \sum_{k=1}^{N} A_k x(t - \bar{k}) \) where the \( A_k \) are constant matrices, then the operator is stable if and only if the roots of the equation
\[
\det[I - A_k p^{-\bar{k}}] = 0
\]
have modulii less than 1.

2. Notation

Let \( R^n \) be a real or complex \( n \)-dimensional linear vector space with norm \( \| \cdot \| \). For \( r \geq 0 \), let \( C = C([[-r, 0], R^n) \) be the space of continuous functions taking \([-r, 0]\) into \( R^n \) with \( \| \varphi \|, \varphi \in C \), defined by \( \| \varphi \| = \sup_{-r \leq \theta \leq 0} \| \varphi(\theta) \| \).

Suppose \( \tau \) is a real number and \( g, f \) are continuous functions taking \([\tau, \infty) \times C \rightarrow R^n \) and define the functional difference operator
\[
D(\cdot) : [\tau, \infty) \times C \rightarrow R^n
\]
by
\[
D(t) \varphi = \varphi(0) - g(t, \varphi), \tag{2.1}
\]
for \( t \in [\tau, \infty), \varphi \in C \). A functional differential equation is a system of the form
\[
\frac{d}{dt} D(t) x_t = f(t, x_t), \tag{2.2}
\]
where \( x_t \in C \) is defined by \( x_t(\theta) = x(t + \theta), -r \leq \theta \leq 0 \).

For any \( \varphi \in C, \sigma \in [\tau, \infty) \), a function \( x = x(\sigma, \varphi) \) defined on \([\sigma - r, \sigma + A] \) is said to be a solution of (2.2) on \((\sigma, \sigma + A) \) with initial value \( \varphi \) at \( \sigma \) if \( x \) is continuous on \([\sigma - r, \sigma + A] \), \( x_\sigma = \varphi \), \( D(t)x_t \) is continuously differentiable on \((\sigma, \sigma + A) \) and relation (2.2) is satisfied on \((\sigma, \sigma + A) \).

The initial value problem
\[
x_\sigma = \varphi, \tag{2.3}
\]
\[
\frac{d}{dt} D(t) x_t = f(t, x_t), \quad t > \sigma,
\]
is equivalent to the integral equation
\[
D(t) x_t = D(\sigma) \varphi + \int_{\sigma}^{t} f(s, x_s) \, ds, \quad t \geq \sigma \tag{2.4}
\]

For \( g = 0 \), \( D(t) \varphi = \varphi(0) \) and equation (2.2) is the standard functional differential equation of retarded type. For \( f = 0 \), equation (2.2) is a type of functional difference equation. If (2.2) has a solution which has a smooth derivative and \( g \) has continuous derivatives in \( t, \varphi \), then the function \( x_t \) satisfies
\[
\dot{x}(t) - g_\varphi(t, x_t) \dot{x}_t - g'_t(t, x_t) = f(t, x_t).
\]
This is a functional differential equation in which the rate of change of the system at time $t$ depends not only upon the history of $[t - r, t]$ but the derivative of the history on $[t - r, t]$. As we see, the main restriction imposed by writing the equation in the form (2.2) is the restriction to equations which are linear in $\dot{x}$, together with some smoothness conditions on the coefficients.

System (2.2) is said to be a functional differential equation of neutral type if $g(t, \varphi)$ is non-atomic at zero. The precise definition of this property is not important to us in this paper, but the concept insures that the function $g(t, \varphi)$ does not depend very strongly upon $\varphi(0)$. In particular, if $g(t, \varphi)$ depends only upon values of $\varphi(\theta)$ for $-r \leq \theta \leq -\epsilon < 0$ then $g(t, \varphi)$ is non-atomic at zero. General existence, uniqueness and continuous dependence theorems have been given in [2] for system (2.3) under the hypothesis that $g(t, \varphi)$ is non-atomic at zero.

In this paper, it is always assumed that $g(t, \varphi)$ is linear in $\varphi$ and the functions $g, f$ satisfy enough additional properties to ensure the existence, uniqueness and continuous dependence of the solution $x(\sigma, \varphi)$ of (2.3) on the initial data. These properties of solutions will prevail if the function $g$ satisfies conditions (3.1) of the next section and the function $f(t, \varphi)$ is continuous in $t, \varphi$ and locally lipschitzian in $\varphi$.

### 3. Stable Functional Difference Operators

In this section, we study a class of functional difference operators which are very useful in stability theory and the asymptotic behavior of solutions of functional differential equations of neutral type. Suppose $\tau \in \mathbb{R}$,

$$ g : [\tau, \infty) \times C \rightarrow \mathbb{R}^n $$

is continuous, $g(t, \varphi)$ is linear in $\varphi$ and there are an $n \times n$ matrix $\mu(t, \theta)$, $t \in [\tau, \infty)$, $\theta \in [-r, 0]$, of bounded variation in $\theta$ and a scalar function $l(s)$ continuous nondecreasing for $s \in [0, r]$, $l(0) = 0$, such that

$$ g(t, \varphi) = \int_{-r}^{0} [d\mu(t, \theta)] \varphi(\theta), $$

$$ |\int_{-s}^{0} [d\mu(t, \theta)] \varphi(\theta)| \leq l(s) \sup_{-s \leq \theta \leq 0} |\varphi(\theta)|, $$

for all $t$ in $[\tau, \infty)$, $\varphi$ in $C$. For the linear operator $D(t)$ defined by (2.1) and for any $H \in C([\tau, \infty), \mathbb{R}^n)$, the space of continuous functions taking $[\tau, \infty)$ into $\mathbb{R}^n$, $\sigma \in [\tau, \infty)$, $\varphi \in C$, consider the equation

$$ D(t) x_t = D(\sigma) \varphi + H(t) - H(\sigma), \quad t \geq \sigma, $$

$$ x_\sigma = \varphi. $$
DEFINITION 3.1. Suppose \( \mathcal{H} \) is a subset of \( C([\tau, \infty), R^n) \). We say the operator \( D(t) \) is uniformly stable with respect to \( \mathcal{H} \) if there are constants \( K, M \) such that for any \( \varphi \in C, \sigma \in [\tau, \infty) \) and \( H \in \mathcal{H} \), the solution \( x(\sigma, \varphi, H) \) of (3.2) satisfies

\[
|x_t(\sigma, \varphi, H)| \leq K|\varphi| + M \sup_{\sigma \leq u \leq t} |H(u) - H(\sigma)|, \quad t \geq \sigma. \quad (3.3)
\]

A trivial example of an operator uniformly stable with respect to \( C([\tau, \infty), R^n) \) is given by \( D(t)\varphi = \varphi(0) - g(t, \varphi) \) where for some \( \delta, 0 < \delta < 1 \) and for some \( t_0 \in [\tau, \infty) \), \( |g(t, \varphi)| \leq (1 - \delta)|\varphi| \) for all \( \varphi \in C, t \geq t_0 \). When \( g \) is independent of \( t \), this condition implies the roots of the equation

\[
\det \left( I - \int_{\tau}^{t} [d\mu(\theta)] \rho^{\theta} \right) = 0
\]

have modulii \( \leq 1 - \gamma, \gamma > 0 \). If one only assumes that the roots of this equation have modulii \( \leq 1 - \gamma, \gamma > 0 \), then it is not known that \( D(t) \) is uniformly stable with respect to \( C([\tau, \infty), R^n) \). On the other hand, with further restrictions on \( g \), we can prove

**Lemma 3.1.** Suppose the \( A_k, k = 1, 2, \ldots, N \) are \( n \times n \) constant matrices, \( \tau_k, 0 \leq \tau_k \leq \tau \), are real numbers such that the ratios \( \tau_j/\tau_k \) are rational if \( N > 1 \). If

\[
D(\varphi) = \varphi(0) - \sum_{k=1}^{N} A_k \varphi(-\tau_k), \quad (3.4)
\]

and all roots of the equation

\[
\det \left[ I - \sum_{k=1}^{N} A_k \rho^{-\tau_k} \right] = 0 \quad (3.5)
\]

have modulii less than 1, then \( D \) is uniformly stable with respect to \( C([\tau, \infty), R^n) \).

**Proof.** By a change in the time scale in (3.2), we may assume that the \( \tau_k \) are integers, say \( \tau_k = k, k = 1, 2, \ldots, N \), and we may take \( \sigma = 0 \). Furthermore, the matrix \( I - \sum_{k=1}^{N} A_k \) is nonsingular so that the constant function

\[
\zeta = \left[ I - \sum_{k=1}^{N} A_k \right]^{-1} D\varphi,
\]

is well defined. If \( h(t) = H(t) - H(0), y_t = x_t - \zeta, \psi(\theta) = \varphi(0) - \zeta \) in (3.2), then

\[
Dy_t = h(t), \quad y_0 = \psi, \quad D\psi = 0.
\]
Letting $z = (z^{(1)}, ..., z^{(N)})$, $z^{(k+1)}(t) = y(t - k)$, $k = 0, 1, ..., N - 1$, this system can be written as

$$z(t) = Az(t - 1) + h^*(t), \quad t \geq 0, \quad z_0 = \Psi,$$

where $|\Psi| \leq L |\psi|$ for some constant $L$, $h^* = (h, 0, ..., 0, 0)$ and the eigenvalues of $A$ have modulii less than 1. If $k$ is the greatest integer $< t + \theta$, then

$$z(t + \theta) = A^{k+1} \Psi(t + \theta - k - 1) + h^*(t + \theta) + A^k h^*(t - 1) + \cdots + A h^*(t + \theta - k),$$

for $-1 < \theta < 0$. Therefore,

$$|z(t + \theta)| \leq |A^{k+1}| |\Psi| + [1 + |A| + \cdots + |A^k|] \sup_{0 \leq u \leq t} |h^*(u)|.$$

Also, as $k \to \infty$, $|A^k|^{1/k} \to \mu < 1$ since the eigenvalues of $A$ have modulii less than 1. This implies the series converge and proves the lemma.

An important special case of a uniformly stable operator is $D\varphi = \varphi(0)$; that is, the "difference" operator associated with retarded functional differential equations. This is obviously stable from Lemma 3.1 since the equation (3.5) has no roots.

Using the same proof as in Lemma 3.1, one can generalize the lemma to the case in which $A_k = A_k(t)$, $t \geq \tau$, provided there is a $\delta > 0$ such that the roots $\rho(t)$ of the equation

$$\det \left[ I - \sum_{k=1}^{N} A_k(t) \rho^{-\tau_k} \right] = 0,$$

satisfy $|\rho(t)| \leq 1 - \delta$ for $t \in [\tau, \infty)$.

If $\mathcal{H} = \{0\}$ and $D(t)$ is uniformly stable with respect to $\{0\}$, then relation (3.3) implies in particular that the solutions of the homogeneous functional difference equation

$$D(t) x_t = 0, \quad t \geq \sigma,$$

$$x_\sigma = \varphi, \quad D(\sigma)\varphi = 0,$$

are uniformly stable. If $\mathcal{H} = C([\tau, \infty), \mathbb{R}^n)$, much more is implied about the solutions of (3.6). A discussion of the properties of such stable operators is the subject of the next lemmas.

**Lemma 3.2.** If $D(t)$ is uniformly stable with respect to $C([\tau, \infty), \mathbb{R}^n)$, then
there are constants $\beta, \alpha > 0$ such that for every $\sigma \in [\tau, \infty)$, $\varphi \in C$, the solution $x(\sigma, \varphi)$ of (3.6) satisfies

$$| x_t(\sigma, \varphi) | \leq \beta e^{-\alpha(t-\sigma)} | \varphi |, \quad t \geq \sigma. $$

**Proof.** Suppose $D(t)x_t = 0$, $x_\tau = \varphi$, $l(s)$ is defined in (3.1) and $\alpha > 0$ is any positive constant chosen so that $2l(r)(e^{\alpha r} - 1) e^{\alpha r} < 1$. There is no loss in generality to take $M = K$ in (3.3). If

$$y_t(\theta) = e^{\alpha(t+\theta-\sigma)}x_t(\theta), \quad f(\theta) = e^{\alpha \varphi}(\theta), \quad -\tau \leq \theta \leq 0,$$

then $y_\sigma = \varphi$ and it is easy to see that

$$D(t)y_t = D(\sigma)y_t + h(t, y_t) - h(\sigma, \varphi),$$

$$h(t, y) = \int_{-\tau}^{0} [d\alpha(t, \theta)](e^{-\alpha \theta} - 1) y(\theta),$$

for every $y \in C$. The above choice of $\alpha$ implies $| h(t, y) | \leq | y | e^{-\alpha t}/K^2$. Since $| \varphi | \leq | \varphi |$ and $D(t)$ is a uniformly stable operator, $| x_t | \leq K | \varphi |$, relation (3.3) and the definition of $y_t$ imply

$$| y_t | \leq \left( K + \frac{1}{2} \right) | \varphi | + \frac{e^{-\alpha t}}{2K} \sup_{\sigma \leq u \leq t} | x_u |$$

$$\leq \left( K + \frac{1}{2} \right) | \varphi | + \frac{e^{-\alpha t}}{2K} e^{\alpha(t-\sigma)} \sup_{\sigma \leq u \leq t} | x_u |$$

$$\leq \left( K + \frac{1}{2} \right) | \varphi | + \frac{e^{-\alpha t}}{2} e^{\alpha(t-\sigma)} | \varphi |, \quad t \geq \sigma. \tag{3.7}$$

Since

$$| y_t | \geq e^{-\alpha t} e^{\alpha(t-\sigma)} | x_t |,$$

this latter inequality yields

$$| x_t | \leq \beta' e^{-\alpha(t-\sigma)} | \varphi | + \frac{1}{2} | \varphi |, \quad t \geq \sigma, \tag{3.8}$$

where $2\beta' = (2K + 1)e^{\alpha r}$.

Reapplying estimate (3.7) using estimate (3.8), we have

$$e^{\alpha r} | y_t | \leq \beta' | \varphi | + \frac{1}{2K} \sup_{\sigma \leq u \leq t} \left( \beta' | \varphi | + \frac{1}{2} e^{\alpha(u-\sigma)} | \varphi | \right)$$

$$\leq \beta' \left( 1 + \frac{1}{2K} \right) | \varphi | + \frac{1}{2K} e^{\alpha(t-\sigma)} | \varphi |.$$

Therefore

$$| x_t | \leq \beta' \left( 1 + \frac{1}{2K} \right) e^{-\alpha(t-\sigma)} | \varphi | + \frac{1}{2(2K)} | \varphi |.$$
A repetition of this process yields
\[ |x_t| \leq \beta \left( 1 + \frac{1}{2K} + \frac{1}{(2K)^2} + \cdots + \frac{1}{(2K)^n} \right) e^{-\alpha(t-u)} |\varphi| + \frac{1}{2(2K)^n} |\varphi|, \]
for all \( t \geq a \) and every positive integer \( n \). Since \( K > 1 \), this implies
\[ |x_t| \leq \frac{2K\beta}{2K - 1} e^{-\alpha(t-u)} |\varphi|, \quad t \geq a, \]
and proves the lemma.

**Lemma 3.3.** There is a positive constant \( N \) such that for any \( t \) in \([\tau, \infty)\), there are \( \varphi_j \in C, |\varphi_j| \leq N, j = 1, 2, \ldots, n \), such that \( \Phi = (\varphi_1, \ldots, \varphi_n) \) satisfies \( D(t)\Phi = I \), the identity matrix. In particular, \( D(t) \) maps \( C \) onto \( \mathbb{R}^n \) for each \( t \in [\tau, \infty) \).

**Proof.** The function \( \det \ A \) is a continuous mapping of the \( n \times n \) matrices \( A \in \mathbb{R}^{n \times n} \) into \( \mathbb{R} \). The fact that \( \det I = 1 \) implies the existence of a \( \epsilon > 0 \) such that \( \det A \neq 0 \) if \( |A - I| \leq \epsilon \). For any \( s \) in \([0, r] \), let the function \( \psi \in C([-\epsilon, 0], \mathbb{R}) \) be defined by
\[ \psi(\theta) = \begin{cases} 0, & -r \leq \theta \leq -s, \\ 1 + \frac{\theta}{s}, & -s \leq \theta \leq 0. \end{cases} \]
Then
\[ D(t) \psi I = I - \int_{-s}^{0} [d\psi(t, \theta)] \left( 1 + \frac{\theta}{s} \right). \]
From (3.1), there is an \( s > 0 \) such that \( |D(t)\psi I - I| \leq \epsilon \) for all \( t \in [\tau, \infty) \).
Thus, the matrix \( D(t)\psi I \) forms a basis in \( \mathbb{R}^n \) and each column of the matrix \( \psi I \) in \( C \) is bounded by a constant independent of \( t \in [\tau, \infty) \). A change of basis yields the conclusion of the lemma.

**Lemma 3.4.** If \( D(t) \) is a uniformly stable operator with respect to \( C([\tau, \infty), \mathbb{R}^n) \), then there are positive constants \( a, b, c, d \) such that for any \( h \in C([\tau, \infty), \mathbb{R}^n) \), \( \sigma \in [\tau, \infty) \), the solution \( x(\sigma, \varphi, h) \) of the equation
\[ D(t)x_t = h(t), \quad t \geq \sigma, \quad x_\sigma = \varphi, \quad (3.9) \]
satisfies
\[ |x_t(\sigma, \varphi, h)| \leq e^{-\alpha(t-\sigma)} (b|\varphi| + c \sup_{\sigma \leq u \leq t} |h(u)|) + d \sup_{\sigma \leq u \leq t} |h(u)|, \quad (3.10) \]
for all \( t \geq \sigma \). Furthermore, the constants \( a, b, c, d \) can be chosen so that for any \( s \in [\sigma, \infty) \)

\[
| x_t(\sigma, \varphi, h) | \leq e^{-a(t-s)}(b|\varphi| + c \sup_{\sigma \leq u \leq t} | h(u) |) + d \sup_{s \leq u \leq t} | h(u) |
\]  

(3.11)

for \( t \geq s + r \).

**Proof.** For any \( s \) in \([\sigma, \infty)\), we first make a transformation of variables in (3.9) which will ensure that \( h(s) = 0 \). From Lemma 3.3, there is a constant \( N \), independent of \( s, \sigma \) and an \( n \times n \) matrix \( \Phi \) depending on \( s, \Phi = (\varphi_1, \ldots, \varphi_n) \), \( \varphi_j \in \mathbb{C}, |\varphi_j| < N \), such that \( D(s)\Phi = I \). If \( y : [s - \tau, s) \rightarrow \mathbb{R}^n \) is defined by

\[
y(t) = \begin{cases} 
\Phi(t) h(s), & s - r \leq t \leq s, \\
\Phi(s) h(t), & t \geq s,
\end{cases}
\]

(3.12)

then \( D(s) y_s = D(s)\Phi h(s) = h(s) \). Therefore, for \( t \geq s \),

\[
D(t)(x_t - y_t) = h^*(t)
\]

(3.13)

where \( h^*(t) \) also depends upon \( s \) and satisfies

\[
h^*(s) = 0,
\]

(3.14)

\[
|h^*(t)| \leq L \sup_{[\max(s, t-\tau), s] \leq u \leq t} | h(u) |,
\]

(3.15)

for some constant \( L \) independent of \( s \) and \( \sigma \). Also, from the definition of \( y \) in (3.12), we have

\[
|y_t| \leq N \sup_{[\max(s, t-\tau), s] \leq u \leq t} | h(u) |.
\]

If \( z_t = x_t - y_t \), then \( D(s)z_s = 0 \). Our next objective is therefore to estimate the function \( z_t \) satisfying the equation

\[
D(t)z_t = h^*(t), \quad t \geq s, \quad z_s = \psi, \quad D(s)\psi = 0,
\]

(3.16)

in terms of its value \( \psi \) at \( s \) and \( |h^*(u)| \) for \( u \geq s \).

The solution \( z(s, \psi, h^*) \) of (3.16) can be written as

\[
z_t(s, \psi, h^*) = z_t(s, \psi, 0) + z_t(s, 0, h^*), \quad t \geq s.
\]

Since \( D(t) \) is uniformly stable with respect to \( C([\tau, \infty), \mathbb{R}^n) \), we know that

\[
|z_t(s, \psi, h^*)| \leq K|\psi| + M \sup_{s \leq u \leq t} |h^*(u)|, \quad t \geq s
\]

and

\[
|z_t(s, 0, h^*)| \leq M \sup_{s \leq u \leq t} |h^*(u)|, \quad t \geq s.
\]
Also, from Lemma 3.2, there are $\beta > 0$, $\alpha > 0$, such that
\[
| z_t(s, \psi, 0) | \leq \beta e^{-\alpha(t-s)} | \psi |, \quad t \geq s.
\]
Consequently,
\[
| z_t(s, \psi, h^*) | \leq \beta e^{-\alpha(t-s)} | \psi | + M \sup_{s \leq u \leq t} | h^*(u) |, \quad t \geq s. \quad (3.17)
\]
If we now let $z_t = x_t - y_t$, $\psi = x_s - y_s$ and use (3.13), (3.14), (3.15), we see there are positive constants $c', d$ such that
\[
| x_t(\sigma, \varphi, h) | \leq e^{-\alpha(t-s)} [ \beta | x_s(\sigma, \varphi, h) | + c' \sup_{s \leq u \leq t} | h(u) | ]
\]
\[
+ d \sup_{[\max(t, t-r)] \leq u \leq t} | h(u) |, \quad t \geq s.
\]
Since $D(t)$ is uniformly stable with respect to $C([\tau, \infty), \mathbb{R}^n)$, we can estimate $| x_t(\sigma, \varphi, h) |$ uniformly in terms of $| \varphi |$ and $\sup_{s \leq u \leq t} | h(u) |$ to obtain constants $b, c$ such that
\[
| x_t(\sigma, \varphi, h) | \leq e^{-\alpha(t-s)} [ b | \varphi | + c \sup_{s \leq u \leq t} | h(u) | ]
\]
\[
+ d \sup_{[\max(t, t-r)] \leq u \leq t} | h(u) |, \quad t \geq s.
\]
For $s = \sigma$, this gives (3.10) and for $t \geq s + r$, this gives (3.11). This completes the proof of the lemma.

**Lemma 3.5.** Suppose $D(t)$ is a uniformly stable operator with respect to $C([\tau, \infty), \mathbb{R}^n)$ and $h \in C([\tau, \infty), \mathbb{R}^n)$ is such that $h(t) \to 0$ as $t \to \infty$. Then the solution $x(\sigma, \varphi, h)$ of (3.9) approaches zero as $t \to \infty$ uniformly with respect to $\sigma$ in $[\tau, \infty)$ and $\varphi$ in closed bounded sets.

**Proof.** Suppose $a, b, c, d$ are the positive constants given in Lemma 3.4 and $\delta, \epsilon$ are arbitrary positive numbers. Choose $T = T(\epsilon)$ so that
\[
d \sup_{T(\epsilon) \leq u} | h(u) | < \epsilon/2
\]
and choose $t_0 = t_0(\delta, \epsilon) \geq T - \tau + r$ so that
\[
[\exp[-a(t_0 - T + r)]][b \delta + c \sup_{\tau \leq u} | h(u) |] < \epsilon/2.
\]
For any $\sigma$ in $[\tau, \infty)$ and $s = \sigma + T - \tau$, relation (3.11) implies that the solution $x(\sigma, \varphi, h)$ of (3.9), $| \varphi | \leq \delta$ satisfies

$$| x_t(\sigma, \varphi, h) | \leq \left\{ \exp[-a(t - \sigma - T + \tau)] \sup_{\tau \leq u} | h(u)| \right\}$$

$$+ d \sup_{\sigma + T - \tau \leq u} | h(u)|$$

$$\leq \frac{\epsilon}{2} \exp[-a(t - t_0 - \sigma)] + \frac{\epsilon}{2},$$

for $t \geq \sigma + T - \tau + r$. Therefore,

$$| x_t(\sigma, \varphi, h) | < \epsilon, \quad t \geq t_0 + \sigma.$$

This implies the conclusion of the lemma.

4. STABILITY FOR NONAUTONOMOUS SYSTEMS

In this section, we consider the equation

$$D(t) x_t = D(\sigma) \varphi + \int_0^t f(s, x_s) \, ds, \quad t \geq \sigma, \quad x_\sigma = \varphi, \quad (4.1)$$

where $D(t)$ is defined in (2.1) with $g$ satisfying (3.1), $f : [\tau, \infty) \times C \to \mathbb{R}^n$ is continuous, takes closed bounded sets into bounded sets and system (4.1) has a unique solution $x(\sigma, \varphi)$ which depends continuously upon $\sigma, \varphi$. If $V : [\tau, \infty) \times C \to \mathbb{R}$ is continuous, we define the “derivative” $\dot{V}(t, \varphi)$ along the solutions of (4.1) as

$$\dot{V}(t, \varphi) \equiv \dot{V}_{(4.1)}(t, \varphi) = \lim_{h \to 0^+} \frac{1}{h} \left[ V(t, x_{t+h}(t, \varphi)) - V(t, \varphi) \right].$$

DEFINITION 4.1. We say that the solution $x = 0$ of (4.1) is uniformly stable if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that for all $\sigma \in [\tau, \infty)$, $\tau > -\infty$, any solution $x(\sigma, \varphi)$ of (4.1) with initial value $\varphi$ at $\sigma$, $| \varphi | \leq \delta$ satisfies $| x_t(\sigma, \varphi) | < \epsilon$ for $t \geq \sigma$. It is uniformly asymptotically stable if it is uniformly stable and for some fixed $\delta > 0$, for any $\eta > 0$, there exists a $T = T(\eta) > 0$ such that $| \varphi | \leq \delta$ implies $| x_t(\sigma, \varphi) | < \eta$ for $t \geq \sigma + T$.

THEOREM 4.1. Suppose $u(s), \nu(s), \psi(s)$ are continuous functions for $s$ in $[0, \infty)$, $u(s), \nu(s)$ are positive, nondecreasing for $s > 0$, $u(0), \nu(0) = 0$, $\psi(s)$ is nonnegative, nondecreasing and $V : [\tau, \infty) \times C \to \mathbb{R}$ is a continuous function satisfying

$$u(|D(t)\varphi|) \leq V(t, \varphi) \leq v(|\varphi|)$$

$$\dot{V}(t, \varphi) \leq -\nu(|D(t)\varphi|). \quad (4.2)$$
If \( D(t) \) is uniformly stable with respect to \( C([\tau, \infty), f^n) \), then the solution \( x = 0 \) of (4.1) is uniformly stable. If, in addition, \( \omega(s) > 0 \) for \( s > 0 \), then the solution \( x = 0 \) is uniformly asymptotically stable.

**Proof.** Suppose the constants \( b, c, d \) are defined as in Lemma 3.4. For any \( \epsilon > 0 \), choose \( \delta \) so that \( b\delta < \epsilon/2, \nu(\delta) < u(\epsilon/2(c + d)) \). If \( \varphi \) is in \( C \), \( |\varphi| < \delta \), then (4.2) implies \( V(t, x_\ell(\sigma, \varphi)) \) nonincreasing and

\[
\nu(|D(t)x_\ell(\sigma, \varphi)|) \leq V(t, x_\ell(\sigma, \varphi)) \leq V(\sigma, \varphi) \leq \nu(\delta) < u(\epsilon/2(c + d)).
\]

Consequently, \( |D(t)x_\ell(\sigma, \varphi)| < \epsilon/2(c + d) \) for all \( t \geq \sigma \). Since \( D(t) \) is uniformly stable with respect to \( C([\tau, \infty), f^n) \), relation (3.10) implies

\[
|x_\ell(\sigma, \varphi)| \leq b |\varphi| + (c + d) \epsilon/2(c + d) \leq b\delta + \epsilon/2 < \epsilon.
\]

Therefore, the solution \( x = 0 \) is uniformly stable.

For \( \epsilon = 1 \), choose \( \delta_0 = \delta(1) \) as the above constant for uniform stability. Then, for any \( \sigma \geq \tau, |\varphi| < \delta_0 \) implies

\[
|x_\ell(\sigma, \varphi)| \leq 1/2(c + d), \quad t \geq \sigma.
\]

For any \( \epsilon > 0 \), we wish to show there is a \( T(\delta_0, \epsilon) \) such that any solution \( x(\sigma, \varphi) \) of (4.1) with \( |\varphi| < \delta_0 \) satisfies \( |x_\ell(\sigma, \varphi)| < \epsilon \) for \( t \geq \sigma + T(\delta_0, \epsilon) \). To do this, we show there is a \( T(\delta_0, \epsilon) \) and \( t' \) in \([\sigma, \sigma + T(\delta_0, \epsilon)]\) such that \( |x_\ell(\sigma, \varphi)| < \delta \), where \( \delta = \delta(\epsilon) \) is the above constant for uniform stability. The uniform stability then implies that \( |x_\ell(\sigma, \varphi)| < \epsilon \) for \( t \geq t' \) and, in particular, for \( t \geq \sigma + T(\delta_0, \epsilon) \).

For \( \alpha, b, c, d \) as in Lemma 3.4, choose \( \alpha = a(\delta_0, \epsilon) > 0 \) so that

\[
e^{-\alpha(\delta_0, \epsilon)} \left( b\delta_0 + c \frac{1}{2(c + d)} \right) \leq \frac{\delta}{2}.
\]

Since \( f \) takes closed bounded sets into bounded sets, there is a constant \( L \) such that \( |f(s, x_\ell(\sigma, \varphi))| < L \) for \( s \geq \sigma, |\varphi| \leq \delta_0 \). Let \( K_0 = K(\delta_0, \epsilon) \) be the smallest integer such that \( K_0 > \nu(\delta_0)/(\nu(\delta_2)/(2d)) \). Suppose there is a solution \( x = x(\sigma, \varphi) \) of (4.1) with \( |\varphi| < \delta_0 \) and \( |x_\ell(\sigma, \varphi)| \geq \delta \) for \( \sigma \leq t \leq \sigma + 2(1 + K_0\alpha) \). Consider the sequences \( s_k = \sigma + (2k - 1)\alpha, s_k' = \sigma + 2k\alpha, k = 1, 2, ..., K_0 + 1 \). Taking \( s = s_k, t = s_k' \) in relation (3.11), we have

\[
\delta < |x_k'| \leq \{\exp[-a(s_k' - s_k)] \} (b |\varphi| + c \sup_{\sigma < u \leq s_k'} |D(u) x_u|)
\]

\[
+ d \sup_{s_k \leq u \leq s_k'} |D(u) x_u| \leq e^{-\alpha s} \left( b\delta_0 + \frac{c}{2(c + d)} \right) + d \sup_{s_k \leq u \leq s_k'} |D(u) x_u| \leq \frac{\delta}{2} + d \sup_{s_k \leq u \leq s_k'} |D(u) x_u|.
\]
therefore, there must exist a \( t_k \) in \([s_k, s'_k]\) such that \(|D(t_k)x_{t_k}| \geq \delta/2d\), \( k = 1, 2, \ldots, K_0 + 1\).

Since \(|f(s, x_s(\sigma, \varphi))| < L\) for all \( s \geq \sigma\), \( |\varphi| < \delta_0\), on the intervals \( I_k = [t_k - \delta/4dL, t_k + \delta/4dL]\) we have \(|D(t)x_t| \geq \delta/4d\). Consequently, \( \dot{V}(t, x_t) < -w(\delta/4d)\), \( t \in I_k\), \( k = 1, 2, \ldots, K_0 + 1\). By taking \( L \) sufficiently large if necessary, we may assume the intervals \( I_k \) are non-overlapping. Therefore,

\[
\dot{V}(t_k, x_{t_k}) < V(\sigma, \varphi) - w\left(\frac{\delta}{4d}\right)\left(\frac{\delta}{2d}\right)(k - 1)
\]

If \( k - 1 = K_0 \), then \( \dot{V}(t_k, x_{t_k}(\sigma, \varphi)) < 0\), which is a contradiction. Therefore, there must be a \( t' \) in the interval

\[
[\sigma, \sigma + 2(1 + K(\delta_0, \epsilon) + \delta(\delta_0, \epsilon))]
\]
such that \( |x_{t'}(\sigma, \varphi)| < \delta\). This completes the proof of the theorem.

**Theorem 4.2.** Suppose \( D \) in (4.1) is independent of \( t \), \( V: C \to R \) is continuous, maps bounded sets into bounded sets and there exist continuous, nondecreasing nonnegative functions \( a(s), b(s), s \geq 0 \), positive for \( s > 0 \), an open set \( U \) in \( C \) and a bounded open neighborhood \( N \) of zero in \( C \) such that \( V(\varphi) \) satisfies

(i) \( V(\varphi) > 0 \) on \( U \), \( V(\varphi) = 0 \) on the boundary \( \partial U \) of \( U \).

(ii) \( 0 \) belongs to the closure of \( U \cap N \);

(iii) \( V(\varphi) \leq a(|D\varphi|) \) on \([\tau, \infty) \times (U \cap N)\)

(iv) \( \dot{V}(\varphi) \geq b(|D\varphi|) \) on \([\tau, \infty) \times (U \cap N)\).

Under these conditions, the solution \( x = 0 \) of (4.1) is unstable. More specifically, each solution \( x(\sigma, \varphi) \) of (4.1) with initial value \( \varphi \) in \( U \cap N \) at \( \sigma \in [\tau, \infty) \) must reach the boundary of \( N \) in finite time.

**Proof.** Suppose \( \varphi_0 \in U \cap N, \varphi \in [\tau, \infty) \). Then \( V(\varphi_0) > 0 \). From (iv), the solution \( x = x(\sigma, \varphi_0) \) satisfies \( \dot{V}(x_t) \geq V(\varphi_0) \) as long as \( x_t \in U \cap N \). From (iii) and (iv), this implies

\[
\dot{V}(x_t) \geq b(|Dx_t|) \geq b(a^{-1}(V(x_t))) \geq b(a^{-1}(V(\varphi_0))) > 0
\]
as long as \( x_t \in U \cap N \). If \( \eta = b(a^{-1}(V(\varphi_0))) \), then this relation implies

\[
V(x_t) \geq V(\varphi_0) + \eta(t - \sigma),
\]
as long as \( x_t \in U \cap N \). Since \( N \) is bounded and \( V \) is bounded on \( N \), there must be a \( t_1 \) such that \( x_{t_1} \in \partial(U \cap N) \). But hypothesis (i) implies that \( x_{t_1} \in \partial N \). This proves the last assertion of the theorem. Hypothesis (ii) implies that each neighborhood of zero contains a \( \varphi_0 \) in \( U \cap N \). Thus, zero is unstable and the theorem is proved.

5. STABILITY FOR AUTONOMOUS SYSTEMS

In this section, we consider the autonomous system

\[
Dx_t = D\varphi + \int_0^t f(x_s) \, ds, \quad t \geq 0, \quad x_0 = \varphi,
\]

(5.1)

where \( D\varphi = \varphi(0) - g(\varphi) \), \( g : C \to \mathbb{R}^n \) is continuous and linear, \( f : C \to \mathbb{R}^n \) is continuous and takes closed bounded sets into bounded sets. We also assume (5.1) has a unique solution but this condition is not essential. If \( x(\varphi) \) is a solution of (5.1) on \([0, \infty)\), then the orbit \( y(\varphi) \) of \( x(\varphi) \) is \( \{x_t(\varphi), t \in [0, \infty)\} \).

**Lemma 5.1.** If \( D \) is stable with respect to \( C([0, \infty), \mathbb{R}^n) \), then the orbit of any solution \( x(\varphi) \) of (5.1) for which \( Dx_t(\varphi) \) is bounded for \( t \geq 0 \) must belong to a compact set of \( C \).

**Proof.** Let \( x = x(\varphi) \). Since \( Dx_t \) is bounded, it follows that \( \int_0^t f(x_s) \, ds \) is bounded. Since \( D \) is stable with respect to \( C([0, \infty), \mathbb{R}^n) \), this implies \( x_t \) is bounded. As a result of the fact that \( f \) takes closed bounded sets into bounded sets, this implies there is a constant \( N \) such that \( |f(x_t)| < N, \, t \geq 0 \). For any \( \tau \geq 0 \), one has

\[
Dx_{t+\tau} - Dx_\tau = \int_0^{\tau} f(x_s) \, ds, \quad t \geq 0,
\]

\[
Dx_t = D\varphi \cdot \int_0^t f(x_s) \, ds, \quad t \geq 0.
\]

Therefore,

\[
D(x_{t+\tau} - x_t) = D(x_\tau - \varphi) + \left( \int_t^{t+\tau} f(x_s) \, ds - \int_0^\tau f(x_s) \, ds \right).
\]
The fact that $D$ is stable with respect to $C([0, \infty), R^n)$ implies from (3.3) that

$$|x_{t+\tau} - x_t| \leq K|x_{\tau} - \varphi| + 2\pi NM$$

for all $t \geq 0, \tau \geq 0$. For any $\epsilon > 0$, there is a $\delta > 0$ such that $|x_{\tau} - \varphi| < \epsilon/2K$, $2\pi N < \epsilon/2M$ for $0 \leq \tau \leq \delta$. Therefore, $|x_{t+\tau} - x_t| < \epsilon$ for $0 \leq \tau \leq \delta$ and all $t \geq 0$ and the function $x_t$ is uniformly continuous in $t$ for $t$ in $[0, \infty)$. This plus the fact that $x_t$ is uniformly bounded for $t \geq 0$ implies $\{x_t\}$ belongs to a compact set of $C$. This proves the lemma.

**Definition 5.1.** An element $\psi$ of $C$ is said to belong to the $\omega$-limit set $\omega(\gamma(\varphi))$ of an orbit $\gamma(\varphi)$ of (5.1) if there is a sequence of real numbers $t_k \to \infty$ as $k \to \infty$ such that $x_{t_k}(\varphi) \to \psi$ as $k \to \infty$. A subset $\Gamma$ of $C$ is said to be invariant with respect to solutions of (5.1) if for any $\varphi \in \Gamma$, there is a function $g(\varphi) : (-\infty, \infty) \to R^n$ which satisfies $g(\varphi) = \varphi$ and

$$Dg_{x_{t+\sigma}}(\varphi) = Dg_\varphi(\varphi) + \int_0^t f(g_{x_{\tau}}, g(\varphi)) \, ds, \quad t \geq 0$$

for all $\sigma$ in $(-\infty, \infty)$.

**Lemma 5.2.** If $D$ is stable with respect to $C([0, \infty), R^n)$ and $x(\varphi)$ is a solution of (5.1) with $Dx_\varphi(\varphi)$ bounded for $t \geq 0$, then $\omega(\gamma(\varphi))$ is a non-empty, compact, connected invariant set of (5.1).

The proof is not given since Lemma 5.1 and Lemma 3 of [4] imply the conclusion. Following LaSalle [5], we give

**Definition 5.2.** We say $V : C \to R$ is a Liapunov function on a set $G$ in $C$ if $V$ is continuous on $\bar{G}$, the closure of $G$, and $V \leq 0$ on $G$. Let

$$S = \{\varphi \in \bar{G} : V(\varphi) = 0\}$$

$$\Gamma = \text{largest set in } S \text{ which is invariant with respect to (5.1).}$$

**Theorem 5.1.** Suppose $D$ is stable with respect to $C([0, \infty), R^n)$ and $V$ is a Liapunov function on $G$. If $x(\varphi)$ is a solution of (5.1) which remains in $G$ and has $Dx_\varphi(\varphi)$ bounded for $t \geq 0$, then $x_t(\varphi) \to \Gamma$ as $t \to \infty$.

**Corollary 5.1.** Suppose $D$ is stable with respect to $C([0, \infty), R^n)$ and $V$ is a Liapunov function on $G = G_1 = \{\varphi \in C : V(\varphi) < I\}$. If there is a constant $K = K(I)$ such that $\varphi$ in $G_1$ implies $|D\varphi| < K$, then any solution $x_t(\varphi)$ of (5.1) with $\varphi$ in $G_1$ approaches $\Gamma$ as $t \to \infty$.

With Lemma 5.2, these results are special cases of Theorem 1 of [4].
COROLLARY 5.2. If the condition of Corollary 5.1 are satisfied and, in addition,

\[ \dot{V}(\varphi) \leq -\omega( |D\varphi| ) \leq 0, \]

for some continuous function \( \omega : [0, \infty) \to \mathbb{R} \), then every solution of (5.1) with \( \varphi \in G_t \) approaches \( \{ \varphi \in G_t : \omega(|D\varphi|) = 0 \} \). In particular, if \( \omega(s) > 0 \) for \( s > 0 \), then every solution of (5.1) with \( \varphi \) in \( G_t \) approaches zero as \( t \to \infty \).

Proof. The first part is obvious since \( \Gamma \subset \{ \varphi \in G_t : \omega(|D\varphi|) = 0 \} \). If \( \omega(s) > 0 \) for \( s > 0 \) then \( \Gamma \subset \{ \varphi : D\varphi = 0 \} \). If \( x_t \) is a solution of (5.1) in \( \Gamma \) for \( t \in (-\infty, \infty) \), then \( x_t \equiv 0 \). In fact, there is an \( L \) such that \( |x_t| < L \), \( t \in (-\infty, \infty) \). If there is a \( \tau \) such that \( |x_t| \neq 0 \), choose \( \sigma \) so that

\[ \beta e^{-\alpha(t-\sigma)} L < |x_\tau|/2, \]

where \( \beta, \alpha \) are given in Lemma 3.2. Then

\[ |x_t(\sigma, x_\sigma)| \leq \beta e^{-\alpha(t-\sigma)} x_\sigma | \leq \beta e^{-\alpha(t-\sigma)} L < \frac{1}{2} e^{-\alpha(t-\sigma)} |x_\tau|, \]

for all \( t \geq \sigma \). For \( t = \tau \), this gives a contradiction and the lemma is proved.

THEOREM 5.2. Suppose \( D \) is stable with respect to \( C([0, \infty), \mathbb{R}^n) \). Suppose zero belongs to the closure of an open set \( U \) in \( C \) and \( N \) is an open neighborhood of zero in \( C \). Assume that

(i) \( V \) is a Liapunov function on \( G = N \cap U \).

(ii) \( \Gamma \cap \overline{G} \) consists of points belonging to \( \partial G \cap N \) plus possibly zero.

(iii) \( V(\varphi) < \eta \) on \( G \) when \( \varphi \neq 0 \)

(iv) \( V(0) = \eta \) and \( V(\varphi) = \eta \) when \( \varphi \in \partial G \cap N \).

If \( N_0 \) is a bounded neighborhood of zero properly contained in \( N \), then \( \varphi \neq 0 \) in \( G \cap N_0 \) implies there exists a \( \tau \) such that \( x_t(\varphi) \in \partial N_0 \).

Proof. Suppose \( \varphi \in G \cap N_0 \). Then \( V(x_t(\varphi)) \leq V(\varphi) < \eta \) for all \( t > 0 \) as long as \( x_t(\varphi) \) remains in \( G \cap N_0 \). If \( x_t(\varphi) \) remains in the bounded set \( G \cap N_0 \) for all \( t > 0 \), then \( D x_t(\varphi) \) is bounded for \( t > 0 \). Lemma 5.2 implies \( \omega(\varphi) \) is a non-empty, invariant set of (5.1) and belongs to \( \overline{G \cap N_0} \). Hypothesis (ii) implies \( \omega(\varphi(\varphi)) \subset \partial G \cap N \). On the other hand, \( V(\psi) = \eta \) for \( \psi \in \partial G \cap N \), which is a contradiction. Thus, there is a \( \tau > 0 \) such that \( x_t(\varphi) \in \partial(N_0 \cap G) \subset (\partial N_0) \cap (\partial G) \). Condition (iv) implies \( x_t(\varphi) \in \partial N_0 \). This proves the theorem.
6. Examples

6.1. Consider the scalar equation

$$\frac{d}{dt} [x(t) + cx(t - r)] + ax(t) = 0 \quad (6.1)$$

where $a, c$ are constants with $a > 0, |c| < 1$. Lemma 3.1 implies the operator $Dg, = g(0) + cg(-r)$ is stable with respect to $C([0, \infty), R)$. If

$$V(\varphi) = (D\varphi)^2 + a e^2 \int_{-r}^{0} \varphi^2(\theta) d\theta,$$

then

$$\dot{V}(\varphi) = -a(D\varphi)^2 - a(1 - c^2) \varphi^2(0) \leq -a(D\varphi)^2.$$

Theorem 4.1 implies the solution $x = 0$ of (6.1) is uniformly asymptotically stable.

6.2. Consider the equation

$$\frac{d}{dt} [x(t) + cx(t - r)] + u(t) x(t) = 0 \quad (6.2)$$

where $c$ is constant, $|c| < 1$, $a(t)$ is continuous for $t \geq 0, a(t) \geq \delta > 0, t \geq 0$, $a(t) \geq c^2 a(t + r), t \geq 0$. This latter condition is satisfied for example if $a(t)$ has a derivative $\dot{a}(t) \leq 0$. If $D\varphi = \varphi(0) + c\varphi(-r)$, then $D$ is stable with respect to $C([0, \infty), R)$. If

$$V(x_i) = (Dx_i)^2 + \int_{t-r}^{t} e^2 a(u + r) x^2(u) du,$$

then

$$\dot{V}(x_i) = -a(t)(Dx_i)^2 - [a(t) - c^2 a(t + r)] x^2(t) \leq -\delta(Dx_i)^2.$$

The conditions of Theorem 4.1 are satisfied and, thus, the solution $x = 0$ of (6.2) is uniformly asymptotically stable.

6.3. Consider the equation

$$\frac{d}{dt} [x(t) + c(t) x(t - r)] + ax(t) + b(t) x(t - r) = 0 \quad (6.3)$$

where $a > 0$ is constant, $c(t), b(t)$ are continuous for $t \geq 0$ and there is a $\delta > 0$ such that $c^2(t) \leq 1 - \delta, t \geq 0$. If $D(t)\varphi = \varphi(0) + c(t) \varphi(-r)$, then
it follows from the remark following Lemma 3.1 that $D(t)\varphi$ is uniformly stable with respect to $C([0, \infty), R)$. If

$$V(t, \varphi) = \frac{1}{2}[D(t)\varphi]^2 + \alpha \int_{-r}^{0} \varphi^2(\theta) \, d\theta,$$

where $\alpha \geq 0$ is a constant, then

$$\dot{V}(t, \varphi) = -(a - \alpha) \varphi^2(0) - (b + ac) \varphi(0) \varphi(-r) - (bc + \alpha) \varphi^2(-r), \quad (6.4)$$

where $b, c$ are evaluated at $t$. Let us consider $\dot{V}(t, \varphi)$ as a quadratic form in $D(t)\varphi, \varphi(0), \varphi(-r)$ and write it in the form

$$\dot{V}(t, \varphi) = -A[D(t)\varphi]^2 - B\varphi^2(0) - 2C\varphi(0) \varphi(-r) - D\varphi^2(-r). \quad (6.5)$$

If we determine a constant $A$ and functions $B, C, D$ with $A > 0, B > 0, D > 0, BD \geq C^2$, then $\dot{V}(t, \varphi) \leq -A[D(t)\varphi]^2$ and the conditions of Theorem 4.1 are satisfied. Therefore, the solution $x = 0$ of (6.3) is uniformly asymptotically stable.

Identification of (6.4) with (6.5) yields

$$A + B = a - \alpha, \quad cA + C = \frac{b + ac}{2}, \quad c^2A + D = bc + \alpha.$$  

If we arbitrarily choose $B = a/2$, then

$$A = a/2 - \alpha, \quad 2C = b + 2\alpha,$$

$$D = c \left( b - \frac{ca}{2} \right) + (1 + c^2)\alpha.$$

Notice that $A$ is constant whereas $C, D$ are functions of $t$. Let $\beta = 2\alpha/a$ and then it is easy to check that $x \geq 0, A > 0, BD \geq C^2$ will be satisfied if we can find a constant $\beta, 0 \leq \beta < 1$, such that

$$[2\gamma(t)c(t) - 1]\beta + [\gamma(t) - c(t)]^2 < 0, \quad \gamma(t) = b(t)/a. \quad (6.6)$$

Any condition on $\gamma(t), c(t)$ which will ensure that (6.6) has a constant solution $\beta$ in $[0, 1)$ will imply that the solution $x = 0$ of (6.3) is uniformly asymptotically stable.

In order for (6.6) to have a solution it is necessary that

$$2\gamma(t)c(t) - 1 < 0, \quad t \geq 0. \quad (6.7)$$
If (6.7) is satisfied, then there is a constant solution $\beta$ of (6.6) in $[0, 1)$ if there is an $\epsilon > 0$ such that

$$\gamma^2(t) + c^2(t) < 1 - \epsilon, \quad t \geq 0$$

(6.8)

6.4. Consider the equation

$$\dot{x}(t) + c(t) \dot{x}(t - r) + ax(t) = 0$$

(6.9)

where $a > 0$ is a constant, $c(t)$, $t \geq 0$, has a continuous first derivative and $|c(t)| \leq 1 - \delta$, $\delta > 0$ a constant. Equation (6.9) can be written in the form

$$\frac{d}{dt} [x(t) + c(t)x(t - r)] + ax(t) - c(t)x(t - r) = 0,$$

(6.10)

which is a special case of (6.3) with $b(t) = -c(t)$. The results of the previous example give conditions on $c(t)$, $a$, $c(t)$ which imply that the solution $x = 0$ of (6.10) is uniformly asymptotically stable.

6.5. Consider the equation

$$\frac{d}{dt} (x(t) + cx(t - r)) + ax(t) = 0,$$

(6.11)

where $|c| < 1$, $a < 0$. This is a special case of system (2.2) with $D\varphi = \varphi(0) - c\varphi(-r)$. If

$$V(\varphi) = \frac{1}{2} (D\varphi)^2 + \frac{a}{2} \int_{-r}^{0} \varphi^2(\theta) d\theta \leq \frac{1}{2} (D\varphi)^2,$$

then

$$\dot{V}(\varphi) = -\frac{a}{2} (D\varphi)^2 - \frac{a}{2} (1 - c^2) \varphi^2(-r) \geq -\frac{a}{2} (D\varphi)^2.$$

If $U = \{\varphi : V(\varphi) > 0\}$, then the conditions of Theorem 4.2 are satisfied and the zero solution of (6.11) is unstable.

6.6. Consider the equation

$$\frac{d}{dt} [x(t) + cx(t - r)] + ax(t) + bx^2(t - r) = 0,$$

(6.12)

where $a > 0$, $0 < c^2 < 1$ and $b$ is any real number. If

$$V(\varphi) = (D\varphi)^2 + ac^2 \int_{-r}^{0} \varphi^2(\theta) d\theta,$$
From the definition of a stable operator, for any \( \delta_0, \epsilon_0 \), it is easily verified that \( |\varphi| \leq \delta_0, \ |Dx_t(\varphi)| \leq \epsilon_0 \), implies \( |x_t(\varphi)| \leq \eta(\delta_0, \epsilon_0) \) for all \( t > 0 \) where \( \eta(0, 0) = 0 \), and \( \eta \) is a nondecreasing continuous function of \( \delta_0, \epsilon_0 \). Choose \( \delta_0, \epsilon_0 \) such that \( \eta_0 = \eta(\delta_0, \epsilon_0), \epsilon_0 \) satisfy

\[
\begin{bmatrix}
\frac{6b}{c^2} \\
\frac{2b}{c^3}
\end{bmatrix} \eta_0^2 + \begin{bmatrix}
\frac{6b}{c^2} \\
\frac{2b}{c^3}
\end{bmatrix} \epsilon_0^2 < a,
\]

\[
\begin{bmatrix}
\frac{2b}{c^3}
\end{bmatrix} \eta_0 \epsilon_0 < a(1 - c^2).
\]

Clearly \( \eta_0 \geq \delta_0 \).

If \( |D| \leq L \) in the operator norm, choose \( 0 < \delta \leq \delta_0 \) such that

\[
L^2 \delta^2 + a c^2 \epsilon_0^2 \leq \epsilon_0^2.
\]

Then, the solution of (6.12) for \( |\varphi| \leq \delta, \varphi \) in \( C \) satisfies

\[
V(x_t(\varphi)) \leq \epsilon_0^2, \quad \dot{V}(x_t(\varphi)) \leq -u(Dx_t(\varphi)) \leq 0
\]

for some positive definite \( u \). This shows that the null solution of (6.12) is asymptotically stable.

7. THE CONVERSE THEOREMS FOR UNIFORM ASYMPTOTIC STABILITY

Consider the system

\[
\frac{d}{dt} [D(t) x_t] = f(t, x_t)
\]

\[
x_\sigma = \varphi, \quad \sigma \text{ in } [\tau, \infty).
\]

(7.1)

In this section, we will show that a converse theorem of Liapunov type exists when (7.1) is uniformly asymptotically stable.

The presentation below follows closely the one in Hahn [6] and Hale [7] for ordinary differential equations. The details of the proofs are not given since they follow as in ordinary differential equations once the essential difference is specified.
Theorem 7.1. Let $D(t)$ and $f(t, \cdot)$ be bounded linear operators from $C$ into $\mathbb{R}^n$ such that for all $\varphi$ in $C$

$$|D(t)\varphi| \leq K|\varphi| \quad \text{for} \quad t \geq \tau. \quad (7.2)$$

If (7.1) is uniformly asymptotically stable, then there are positive constants $M$, $a$ and a continuous scalar function $V$ on $[\tau, \infty) \times C$ such that

(a) $|D(t)\varphi| \leq V(t, \varphi) \leq M|\varphi|$

(b) $\dot{V}(t, \varphi) \leq -aV(t)$

(c) $|V(t, \varphi) - V(t, \psi)| \leq M|\varphi - \psi|$

for all $t \geq \tau$, $\varphi$, $\psi$ in $C$. $\dot{V}$ is the usual upper right hand derivative along the solutions of (7.1).

Proof. Uniform asymptotic stability implies there are positive constants $K_0$ and $a$ such that the solution $x(\sigma, \varphi)$ of (7.1), $\sigma \geq \tau$, $x_0(\sigma, \varphi) = \varphi$ satisfies

$$|x_\sigma(\sigma, \varphi)| \leq K_0 e^{-a(t-\sigma)} |\varphi|,$$

for all $t \geq \sigma$, $\varphi$ in $C$. Define

$$V(t, \varphi) = \sup_{s \geq 0} |D(t + s) x_{t+s}(t, \varphi)| e^{as}, \quad t \geq \sigma.$$

We refer the reader for the details in the verification of (7.2) to [7]. The constant $M$ is given by $M = KK_0$.

When $f(t, \varphi)$ is not linear in $\varphi$, the lemma below may be used to prove the theorem that follows it. The proofs given in [6] and [7] apply with only a slight modification.

Lemma 7.1. The null solution of (7.1) is uniformly asymptotically stable if and only if there exist functions $\rho(u)$, $\nu(u)$ with

(a) $\rho(u)$ defined, continuous and monotonically increasing in an interval $0 \leq u \leq \delta_0$, $\rho(0) = 0$.

(b) $\nu(u)$ defined, continuous and monotonically decreasing in $0 \leq u < \infty$, $\nu(u) \to 0$ as $u \to \infty$

such that for any $\varphi$ in $C$, $|\varphi| \leq \delta_0$ and $\sigma \geq \tau$, the solution $x(\sigma, \varphi)$ of (7.1) satisfies

$$|x_\sigma(\sigma, \varphi)| \leq \rho(|\varphi|) \nu(t-\sigma), \quad t \geq \sigma. \quad (7.4)$$

Theorem 7.2. Let $f(t, 0) = 0$, $f(t, \varphi)$ be locally Lipschitzian in $\varphi$ uniformly with respect to $t$, $g$ satisfy (7.2) and the null solution of (7.1) be uniformly asymptotically stable. Then, there exists a $\delta_0 > 0$, $K = K(\delta_0) > 0$, positive
definite functions \( b(u) \), \( c(u) \) on \( 0 \leq u \leq \delta_0 \) and a scalar function \( V(t, \varphi) \) defined and continuous for \( t \) in \( [\tau, \infty) \), \( \varphi \) in \( C \), \( |\varphi| \leq \delta_0 \) such that

(a) \( |D(t) \varphi| \leq V(t, \varphi) \leq b(|\varphi|) \)

(b) \( \dot{V}(t, \varphi) \leq -c(|D(t)\varphi|) \) \hspace{1cm} (7.5)

(c) \( |V(t, \varphi) - V(t, \psi)| \leq K|\varphi - \psi| \)

for all \( t \geq \tau \), \( \varphi, \psi \) in \( C \), \( |\varphi|, |\psi| \leq \delta_0 \).

The proof of this last theorem follows step by step the argument given in [7] with the only change being that \( D(t)x(t, \sigma, \varphi) \) as in the proof of Theorem 7.1 is used to define the function \( V(t, \varphi) \).

Theorems 7.1 and 7.2 can be used to discuss the effect of certain types of perturbation on systems for which the zero solution is uniformly asymptotically stable. More precisely, suppose \( D(t) \) is the same as in (7.1),

\[ F: [\tau, \infty) \times C \to \mathbb{R}^n \]

is continuous and consider the system

\[
\frac{d}{dt} [D(t)y_1] = F(t, y_1). \tag{7.6}
\]

If \( y = y(\sigma, \varphi) \), \( x = x(\sigma, \varphi) \) are the solutions of (7.6), (7.1), respectively, with initial value \( \varphi \) at \( \sigma \) and the zero solution of (7.1) is uniformly asymptotically stable, then relations (7.5) imply that

\[
\dot{V}_{(\sigma, \varphi)}(\sigma, \varphi) \leq \dot{V}_{(\sigma, \varphi)}(\sigma, \varphi) + K \lim_{h \to 0^+} \frac{1}{h} |y_{\sigma+h}(\sigma, \varphi) - x_{\sigma+h}(\sigma, \varphi)|. \tag{7.7}
\]

On the other hand,

\[
D(\sigma + h)(y_{\sigma+h} - x_{\sigma+h}) = \int_{\sigma}^{\sigma+h} [F(s, y_s) - f(s, x_s)] \, ds,
\]

for any \( h \geq 0 \). Since \( g(t, \varphi) \) satisfies (3.1), it follows there is an \( h_0 > 0 \) such that

\[
|y_{\sigma+h} - x_{\sigma+h}| \leq \frac{1}{1 - l(h_0)} \int_{\sigma}^{\sigma+h} |F(s, y_s) - f(s, x_s)| \, ds,
\]

for \( 0 \leq h \leq h_0 \). If this relation is used in (7.7), one obtains

\[
\dot{V}_{(\sigma, \varphi)}(t, \varphi) \leq \dot{V}_{(\sigma, \varphi)}(t, \varphi) + \frac{K}{1 - l(h_0)} |F(t, \varphi) - f(t, \varphi)|, \tag{7.8}
\]

for all \( t \geq \tau \), \( |\varphi| \leq \delta_0 \).
As in ordinary differential equations, relation (7.8), the known upper bound for $\dot{V}_{(\gamma,\alpha)}(t, \varphi)$ and the inequalities for $V$ can now be used to prove asymptotic properties of the perturbed equation (7.6).

REFERENCES