We show that the structure of multidimensional systems of conservation laws, equipped with a single, convex entropy, and with both rotation and Galilean symmetries, is perhaps surprisingly limited. The following are established.

The simplest examples of such systems, with only one vector field, are at most simple extensions of the familiar Euler systems. In such models of magnetohydrodynamic flow, including a solenoidal, Galilean invariant magnetic field, the property of finite signal propagation speeds is often lost.

The only such system describing adiabatic multi-fluid flow, with the mass of each species conserved separately, corresponds to independent flow of each species. Whatever interaction between the different species occurs in an energy equation or through some other dependent variable, and not simply through expressions for the pressures in terms of the species’ densities. A similar conclusion holds for incompressible flow.

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1. Introduction and summary

Nonlinear, multidimensional systems of conservation laws are routinely adopted as models of fluid flow. Two litmus tests commonly applied to such proposed systems are the existence of a symmetry group containing both rotation and Galilean symmetries [1], and the existence of a convex entropy density [9]. Indeed, lack of such a symmetry group undermines the interpretation of such a system as a model of nonrelativistic fluid flow. Existence of a convex entropy implies hyperbolicity [6] and provides a crude bound on weak solutions. We acknowledge, nonetheless, that in higher dimensions the corresponding entropy inequality appears unlikely to be sufficient to recover uniqueness in the class of admissible weak solutions [3,14].

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It is well known that the Euler systems have such a structure. As against this, models of multi-fluid flow [4,12] remain problematic in this regard. Here we provide some explanation, emphasizing some simple examples. In particular, we find that within this class of systems, the flux functions are largely determined by specification of the primitive variables, including the entropy density, and their properties under rotation and Galilean transformations.

A brief summary of our conclusions follows. Throughout we assume at least three space dimensions and the primitive variables assembled in minimal, rotation invariant subsets, and use the term vector field to describe such a subset with more than one element.

For such systems with only one vector field, possibly after a change of basis the vector field transforms as a fluid momentum. Such systems correspond to Euler systems, possibly with additional equations of “conservation of mass” form.

We consider systems with two vector fields, a fluid momentum and a “magnetic field” $B$, assumed solenoidal, Galilean invariant, and contributing $\frac{1}{2} |B|^2$ to the local energy density. We show that for multidimensional systems, if the solenoidal magnetic field condition is explicitly included in the primitive equations, then the property of finite signal propagation speed is lost. This is avoided in various forms of the Lundquist system [2,5,8,10] by making $\partial B/\partial t$ solenoidal, tacitly assuming solenoidal initial data for $B$.

For such systems corresponding to multi-fluid flow without chemical interaction, conservation of mass of each separate species implies conservation of momentum of each separate species, using a pressure for each species which is independent of the density of the other species. Whatever coupling of the flows is through the energy equation(s) or some other variable, which may nonetheless admit interpretation as a void fraction.

The condition of incompressibility is also shown to be incompatible with coupled multi-fluid flow.

2. Notation

Our point of departure is the observation that systems of conservation laws with convex entropies admit representation in symmetric or gradient form [7,11], and that the symmetric form is useful for describing symmetries [13,15].

Thus we consider a system

$$w_t + \nabla_x \cdot q(w) = 0, \quad (2.1)$$

with

$$x \in \mathbb{R}^m, \quad (2.2)$$

$$w(x, t) \in D_w \subset \mathbb{R}^n, \quad (2.3)$$

regarding the primitive variables $w$ as forming an $n$-row vector for convenience in subsequent notation. In (2.1), $q$ is a smooth map of $D_w$ into the space of $m \times n$ matrices.

Our systems are assumed equipped with a convex entropy $W : D_w \to \mathbb{R}$ and corresponding flux $Q : D_w \to \mathbb{R}^m$, so admissible weak solutions of (2.1) satisfy [9]

$$W(w)_t + \nabla_x \cdot Q(w) \leq 0 \quad (2.4)$$

in the sense of distributions. It is assumed that $(W, Q)$ is unique (up to additive constants or constant multiples of $(w, q)$, or multiplication by a positive constant).

Such systems admit symmetric or gradient form using “symmetric dependent variables” $z$

$$z(x, t) \in D_z \subset \mathbb{R}^n, \quad (2.5)$$

obtained implicitly from
which must hold for all sufficiently smooth \( w \), whether or not (2.1) is satisfied.

We assume throughout that the components of \( z \), determined from (2.6), are independent, so \( \{ z \} \) are local coordinates for phase space.

Then there exist potential functions \( \phi : D_z \to \mathbb{R}, \psi : D_z \to \mathbb{R}^m \), such that

\[
\begin{align*}
\phi(z) &= w \cdot z - W(w), \\
\psi(z) &= q(w) \cdot z - Q(w).
\end{align*}
\] (2.10)

For any \( j = 1, \ldots, n \) such that \( w_j \) in (2.1) is not identically constant, the corresponding \( z_j \) is given by

\[
z_j = W_{w_j}(w). \] (2.9)

However in the following, to allow for incompressible flow or solenoidal magnetic fields, we allow for constant \( w_j \), independent of \( x, t \). The corresponding \( z_j \) are not in general constants, and are not determined as functions of \( w \) by (2.9).

The potential functions \( \phi, \psi \) are Lagrange duals of \( W, Q \), satisfying

\[
\phi(z) = w \cdot z - W(w), \quad \psi(z) = q(w) \cdot z - Q(w). \] (2.11)

Anticipating that the entropy density \( W \) need not be invariant under Galilean transformations, we express the symmetries in terms of trivially extended dependent variables. We denote by

\[
\begin{align*}
\bar{W} &\equiv (w \ W(w)), \\
\bar{q}(w) &\equiv (q(w) \ Q(w)), \\
\bar{z} &\equiv \begin{pmatrix} \xi z \\ -\xi \end{pmatrix},
\end{align*}
\] (2.12)

a row vector of dimension \( n + 1 \);

\[
\bar{q}(w) \equiv (q(w) \ Q(w)), \] (2.13)

a matrix of dimension \( m \times (n + 1) \);

\[
\bar{z} \equiv \begin{pmatrix} \xi z \\ -\xi \end{pmatrix}, \] (2.14)

with

\[
\xi > 0 \] (2.15)

regarded throughout as a supplemental dependent variable, the dependence of which on \( x, t \) is arbitrary, unimportant and ignored.
Additionally, we denote by

\[
\bar{\phi}(\bar{z}) \overset{\text{def}}{=} \xi \phi(z); \quad (2.16)
\]

\[
\bar{\psi}(\bar{z}) \overset{\text{def}}{=} \xi \psi(z); \quad (2.17)
\]

\[
\bar{x} \overset{\text{def}}{=} \begin{pmatrix} t \\ x \end{pmatrix}; \quad (2.18)
\]

\[
\theta(\bar{z}) \overset{\text{def}}{=} \begin{pmatrix} \bar{\phi}(\bar{z}) \\ \bar{\psi}(\bar{z}) \end{pmatrix}. \quad (2.19)
\]

Then from (2.12), (2.13), (2.10), (2.11), (2.14), (2.16), (2.17)

\[
\bar{w}(\bar{z}) = \bar{\phi}_2(\bar{z}), \quad (2.20)
\]

and

\[
\bar{q}(w) = \bar{\psi}_2(\bar{z}). \quad (2.21)
\]

From (2.18), (2.19), (2.20), (2.21), we combine the left sides of (2.1), (2.4) in the notation

\[
\left( w_t + \nabla_x \cdot q(w) \quad W_t + \nabla_x \cdot Q(w) \right) = \nabla_{\bar{x}} \cdot \theta_2(\bar{z}). \quad (2.22)
\]

### 3. Symmetries

The symmetries of interest here form one-parameter Lie groups with infinitesimal generators determined from two constant, dimensionless square matrices, \( X \) of dimension \( m + 1 \) and \( A \) of dimension \( n + 1 \).

We require that

\[
A_{n+1,j} = 0, \quad j = 1, \ldots, n, \quad (3.1)
\]

and that for any \( \bar{z} \in D_{\bar{z}} \)

\[
\theta_2(\bar{z}) A \bar{z} = X \theta(\bar{z}). \quad (3.2)
\]

It is in the condition (3.2) where the assumption of a single entropy is needed. Generalization to systems with multiple entropies is complicated, and will be treated in a forthcoming paper.

In the following, we depart from conventional practice and express symmetry transformations as finite transformations as opposed to familiar infinitesimal notation. This facilitates presentation of the differences between rotations and Galilean transformations, essential in the present discussion, particularly in Lemma 3.2 below.

From (3.2), it follows that for any \( \lambda \in \mathbb{R} \) (not necessarily dimensionless) and any \( \bar{z} \in D_{\bar{z}} \),

\[
\theta(e^{\lambda A} \bar{z}) = e^{\lambda X} \theta(\bar{z}). \quad (3.3)
\]

Then identifying

\[
\bar{x}(\lambda) = e^{\lambda X} \bar{x}, \quad (3.4)
\]

\[
\bar{z}(\lambda) = e^{\lambda A} \bar{z}, \quad (3.5)
\]
from (3.3), (3.4), (3.5)

\[ \nabla \tilde{x}(\lambda) \cdot \theta(\tilde{z}(\lambda)) = \nabla \tilde{x} \cdot \theta(\tilde{z}) e^{-\lambda A}. \quad (3.6) \]

From (3.1), for all \( \lambda \in \mathbb{R} \)

\[ (e^{\lambda A})_{n+1,j} = 0, \quad j = 1, \ldots, n, \quad (3.7) \]
\[ (e^{\lambda A})_{n+1,n+1} = e^{\lambda A_{n+1,n+1}} \quad (3.8) \]

so from (2.22), (3.6), (3.7), (3.8), admissible weak solutions of (2.1), (2.4) survive such transformations. From (2.14), (3.5), (3.7), (3.8), we have

\[ \xi(\lambda) = \xi e^{\lambda A_{n+1,n+1}} \quad (3.9) \]

so (2.15) also survives.

Under a change of basis, of the form

\[ \hat{z}_\Lambda = \Lambda \hat{z} \quad (3.10) \]

with \( \Lambda \) a dimensionless nonsingular matrix of dimension \( n + 1 \) and satisfying

\[ \Lambda_{j,n+1} = 0, \quad j = 1, \ldots, n, \quad (3.11) \]

the conditions (3.1), (3.3) survive with

\[ X_\Lambda = X, \quad (3.12) \]
\[ A_\Lambda = \Lambda A \Lambda^{-1}. \quad (3.13) \]

The simplest examples of such symmetries correspond to scaling, with \( \lambda \) dimensionless and \( X, A \) diagonal. A trivial example is with \( X, A \) identity matrices, obtaining

\[ \tilde{x}(\lambda) = e^{\lambda} \tilde{x}, \quad (3.14) \]
\[ \tilde{z}(\lambda) = e^{\lambda} \tilde{z} \quad (3.15) \]

from (3.4), (3.5). From (2.16), (2.17), (2.19), (2.14), \( \theta \) is homogeneous of degree one in \( \tilde{z} \), so (3.3), assuming the form

\[ \theta(e^{\lambda} \tilde{z}) = e^{\lambda} \theta(\tilde{z}) \quad (3.16) \]

is necessarily satisfied. Then from (2.19), (2.20), \( \tilde{w} \) is invariant

\[ \tilde{w}(e^{\lambda} \tilde{z}) = \tilde{w}(\tilde{z}) \quad (3.17) \]

and solutions of (2.1), (2.4) obviously survive.

The inclusion of such symmetries, however, aids in describing the region \( D_{\tilde{z}} \). In particular, from (2.9), the arbitrary linear multiple of \( w \) in \( W \) implies arbitrary additive constants in \( z \). This is largely removed by the following.
Lemma 3.1. Assume $W$, $z$ determined such that the symmetries with $X$, $A$ identity matrices are included in the symmetry group for a system (2.1), and that for some $j = 1, \ldots, n$ that $\bar{z}_j$ does not vanish within $\bar{D}_z$. Then without loss of generality, we may assume that $\bar{z}_j$ assumes values only in $(-\infty, 0)$.

Proof. As the range of $\bar{z}_j$ has to be connected, using (3.15) it follows that if the range of $\bar{z}_j$ is not all of $\mathbb{R}$, it is either $(-\infty, 0)$ or $(0, \infty)$. Replacing $\bar{z}_j$ by $-\bar{z}_j$ if necessary ($\bar{w}_j$ by $-\bar{w}_j$) it is no loss of generality to assume the former. The entire symmetry group survives by application of (3.12), (3.13). □

Rotation symmetries also correspond to $\lambda$ dimensionless, interpreted as the angle of rotation. The plane of rotation is determined from two orthonormal vectors $\mu, \nu \in \mathbb{R}^m$, setting

$$R_{ij} = \mu_i\nu_j - \mu_j\nu_i, \quad i, j = 1, \ldots, m,$$

(3.18)

with the corresponding matrix $X_R$ given by

$$X_R = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix}$$

(3.19)

We denote by $\{R\}$ the set of all matrices $R$ which are so obtained.

For any $R \in \{R\}$ we denote by $A_R$ the corresponding matrix $A$ satisfying (3.1), (3.3) with $X$ obtained from (3.19). With $\bar{z}$ assembled in minimal rotation invariant subsets, the matrices $A_R$ are necessarily block diagonal, with block structure independent of $R \in \{R\}$.

Galilean symmetries correspond to

$$[\lambda] = [x/t]$$

(3.20)

where here and below, $[ ]$ means “dimensions of”. As the Galilean transformation subgroup is commutative, we may describe such symmetries by a velocity vector $s \in \mathbb{R}^m, [s] = [x/t]$ with $\lambda X, \lambda A$ replaced in (3.3) by

$$X_s = \begin{pmatrix} 0 & 0 \\ \cdots & -s \\ 0 & 0 \end{pmatrix}$$

(3.21)

and

$$A_s = \sum_{i=1}^{m} s_i A_i$$

(3.22)

with the matrices $A_i, i = 1, \ldots, m$ independent of $\bar{z}$ and $s$.

The symmetry group associated with our system (2.1) is assumed throughout to include the trivial scaling symmetry (3.14), (3.15), the rotation symmetry for any $R \in \{R\}$, and the Galilean symmetry for any $s \in \mathbb{R}^m$.

From this assumption we obtain conditions on the corresponding matrices $A_R, A_s$.

From (3.19), (3.21), for rotation or Galilean symmetries, the first row of $X_R$ or $X_s$ vanishes identically. From (2.19), (3.3) this implies the function $\bar{\phi}$ invariant,

$$\bar{\phi}(e^{\lambda A_R} \bar{z}) = \bar{\phi}(\bar{z}),$$

(3.23)

$$\bar{\phi}(e^{A_s} \bar{z}) = \bar{\phi}(\bar{z})$$

(3.24)

for all $R \in \{R\}, s \in \mathbb{R}^m, \bar{z} \in \bar{D}_z$. 

From (2.20), (3.23), (3.24) we find the corresponding transformations of \( \bar{w} \)

\[
\bar{w}(e^{\lambda A_R} \bar{z}) = \bar{w}(\bar{z}) e^{-\lambda A_R},
\]

(3.25)

\[
\bar{w}(e^{A_i} \bar{z}) = \bar{w}(\bar{z}) e^{-A_i}.
\]

(3.26)

Differentiating (3.23) with respect to \( \lambda \) at \( \lambda = 0 \) (respectively differentiating (3.24) with respect to \( s \) at \( s = 0 \)) and using (2.20), we obtain the conditions

\[
\bar{w}(\bar{z}) A_R \bar{z} = 0,
\]

(3.27)

\[
\bar{w}(\bar{z}) A_s \bar{z} = 0
\]

(3.28)

for all \( R \in \{R\}, s \in \mathbb{R}^m, \bar{z} \in D_{\bar{z}}. \)

The matrices \( A_i, A_R \) do not commute, but they satisfy the following.

**Lemma 3.2.** For any \( i = 1, \ldots, m \) and any \( R \in \{R\} \),

\[
A_R A_i - A_i A_R = \sum_{j=1}^{m} R_{ij} A_j.
\]

(3.29)

**Proof.** For any \( \lambda \in \mathbb{R}, R \in \{R\}, s \in \mathbb{R}^m \) we denote by

\[
s(\lambda) = e^{\lambda R} s,
\]

(3.30)

the velocity after a rotation. From (3.5), for any \( \bar{z} \in D_{\bar{z}} \), necessarily

\[
e^{\lambda A_R} e^{A_i} \bar{z} = e^{A_i \lambda} e^{\lambda A_R} \bar{z}.
\]

(3.31)

Since \(|s| > 0\) can be arbitrarily small, (3.22), (3.30), (3.31) imply

\[
\sum_{j=1}^{m} s_j A_j = e^{-\lambda A_R} \left( \sum_{j=1}^{m} (e^{\lambda R} s)_j A_j \right) e^{\lambda A_R}.
\]

(3.32)

The derivative of (3.32) with respect to \( \lambda \) at \( \lambda = 0 \) gives

\[
-A_R \left( \sum_{j=1}^{m} s_j A_j \right) + \sum_{j=1}^{m} (Rs)_j A_j + \left( \sum_{j=1}^{m} s_j A_j \right) A_R = 0.
\]

(3.33)

Setting \( s = s_i \) in (3.33) gives (3.29). \( \square \)

**4. Some examples**

Here we display the dependent variables \( \bar{w}, \bar{z} \) and the nonzero blocks of the matrices \( A_R, A_s \) for some examples of systems (2.1) with rotation and Galilean symmetries. We regard \( \bar{w} \) and its rotation and Galilean transformations as having been specified, thus determining \( \bar{z} \) from (2.9), (2.14), \( A_R \) from (3.25), \( A_s \) from (3.26), and \( \phi \) from (2.10). At this stage the flux functions \( \bar{q} \) remain unspecified.
Consistency requires that the $A_R$, $A_s$, and $\bar{z}$ so obtained satisfy the conditions (3.27), (3.28), (3.29); otherwise such a system does not exist.

We employ standard notation, with $\rho$, $p$, $V$, $P$, $S$, $T$, $E$, $H$, $G$ respectively denoting mass density, $(m\text{-row vector})$ momentum density, specific volume, pressure, entropy, temperature, internal energy, enthalpy, and Gibbs free energy of a fluid, satisfying

$$\rho = 1/V;$$  
$$H = E + PV, \quad HP = V;$$  
$$G = H - TS, \quad GP = V, \quad GT = -S.$$  

Our first example is the adiabatic Euler system, for which

$$\phi(z) = P$$  
and

$$\bar{w}^\dagger = \bar{z} \text{ nonzero blocks of } A_R, A_s$$  

$$\rho \quad \xi(H - \frac{1}{2} \frac{|p|^2}{\rho^2})$$  

$$p \quad \xi \frac{p}{\rho}$$  

$$\frac{1}{2} \frac{|p|^2}{\rho} + \rho E \quad -\xi$$

For the Euler system including the equation of conservation of energy, we have closely related expressions,

$$\phi(z) = P/T,$$

$$\bar{w} = \left( \rho \quad p \quad \frac{1}{2} \frac{|p|^2}{\rho} + \rho E \quad -\rho S \right),$$

and the nonzero blocks of $A_R, A_s$ are the same as in the adiabatic case.

Our next example is a simple magnetohydrodynamics model, with primitive variables corresponding to an adiabatic fluid and a magnetic field $B$, also of dimension $m$, which we specify as transforming like a vector under rotation, Galilean invariant, solenoidal, and contributing $\frac{1}{2} |B|^2$ to the local energy density. (For this system, the local energy density is $W$ in (2.4).)

For such a system, one component of $w$ vanishes identically, corresponding to the anticipated equation

$$\nabla_x \cdot B = 0$$

within the given system, included among the equations obtained from (2.7), (2.8).

The corresponding component of $z$, denoted by $\zeta$, is not determined by $w$ and must be determined using (4.9).
We nonetheless obtain the remaining components of $z$ from (2.9) and $\phi$ from (2.10),

$$\phi(z) = P + \frac{1}{2} |B|^2$$

(4.10)

and

$$\bar{w}^\dagger \quad \bar{z} \quad \text{nonzero blocks of } A_R, A_s$$

$$\rho \quad \xi (H - \frac{1}{2} |p|^2 / \rho^2)$$

$$p^\dagger \quad \xi p^\dagger / \rho$$

$$B \quad \xi B$$

$$0 \quad \xi$$

$$\frac{1}{2} |p|^2 / \rho + \rho E + \frac{1}{2} |B|^2 \quad -\xi$$

(4.11)

Our final example is of two-fluid flow, with a common fluid temperature. This corresponds to primitive variables including the mass and momentum densities of each fluid, and the sum of the energy densities. For each fluid species an equation of state is given, corresponding to expressions $G_1(P_1, T)$ and $G_2(P_2, T)$. For this system

$$\phi(z) = (P_1 + P_2) / T$$

(4.12)

and

$$\bar{w}^\dagger \quad \bar{z} \quad \text{nonzero blocks of } A_R, A_s$$

$$\rho_1 \quad \xi (G_1 - \frac{|p_1|^2}{\rho_1^2}) / T$$

$$p_1^\dagger \quad \xi p_1^\dagger / (\rho_1 T)$$

$$\rho_2 \quad \xi (G_2 - \frac{|p_2|^2}{\rho_2^2}) / T$$

$$p_2^\dagger \quad \xi p_2^\dagger / (\rho_2 T)$$

$$\frac{1}{2} |p_1|^2 / \rho_1 + \rho_1 E_1 + \frac{1}{2} |p_2|^2 / \rho_2 + \rho_2 E_1 \quad -\xi / T$$

$$-\rho_1 S_1 - \rho_2 S_2 \quad -\xi$$

(4.13)

5. The structures of $\bar{z}, A_R, A_s$

The obvious similarities in the above examples are not accidental; here we show how they arise.

The entropy density $W$ convex in $w$ implies $\phi$ convex in $z$. As an additive affine function of $z$ is unimportant in $\phi, \psi$ it is no loss of generality to assume $\phi$ nonnegative,
\[ \phi(D_z) \subset [0, \infty), \]  
\quad (5.1)

and \( \tilde{\psi}(\tilde{z})/\tilde{\phi}(\tilde{z}) \) continuous where \( \tilde{\phi}(\tilde{z}) \) vanishes.

Using (5.1), (3.3), (2.19), (3.23), (3.24), (3.19), (3.21), the \( m \)-vector

\[ v(\bar{z}) \overset{\text{def}}{=} \tilde{\psi}(\bar{z})/\tilde{\phi}(\bar{z}) \]  
\quad (5.2)

is homogeneous of degree zero in \( \bar{z} \), and transforms like a velocity vector for all \( \bar{z} \in D_{\bar{z}} \), satisfying

\[ v(e^{\lambda R} \bar{z}) = e^{\lambda R} v(\bar{z}), \]  
\quad (5.3)

\[ v(e^{A_i \bar{z}}) = v(\bar{z}) - s. \]  
\quad (5.4)

For (5.4) to hold, there must be components of \( \bar{z} \) which are not Galilean invariant. Making a change of basis for \( \bar{z} \) as necessary, for each \( i = 1, \ldots, m \), there are necessarily value(s) \( j_i \in (1, \ldots, n) \) such that

\[ (A_i \bar{z}) \mid_{j_i} \neq 0 \]  
\quad (5.5)

for all \( \bar{z} \in D_{\bar{z}} \). Using (3.5), (3.21), (3.22), this is possible only if there are value(s) \( \ell_i \in (1, \ldots, n + 1) \) such that

\[ A_{i, j_i \ell_i} \neq 0 \]  
\quad (5.6)

and such that for all \( \bar{z} \in D_{\bar{z}} \),

\[ \bar{z}_{\ell_i} \neq 0. \]  
\quad (5.7)

We denote by

\[ J \overset{\text{def}}{=} \bigcup_{i=1}^{m} \{j_i\} \]  
\quad (5.8)

\[ L \overset{\text{def}}{=} \bigcup_{i=1}^{m} \{\ell_i\} \]  
\quad (5.9)

For any \( \bar{z} \in D_{\bar{z}} \) we denote by \( Z_1 \) the subset of \( \bar{z} \) with \( d_1 \) elements, those components \( \bar{z}_j \) with \( j \in J \). The set \( \{Z_1 \mid \bar{z} \in D_{\bar{z}}\} \) is isomorphic to a subspace of \( D_{\bar{z}} \) of dimension \( d_1 \).

Similarly, we denote by \( Z_0 \) the subset of \( \bar{z} \) with \( d_0 \) elements, those components \( \bar{z}_l \) with \( l \in L \). The set \( \{Z_0 \mid \bar{z} \in D_{\bar{z}}\} \) is isomorphic to a solid cone of dimension \( d_0 \) within \( D_{\bar{z}} \).

Then from (5.5), (5.6), (5.7), we have a relation of the form

\[ A_i \bar{z} \mid_{Z_1} = a_i Z_0, \quad i = 1, \ldots, m, \]  
\quad (5.10)

where \( A_i \bar{z} \mid_{Z_1} \) is a \( d_1 \)-vector, those components \( (A_i \bar{z}) \mid_{j} \) with \( j \in J \). Similar notation is used below.

Also in (5.10), each of the \( a_i \) is a \( d_1 \times d_0 \) matrix with the \( A_{i, j_i \ell_i} \) appearing in (5.6) as its nonzero elements.

Assembling the \( Z_1, Z_0 \) as blocks within \( \bar{z} \), we locate the relevant blocks of \( A_R, A_i \) according to the following diagram, in the same notation as used in Section 4.
\[
\begin{pmatrix}
\cdots & Z_1 & Z_0 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\vdots & \vdots & \vdots & \ddots \\
A_R, A_i : \\
Z_1 & & & \\
\vdots & & & \\
\vdots & & & \\
A_R | Z_1, A_i = a_i \\
\vdots & & & \\
Z_0 & & & \\
\vdots & & & \\
\vdots & & & \\
& & & \\
& & & \\
\end{pmatrix}
\]

From (5.5), (5.8), for each \( j = 1, \ldots, d_1 \), we have

\[ a_{i,j\ell} \neq 0 \] (5.12)

for some \( i = 1, \ldots, m \), \( \ell = 1, \ldots, d_0 \) both depending on \( j \). From (5.7), (5.9), and the definition of \( Z_0 \) above, by application of Lemma 3.1, it is no loss of generality to assume

\[ Z_{0,\ell} \in (-\infty, 0), \quad \ell = 1, \ldots, d_0. \] (5.13)

As the subspace \( \{ Z_1 \} \) so constructed is clearly rotation invariant, it is a union of vector fields within \( \tilde{z} \). However in the following, we shall tacitly assume that \( Z_1 \) is a single vector field of dimension \( d_1 \), satisfying (5.10), (5.11), (5.12), (5.13) with some \( Z_0 \) of dimension \( d_0 \), making a change of basis as necessary, reducing the values of \( d_1, d_0 \) and truncating the matrices \( a_i \) as necessary. Clearly this involves no loss of generality, and the results below will apply to each such vector field within \( \tilde{z} \) when uniqueness of \( Z_1, Z_0 \) is lost by this procedure.

In the Euler systems above, \( Z_1 \) is \( \xi p^1/\rho \) or \( \xi p^1/(\rho T) \); \( Z_0 \) is \( -\xi \) or \( -\xi/T \). For the two-fluid model of Section 4, \( Z_1 \) and \( Z_0 \) are not uniquely so determined.

From (5.13) it follows that \( Z_0 \) is necessarily rotation and Galilean invariant, in particular for all \( R \in \{ R \} \).

\[ A_R \mid_{Z_0} = 0. \] (5.14)

Applying (3.29) to the block of \( A_R A_i, A_j A_R \) and \( A_j \) corresponding to \( a_i \), using (5.14), we have for all \( R \in \{ R \} \), \( i = 1, \ldots, m \), \( j = 1, \ldots, d_1 \), \( \ell = 1, \ldots, d_0 \)

\[ \sum_{k=1}^{d_1} (A_R \mid_{Z_1})_{jk} a_{i,k\ell} = \sum_{k=1}^{m} R_{k\ell} a_{k,j\ell}. \] (5.15)

which implies that

\[ d_1 \geq m. \] (5.16)

Making a basis change for \( Z_0 \) as necessary, it is no loss of generality to assume that the first column of at least one of the \( a_i \) does not vanish identically,

\[ a_{i,1} \neq 0. \] (5.17)
Lemma 5.1. Assume that (5.17) holds for some \( i = 1, \ldots, m \). Then the \( d_1 \)-vectors \( a_{1,1}, a_{2,1}, \ldots, a_{m,1} \) are linearly independent.

**Proof.** If not, there exists a unit vector \( \mu \in \mathbb{R}^m \) such that

\[
\sum_{i=1}^{m} \mu_i a_{i,j} = 0, \quad j = 1, \ldots, d_1. \tag{5.18}
\]

For any \( \nu \in \mathbb{R}^m \) orthonormal to \( \mu \), we obtain \( R \) from (3.18) and set \( \ell = 1 \) in (5.15), obtaining

\[
\sum_{k=1}^{d_1} (A_R | Z_1)_{jk} a_{k,j1} = \sum_{k=1}^{m} (\mu_k v_i - \mu_i v_k) a_{k,j1} \tag{5.19}
\]

for any \( i = 1, \ldots, m \), \( j = 1, \ldots, d_1 \).

Multiplying (5.19) by \( \mu_i \) and summing over \( i \), using (5.18), \( \nu \) orthogonal to \( \mu \) and \( \mu \) of unit length, we have for any \( j = 1, \ldots, d_1 \)

\[
0 = \sum_{k=1}^{d_1} (A_R | Z_1)_{jk} \sum_{i=1}^{m} \mu_i a_{i,j1} - \sum_{i=1}^{m} \mu_i v_i \sum_{k=1}^{m} \mu_k a_{k,j1} + \sum_{i=1}^{m} \mu_i^2 \sum_{k=1}^{m} v_k a_{k,j1} \]

\[
= \sum_{k=1}^{m} v_k a_{k,j1}. \tag{5.20}
\]

Since \( \nu \in \mathbb{R}^m \) orthonormal to \( \mu \) is otherwise arbitrary, (5.17) and (5.20) are incompatible. \( \square \)

Using (5.16), (5.12) and Lemma 5.1, making a basis change for \( \{Z_1\} \) as necessary, without loss of generality hereafter we assume that

\[
d_1 = m \tag{5.21}
\]

and

\[
a_{i,j1} = \delta_{ij}, \quad i, j = 1, \ldots, m, \tag{5.22}
\]

with \( \delta \) the Kronecker delta here and throughout.

**Lemma 5.2.** For all \( R \in \mathbb{R} \)

\[
A_R | Z_1 = R. \tag{5.23}
\]

**Proof.** Setting \( \ell = 1 \) in (5.15), this is immediate using (5.22). \( \square \)

**Lemma 5.3.** Assume \( m \geq 3 \). Then each of the matrices \( a_1, \ldots, a_m \), each of dimension \( m \times d_0 \), is of rank one.
Proof. For any \( \beta, \gamma = 1, \ldots, m, \beta \neq \gamma \), we set

\[
\mu_i = \delta_{i\beta}, \quad v_i = \delta_{i\gamma}, \quad i = 1, \ldots, m,
\]

(5.24) in (3.18), obtaining \( R \). Using such \( R \) and (5.23) in (5.15), we obtain for any \( i, j = 1, \ldots, m, \ell = 1, \ldots, d_0 \)

\[
\delta_{j\beta} a_{i,\gamma \ell} - \delta_{j\gamma} a_{i,\beta \ell} = \delta_{i\gamma} a_{\beta, j \ell} - \delta_{i\beta} a_{\gamma, j \ell}.
\]

(5.25)

Setting \( i = \gamma, j = \beta \) in (5.25) we obtain

\[
a_{i,i \ell} = a_{j,j \ell}, \quad i, j = 1, \ldots, m, \ell = 1, \ldots, d_0.
\]

(5.26)

Setting \( i = \gamma, j \neq \beta, j \neq \gamma \) (here needing \( m \geq 3 \)) in (5.25), we find that for all such \( j, \beta, \ell \),

\[
a_{\beta,j \ell} = 0.
\]

(5.27)

Now (5.26) and (5.27) are compatible only if there are constants \( \kappa_{\ell}, \ell = 1, \ldots, d_0 \), such that

\[
a_{i,j \ell} = \delta_{ij} \kappa_{\ell}, \quad i, j = 1, \ldots, m, \ell = 1, \ldots, d_0.
\]

(5.28)

Thus making a basis change for \( \{ Z_0 \} \) as necessary, we assume hereafter that

\[
d_0 = 1,\]

(5.29)

and thus that each of the \( a_i, i = 1, \ldots, m \), is an \( m \)-vector satisfying (5.22).

Assuming that (5.21), (5.22), (5.23), (5.29) hold, we rewrite (5.10) as

\[
A_{\tilde{z}} \bigg|_{Z_1=Z_0} = \delta_{ij} Z_0, \quad i, j = 1, \ldots, m.
\]

(5.30)

Next we denote a (row) \( m \)-vector

\[
p(\tilde{z}) \overset{\text{def}}{=} \phi_{Z_1}(\tilde{z})
\]

(5.31)

which is necessarily components of \( w \) from (2.20), and a (column) \( m \)-vector

\[
u(\tilde{z}) \overset{\text{def}}{=} -Z_1 / Z_0.
\]

(5.32)

The functions \( p, u \) coincide with those appearing in the examples of Section 4.

From (5.23), (5.30), (5.32), the rotation and Galilean symmetry transformations of \( u \) coincide with those of \( v \), (5.3), (5.4). In particular, for all \( \tilde{z} \in D_{\tilde{z}}, R \in \{ R \}, \lambda \in \mathbb{R}, s \in \mathbb{R}^m \)

\[
u(e^{\lambda A} \tilde{z}) = e^{\lambda R} u(\tilde{z}),
\]

(5.33)

and

\[
u(e^{\Delta s} \tilde{z}) = u(\tilde{z}) - s.
\]

(5.34)
As $R$ is always antisymmetric from (3.18) and $A_R$ is block diagonal, from (3.25), using (5.31), (5.23) we have
\[ p^\dagger(e^{\lambda A_R} \bar{z}) = e^{\lambda R} p^\dagger(\bar{z}) \quad (5.35) \]
for all $\lambda \in \mathbb{R}$, $R \in \{R\}$, $\bar{z} \in D_{\bar{z}}$.

**Theorem 5.4.** Assume $Z_1, Z_0$ satisfy (5.21), (5.29), (5.13), (5.23), (5.30) and $p, u$ obtained from (5.31), (5.32). Then the following four conditions are equivalent.

There exists a scalar, rotation and Galilean invariant function $\rho(\bar{z})$ such that
\[ p^\dagger(\bar{z}) = \rho(\bar{z})u(\bar{z}); \quad (5.36) \]

There exists a scalar, rotation invariant component $Z_2$ of $\bar{z}$ such that
\[ A_i\bar{z} \big|_{Z_2} = Z_{1,i}, \quad i = 1, \ldots, m, \quad (5.37) \]
and such that $\bar{\phi}$ depends on $Z_1, Z_2$ only as a function of $\tau$,
\[ \tau \overset{\text{def}}{=} -\frac{Z_2}{Z_0} + \frac{1}{2} \frac{|Z_1|^2}{Z_0^2}; \quad (5.38) \]

For all $\bar{z} \in D_{\bar{z}}, R \in \{R\}$
\[ p(\bar{z})Ru(\bar{z}) = 0, \quad (5.39) \]
and there exists a Galilean invariant function $\tilde{\rho}(\bar{z})$ such that for all $\bar{z} \in D_{\bar{z}}, s \in \mathbb{R}^m$,
\[ p(e^{A_i\bar{z}}) p(\bar{z}) - s^\dagger \tilde{\rho}(\bar{z}); \quad (5.40) \]

There exists $Z_2$ satisfying (5.37) and such that for all $\bar{z} \in D_{\bar{z}}$,
\[ \bar{\phi}_{Z_2}(\bar{z}) A_i\bar{z} \big|_{Z_2} + \bar{\phi}_{Z_1} A_i\bar{z} \big|_{Z_1} = 0, \quad i = 1, \ldots, m. \quad (5.41) \]

**Remarks.** These conditions hold in each of the examples of Section 4, with $\rho$ as appearing therein and $\tau$ one of $H, G/T, G_1/T, G_2/T$.

For a given example, the conditions (5.39) or (5.41) may be expected to follow from (3.27) or (3.28), respectively, and the condition (5.40) by inspection; $\tilde{\rho}$ will always coincide with $\rho$. For systems where Theorem 5.4 holds, there necessarily exist hierarchies within $\bar{w}, \bar{z}$, determined by dimensions,
\[ [\bar{\phi}_{Z_0}] = [x/t][\rho] = [x^2/t^2][\rho], \quad (5.42) \]
\[ [Z_2] = [x/t][Z_1] = [x^2/t^2][Z_0]. \quad (5.43) \]

However the Galilean invariance of $\rho$ precludes continuation thereof. Each of the $(n+1)$-vectors $A_i\bar{z}, i = 1, \ldots, m$, depends on $Z_1, Z_0$, but not on $Z_2$.

Using (5.38), the required convexity of $\phi$ in $\bar{z}$ requires $\bar{\phi}_\tau, \bar{\phi}_{\tau\tau}$, nonnegative. Incompressible flow is recovered with $\phi$ linear in $\tau$. 
Proof of Theorem 5.4. The condition (5.36) follows from (5.37), (5.38), using (5.31). Furthermore, we have the identification

$$\rho(\bar{z}) \overset{\text{def}}{=} \bar{\phi}_{Z_2}(\bar{z}) = -\frac{1}{Z_0} \bar{\phi}_{\tau}(\bar{z}).$$

(5.44)

From (5.30), (5.37), it follows that $\tau$ is rotation and Galilean invariant. The rotation and Galilean invariance of $\rho$ now follows from (5.44), (3.23), (3.24) and the rotation and Galilean invariance of $Z_0$.

Next, we show that (5.37), (5.38) follows from (5.41), (5.37). Using (5.37), (5.30), we rewrite (5.41) as

$$\bar{\phi}_{Z_2}(\bar{z}) Z_1, i + \bar{\phi}_{Z_1, i}(\bar{z}) Z_0 = 0,$$  

(5.45)

from which (5.38) follows.

Finally we show that (5.39), (5.40) imply (5.37), (5.45). From the possible forms of $R \in \{R\}$ obtained from (3.18), it follows that (5.39) and (5.40) imply (5.36). Then from (5.36), (5.34), (5.40) it follows that $\bar{\rho} = \rho$ holds in (5.40).

Thus from (3.26), again $\rho$ is a component of $w$, and (5.44) follows from (2.20). Then (5.37) follows from (3.26), (5.40), (5.44), (3.5). Now (5.45) follows from (5.31), (5.32), (5.36), (5.44). □

6. The flux functions

Here we assume $\bar{w}, \bar{\phi}, \bar{z}$ given or obtained satisfying (2.7), (2.9), (2.10), (2.12), (2.16), (2.14), the symmetry transformation matrices $A_R, A_s$, given satisfying the consistency conditions (3.27), (3.28), (3.29), and $m \geq 3$ so Lemma 5.3 holds. We may assume a basis for $\bar{z}$ such that (5.21), (5.23), (5.29), (5.30) hold for some $Z_1, Z_0$ with

$$Z_0 < 0$$

(6.1)

and $p, u$ determined from (5.31), (5.32). Our objective is to determine whether such a system is possible, and if so to characterize the corresponding flux functions $\bar{q}$.

From (2.20), (2.21), the system (2.1) is rewritten as

$$\left(\bar{\phi}_{Z_j}(\bar{z})\right)_t + \nabla_x \cdot \bar{\psi}_{Z_j}(\bar{z}) = 0, \quad j = 1, \ldots, n,$$

(6.2)

and the entropy inequality (2.4) is

$$\left(\bar{\phi}_\xi(\bar{z})\right)_t + \nabla_x \cdot \bar{\psi}_\xi(\bar{z}) \geq 0.$$

(6.3)

Using (5.2) it is clear that the system is determined to the extent to which the vector $v$ is identified. Even if $Z_1, Z_0, p, u$ are uniquely determined from (5.10), (5.12), (6.1), comparison of (5.3), (5.4) with (5.33), (5.34) generally does not uniquely determine $v$. It is at this stage that whatever additional system specific information is applied; we discuss some examples below.
6.1. The Euler systems

The simplest such examples occur when \( Z_1 \) is the unique vector field within \( \bar{z} \), equivalently \( p \) the unique vector field within \( w \). Then from (5.3), (5.4), (5.33), (5.34), using (5.32), the identification

\[
v(\bar{z}) = u(\bar{z}) = -\frac{Z_1}{Z_0}
\]

is unambiguous.

Thus from (5.2), (6.4)

\[
\bar{\psi}_i(\bar{z}) = u_i(\bar{z})\bar{\phi}(\bar{z}) = -\frac{Z_{1,i}}{Z_0}\bar{\phi}(\bar{z}), \quad i = 1, \ldots, m,
\]

with \( \bar{\psi}_i \) the \( i \)-th component of \( \bar{\psi} \).

**Lemma 6.1.** The conditions (5.39), (5.40) hold for such systems.

**Proof.** Using (5.23), (5.32) and the existence of only one vector field within \( \bar{z} \) or \( w \), the condition (3.27) becomes

\[
0 = pA_R | Z_1 = -Z_0pRu
\]

which implies (5.39), using (6.1). From (6.6), given the possible forms of \( R \), it follows that \( p^\dagger \) and \( u \) are parallel, with

\[
p^\dagger(\bar{z}) = \bar{\rho}(\bar{z})u(\bar{z})
\]

for some scalar function \( \bar{\rho}(\bar{z}) \).

The assumption of only one vector field implies that all nonvanishing elements of \( A_R \) are located within \( A_R | Z_2 \). Using Lemma 3.2, this implies that any nonzero elements of \( A_z \) are in rows or columns corresponding to \( Z_1 \).

Thus the condition (3.28) is equivalent to a relation

\[
0 = \bar{\phi}_{Z_2} A_i \bar{z} | Z_2 + pA_i \bar{z} \bigg| Z_1 = \bar{\phi}_{Z_2} A_i \bar{z} \bigg| Z_2 + p_i Z_0, \quad i = 1, \ldots, m.
\]

In (6.8), \( Z_2 \) is a scalar component of \( \bar{z} \), and \( \bar{\phi}_{Z_2} \) is a scalar component of \( w \). Using (3.26) and the assumption of only one vector field, necessarily \( \bar{\phi}_{Z_2} \) is Galilean invariant, and in addition, using (5.32)

\[
A_i \bar{z} \bigg| Z_2 = \omega_i Z_1 = -Z_0\omega_i u
\]

with each \( \omega_i \) a constant row \( m \)-vector. Now (6.7), (6.8), (6.9) are compatible only if
\[ \tilde{\rho} = c \bar{\phi} Z_2, \]  
\[ \omega_{i,j} = c \delta_{ij}, \quad i, j = 1, \ldots, m, \]  
for some constant \( c \). Thus \( \tilde{\rho} \) is Galilean invariant, and (5.40) follows from (6.7), (6.4), (5.4). \( \square \)

From Theorem 5.4, for any such systems we have \( Z_2 \in \tilde{z}, \rho \in w, \tau \) satisfying (5.36), (5.38), (5.44) and

\[ \tilde{\phi}(\tilde{z}) = \bar{\phi}(\tau, Z_0, \eta, \xi). \]  
(6.12)

In (6.12) and below, \( \xi \eta \) is any remaining components of \( \tilde{z} \), excluding \( Z_0, Z_1, Z_2, -\xi \).

Now using (2.20), (2.21), (6.12), (6.5), we can relate the elements of \( \tilde{q} \) to those of \( \tilde{w} \).

For any \( \tilde{z}_j \) which is not \( Z_0 \) and is not \( Z_1, k \) for some \( k = 1, \ldots, m \), we obtain

\[ \tilde{q}_{ij}(\tilde{z}) = u_i \tilde{w}_j(\tilde{z}), \quad i = 1, \ldots, m. \]  
(6.13)

This includes the cases \( \tilde{z}_j \in \xi \eta, \tilde{z}_j = -\xi \) (if \( Z_0 \neq -\xi \)), and \( \tilde{z}_i = Z_2 \). In this last case, using (5.44), we obtain an equation of "conservation of mass", with \( \tilde{w}_j = \rho \) obtained from (5.44).

For \( \tilde{z}_j = Z_{1,k} \) for some \( k = 1, \ldots, m \), we obtain an equation of "conservation of momentum", with

\[ \tilde{w}_j = p_k \]
\[ = \rho u_k \]  
(6.14)

from (5.31), (5.36), and

\[ \tilde{q}_{ij} = \tilde{w}_j u_i - \delta_{ik} \bar{\phi}(\tilde{z})/Z_0 \]
\[ = \rho u_i u_j + \delta_{ik} P, \quad i = 1, \ldots, m, \]  
(6.15)

using (6.14) and identifying a "pressure" function

\[ P \overset{\text{def}}{=} -\bar{\phi}(\tilde{z})/Z_0. \]  
(6.16)

For \( \tilde{z}_j = Z_0 \), from (6.5), (6.16), (5.32) we obtain

\[ \tilde{q}_{ij} = \tilde{w}_j u_i + \frac{Z_{1,j}}{Z_0} \bar{\phi}(\tilde{z}) \]
\[ = u_i (\tilde{w}_j + P), \quad i = 1, \ldots, m. \]  
(6.17)

This is the equation of "conservation of energy" in the case \( Z_0 \neq -\xi \), and corresponds to the entropy flux in (2.4) or (6.3) otherwise.

These are the familiar Euler systems, possibly with additional equations of the form (6.13) corresponding to \( \eta \). We summarize these results as follows.

**Theorem 6.2.** A system (2.1) with \( m \geq 3 \), equipped with a convex entropy and a symmetry group containing rotation and Galilean symmetries, and containing only one vector field within \( w \) (or \( \tilde{z} \)), is necessarily an Euler system; possibly adiabatic, possibly incompressible, and possibly with supplemental equations of the form (6.13).
6.2. A magnetohydrodynamics model

Our next example is the “adiabatic MHD” model of Section 4, with the primitive variables \( \bar{w} \), the symmetric variables \( \bar{z} \), and the symmetry transformation matrices \( A_R \), \( A_a \), shown in (4.11), (4.10). The thermodynamic variables \( \rho, E, H, P \) satisfy (4.1), (4.2). In this example a solenoidal magnetic field is stipulated (4.9).

**Theorem 6.3.** Assume \( m \geq 3 \). Then the system (2.1), (2.4) corresponding to the model (4.11), (4.9) is of the form

\[
\rho_t + \sum_{i=1}^{m} (\rho u_i + \chi H B_i) x_i = 0; \tag{6.18}
\]

\[
(\rho u_k)_t + \sum_{i=1}^{m} \left( \rho u_i u_k + \delta_{ik} \left( P + \frac{1}{2} |B|^2 \right) + \chi H u_k B_i \right) x_i = 0, \quad k = 1, \ldots, m; \tag{6.19}
\]

\[
B_{k,t} + \sum_{i=1}^{m} \left( u_i B_k + \delta_{ik} (\zeta + \chi) + 2B_i B_k \chi |B|^2 \right) x_i = 0, \quad k = 1, \ldots, m; \tag{6.20}
\]

\[
\sum_{i=1}^{m} B_i x_i = 0; \tag{6.21}
\]

\[
\left( \frac{1}{2} \rho |u|^2 + \rho E + \frac{1}{2} |B|^2 \right)_t + \sum_{i=1}^{m} \left( u_i \left( \frac{1}{2} \rho |u|^2 + \rho H + |B|^2 \right) \right)_t + B_i \left( \left( H + \frac{1}{2} |u|^2 \right) \chi H + 2|B|^2 \chi |B|^2 \right) x_i \leq 0; \tag{6.22}
\]

with some smooth, otherwise arbitrary function

\[ \chi = \chi (H, |B|^2). \]  

**Remarks.** An equation for \( \zeta \) is obtained taking the divergence of (6.20) and using (6.21), obtaining

\[
\Delta \zeta + \Delta \chi + \sum_{i,k=1}^{m} (u_i B_k + 2B_i B_k \chi |B|^2) x_i \leq 0. \tag{6.24}
\]

As (6.24) is elliptic, the property of finite signal propagation speeds is lost in the system (6.18), (6.19), (6.20), (6.21). This result survives removal of the adiabatic approximation, which is made here purely for simplicity of notation.

However, the property of finite signal propagation speed is recovered in the case \( m = 1 \), where (6.24) is integrable and \( \zeta \) is obtained explicitly.

**Proof of Theorem 6.3.** By inspection of (4.11), we identify

\[
Z_1 = \xi p^\dagger / \rho, \tag{6.25}
\]

\[
Z_0 = -\xi \tag{6.26}
\]
satisfying (5.23), (5.30), and obtain \( p, u \) from (5.31), (5.32). As (5.36) holds by inspection, Theorem 5.4 applies.

Using (5.44), by inspection of (4.11) for this system

\[ Z_2 = \xi \left( H - \frac{1}{2} |u|^2 \right), \quad (6.27) \]

and from (5.38), (6.27), (6.25),

\[ \tau = H. \quad (6.28) \]

Again by inspection of (4.11), the rotation and Galilean transformation properties of the magnetic field are

\[ A_{R_{\xi}} \big{|}_{B} = R, \quad (6.29) \]

for all \( R \in \{ R \} \), and

\[ A_{s_{\xi}} \big{|}_{B} = 0, \quad (6.30) \]

for all \( s \in \mathbb{R}^m \).

The condition (6.4) is not valid for this example, but from (5.3), (5.4), (5.33), (5.34), (6.29), (6.30) we have a relation

\[ \nu(\bar{z}) - u(\bar{z}) = \frac{\sigma(\bar{z})}{\phi(\bar{z})} B \quad (6.31) \]

for some scalar function \( \sigma(\bar{z}) \), rotation and Galilean invariant, otherwise arbitrary.

We achieve the solenoidal condition (4.9) by taking

\[ \sigma(\bar{z}) = \xi + \chi(\bar{z}) \quad (6.32) \]

for some scalar function \( \chi \), rotation and Galilean invariant, and independent of \( \xi \). Using (6.28), (6.29), (6.30), necessarily \( \chi \) is of the form (6.23).

Combining (5.2), (5.32), (6.31), (6.32), we have for this system

\[ \bar{\psi}(\bar{z}) = -\frac{Z_1}{Z_0} \phi(\bar{z}) + \xi \psi B + \xi \chi(\psi, |B|^2)B \]

\[ = -\frac{Z_1}{Z_0} \left( p + \frac{1}{2} |B|^2 \right) + \xi \psi B + \xi \chi(\psi, |B|^2)B. \quad (6.33) \]

Using (6.33), (6.28), (5.38), we compute the flux functions \( \bar{q} \) from (2.21).

Choosing \( \bar{z} = Z_2 \), we obtain (6.18).

Choosing \( \bar{z} = Z_{1, k}, k = 1, \ldots, m \), we obtain the corresponding equations (6.19).

Choosing \( \bar{z} = \xi B_k, k = 1, \ldots, m \), we obtain the corresponding equations (6.20).

Choosing \( \bar{z} = \xi \psi \), we obtain (6.21), equivalent to (4.9).

And choosing \( \bar{z} = Z_0 \), we obtain (6.22), identified as the entropy inequality using (6.26). \( \square \)
6.3. Multi-fluid flow

Below we employ the following definitions.

**Definition.** A given system (2.1) is reducible if it contains a closed proper subset of equations. A system for which no such subset exists is irreducible.

**Definition.** For any $K \geq 2$, a $K$-fluid model is an irreducible system (2.1) (equipped with a convex entropy, rotation and Galilean symmetries) with $\bar{z}$ containing exactly $K$ independent vector fields, $Z_1^1, \ldots, Z_1^K$ of the same physical dimensions, each of dimension $m$ and satisfying (5.23), (5.30) with a corresponding scalar, negative-valued $Z_0^1, \ldots, Z_0^K \in \bar{z}$.

We emphasize that the $Z_0^1, \ldots, Z_0^K$ need not be independent; they may be linearly related to each other or to $\xi$.

First we show that the results of Section 5 apply to $K$-fluid models.

**Lemma 6.4.** For a $K$-fluid model with $m \geq 3$, there exist $p_k \in w$, $u_k \in w$, $Z_k^2 \in \bar{z}$, $\tau_k$, $k = 1, \ldots, K$, satisfying (5.31), (5.32), (5.33), (5.34), (5.35), (5.36), (5.37), (5.38), (5.44) separately with each value of $k$, otherwise independent. Furthermore,

$$\bar{\phi}(\bar{z}) = \bar{\phi}(\tau_1, \ldots, \tau_K; Z_0^1, \ldots, Z_0^K; \xi, \eta)$$

(6.34)

where $\xi$, $\eta$ is any remaining components of $\bar{z}$, excluding $Z_0^k$, $Z_1^k$, $Z_2^k$, $k = 1, \ldots, K, \xi$.

**Remark.** Making a basis change for $\bar{z}$ as necessary, it is no loss of generality to assume any such $\xi$, $\eta$ rotation and Galilean invariant.

**Proof of Lemma 6.4.** With each $p_k, u_k$, obtained from (5.31), (5.32), using (5.32), (5.23) for each value of $k$, the condition (3.27) is

$$0 = \sum_{k=1}^K p_k A_{RZ_1} \bigg|_{Z_1^k}$$

$$= \sum_{k=1}^K Z_0^k p_k R u_k. \quad (6.35)$$

As the $Z_1^k$ are independent by assumption, each term in the sum (6.35) must vanish separately. As the $Z_0^k$ are nonvanishing, this establishes (5.39) for each value of $k$.

The remainder of the argument of Lemma 6.1 now establishes (5.40) for each separate value of $k$. In particular, (3.28) can hold for the system (2.1) only if (6.8) holds for each $k$. The conclusions of Lemma 6.4 now follow by application of Theorem 5.4 with each value $k = 1, \ldots, K$.

It remains to show, however, that the $Z_2^1, \ldots, Z_2^K$ so obtained are independent components of $\bar{z}$. Suppose not; then there exists a nontrivial linear relation among the $Z_2^1, \ldots, Z_2^K$, and by a basis change for $\bar{z}$, some $Z_2^k$ is made to vanish identically. For this value of $k$, (6.8) becomes

$$p_k A_1 \bar{z} \bigg|_{Z_1^k} = 0 \quad (6.36)$$

which implies $Z_2^k$ Galilean invariant. This contradicts the assumption of a $K$-fluid model, according to the above definition. □
By assumption, the only vector fields within $\bar{z}$ are $Z^1, \ldots, Z^K$. Thus comparison of (5.2), (5.3), (5.4) with (5.33), (5.34) (for each $k$) implies a decomposition of $\bar{\phi}, \bar{\psi}$ of the form

$$\bar{\phi}(\bar{z}) = \sum_{\ell=1}^{K} \bar{\phi}_\ell(\bar{z}), \quad (6.37)$$

$$\bar{\psi}(\bar{z}) = \sum_{\ell=1}^{K} u_\ell \bar{\phi}_\ell(\bar{z})$$

$$= - \sum_{\ell=1}^{K} \frac{Z^\ell_1}{Z^0_0} \bar{\phi}_\ell(\bar{z}) \quad (6.38)$$

with each $\bar{\phi}_\ell$ of the form (6.34).

Setting $\bar{z}_j = Z^k_2$ in (6.2), using (5.44), (5.38), (6.34), (5.32), the decomposition (6.37), (6.38) implies equations of “conservation of mass” within the system (2.1) of the form

$$\rho_k, t - \sum_{i=1}^{m} \left( \sum_{\ell=1}^{K} \frac{u_\ell, i Z^\ell_0}{Z^0_0} \bar{\phi}_\ell \right) x_i = 0, \quad k = 1, \ldots, K, \quad (6.39)$$

with

$$\rho_k = - \sum_{\ell=1}^{K} \frac{\bar{\phi}_\ell}{Z^\ell_0}. \quad (6.40)$$

Often if not always, however, one stipulates equations of conservation of mass of simpler form

$$\rho_k, t + \sum_{i=1}^{m} (\rho_k u_{k,i}) x_i = 0. \quad (6.41)$$

However, such a requirement strongly affects the form of the system (2.1).

**Theorem 6.5.** For a $K$-fluid model with $m \geq 3$, assume that for some specific value of $k$ the system (2.1) contains an equation of conservation of mass of the form (6.41).

Then the system (2.1) also contains $m$ equations of “conservation of momentum”, of the form

$$(\rho_k u_{k, \beta})_t + \sum_{i=1}^{m} (\rho_k u_{k,i} u_{k, \beta} + \delta_{i\beta} P_k) x_i = 0, \quad \beta = 1, \ldots, m, \quad (6.42)$$

with a “pressure” function given by

$$P_k \overset{\text{def}}{=} \frac{-\bar{\phi}^k}{Z^k_0}. \quad (6.43)$$

Furthermore, for all $\ell \neq k$, $\bar{\phi}_\ell$ is independent of $\tau_k$. 

Proof. For each $\beta = 1, \ldots, m$, successively using (5.36), (5.31), (6.37), (6.34), (5.38) we obtain

$$
\rho_k u_{k, \beta} = p_{k, \beta}
\quad = \tilde{\phi} z_{1, \beta}^k
\quad = \sum_{\ell=1}^{K} \tilde{\phi} \ell z_{k, \beta}^k
\quad = \frac{Z_{k, \beta}^k}{(Z_0^k)^2} \sum_{\ell=1}^{K} \tilde{\phi}_\ell \tau_k.
$$

(6.44)

From (5.44), Eq. (6.41) necessarily corresponds to the choice

$$
\tilde{z}_j = Z_2^k
$$

(6.45)
in (6.2).

Using (6.45) in (2.21), comparison with (6.41) implies

$$
\rho_k u_{k, \beta} = \tilde{\psi}_{\beta, Z_2^k}.
$$

(6.46)

Then using (6.38), (5.38) in (6.46) we obtain

$$
\rho_k u_{k, \beta} = -\left( \sum_{\ell=1}^{K} \frac{Z_{1, \beta}^\ell}{Z_0^\ell} \tilde{\phi} \ell \right) z_{2, \beta}^k
\quad = \sum_{\ell=1}^{K} \frac{Z_{1, \beta}^\ell}{Z_0^\ell Z_0^\ell} \tilde{\phi}_\ell \tau_k.
$$

(6.47)

From the independence of $Z_1^1, \ldots, Z_K^1$, the relations (6.44), (6.47) are compatible only if $\tilde{\phi}^\ell$ is independent of $\tau_k$ for all $\ell \neq k$.

From (5.44), (6.37), (5.38)

$$
\rho_k = \tilde{\phi} Z_2^k
\quad = \phi Z_2^k
\quad = -\phi_{\tau_k}/Z_0^k.
$$

(6.48)

Then from (6.37), (5.38), (6.48), (5.32)

$$
\tilde{\phi} z_{1, \beta}^k = \phi z_{1, \beta}^k
\quad = \frac{Z_{1, \beta}^k}{(Z_0^k)^2} \phi_{\tau_k}^k
\quad = \rho_k u_{k, \beta}.
$$

(6.49)
And from (6.38), (5.38), (5.32), (6.49), (6.43)

\[ \bar{\psi}_{i,k} = \bar{\psi}_{i,k}, \quad \bar{z}_j = \bar{z}_1, \beta \]

\[ = \rho_k u_{k,i}u_{k,\beta} + \delta_{i\beta} P_k. \]  \hfill (6.50)

Now for each \( \beta = 1, \ldots, m \), Eq. (6.42) follows from the choice

\[ \bar{z}_j = \bar{z}_1^k \]  \hfill (6.51)

in (6.2), using (6.49), (6.50). \( \square \)

**Corollary 6.6.** Assume Theorem 6.5 applies for some \( k \) and in addition that \( Z_0^k = -\xi \) and

\[ \bar{\phi}^k(\bar{z}) = \bar{\phi}^k(\tau_k, \xi). \]  \hfill (6.52)

Then the system (2.1) is reducible.

**Proof.** Using (6.52) in (6.43), the subsystem composed of (6.41), (6.42) is a closed Euler system. \( \square \)

**Corollary 6.7.** Suppose Theorem 6.5 applies with every \( k = 1, \ldots, K \), and the system (2.1) is irreducible. Then the dimension \( n \) of the system (2.1) satisfies

\[ n \geq (m + 1)K + 1. \]  \hfill (6.53)

**Proof.** Such a system necessarily includes \( K \) equations (6.41) and \( mK \) equations (6.42). Thus if (6.53) fails, there are no additional equations. In particular, there are no variables \( \eta \) and

\[ Z_0^1 = Z_0^2 = \cdots = Z_0^k = -\xi. \]  \hfill (6.54)

From Theorem 6.5, the condition (6.52) holds for each \( k = 1, \ldots, K \). Thus not only is the system (2.1) reducible in this case, it is simply \( K \) copies of the adiabatic Euler system. \( \square \)

**Corollary 6.8.** Assume that Theorem 6.5 applies with fluid species \( k \) incompressible. Then the system (2.1) is reducible.

**Proof.** For \( \rho_k \) constant, Eqs. (6.41), (6.42) become

\[ \sum_{i=1}^{m} (u_{k,i}) x_i = 0, \]  \hfill (6.55)

\[ (u_{k,\beta})_t + \sum_{i=1}^{m} \left( u_{k,\beta} u_{k,i} + \frac{1}{\rho_k} \delta_{i\beta} P_k \right) x_i = 0, \quad \beta = 1, \ldots, m. \]  \hfill (6.56)

Taking the divergence of (6.56) and using (6.55), we obtain an elliptic equation for \( P_k \), irrespective of the dependence of \( \bar{\phi}^k \) on \( \bar{z} \),

\[ \frac{1}{\rho_k} \Delta P_k = -\sum_{i,\beta=1}^{m} (u_{k,i} u_{k,\beta}) x_i x_{\beta}. \]  \hfill (6.57)

Thus (6.55), (6.56) is a closed proper subset of the system (2.1). \( \square \)
We conclude with a characterization of the simplest models of two-fluid flow.

**Definition.** A system (2.1) is of class \( S \) if it is a model of two-fluid flow with \( m \geq 3 \), is of minimum dimension

\[
n = 2m + 3,
\]

satisfies the assumptions of Theorem 6.5 for \( k = 1 \) and for \( k = 2 \), and is invariant under the interchange of fluids 1 and 2 (including exchange of the equations of state).

Such systems necessarily include two equations of the form (6.41) and \( 2m \) equations of the form (6.42), (6.43). From (6.34), (6.37), (6.43) and Theorem 6.5, the equations of state are expressions of the form

\[
\tilde{\phi}^k = \phi^k (\tau_k, -\tilde{z}_{2m+3}/\xi, \xi), \quad k = 1, 2.
\]

Thus it remains only to characterize Eq. (6.2) with \( j = n = 2m + 3 \) and the entropy inequality (6.3).

**Theorem 6.9.** There are only three forms of systems (2.1) of class \( S \), with the equations of state remaining to be specified.

**Proof.** As components of \( \tilde{z} \), independent of \( Z_1^1, Z_1^2, Z_2^1, Z_2^2 \), there remain \( Z_0^1, Z_0^2 \), and possibly a scalar \( \xi \eta \). Each is a linear combination of \( \tilde{z}_{2m+3} \) and \( \xi \). Thus there must be two independent linear relations of the form

\[
c_1 Z_0^1 + c_2 Z_0^2 + c_3 \xi \eta + c_4 \xi = 0, \tag{6.60}
\]

\[
c_5 Z_0^1 + c_6 Z_0^2 + c_7 \xi \eta + c_8 \xi = 0, \tag{6.61}
\]

holding for all

\[
Z_0^1, Z_0^2 < 0, \quad \xi > 0. \tag{6.62}
\]

If either \( c_3 \) or \( c_7 \) is nonzero, then \( \xi \eta \) may be eliminated from (6.60), (6.61), obtaining a relation

\[
c_9 Z_0^1 + c_{10} Z_0^2 + c_{11} \xi = 0. \tag{6.63}
\]

By assumption, the relation (6.63) must be invariant under interchange of \( Z_0^1, Z_0^2 \). Then using (6.62), as \( Z_0^1, Z_0^2 \) may be scaled as necessary, it is no loss of generality to take (6.63) in one of the two forms

\[
Z_0^1 = Z_0^2, \tag{6.64}
\]

or

\[
Z_0^1 + Z_0^2 = -\xi. \tag{6.65}
\]

In either case, \( \xi \eta \) is obtained explicitly, from (6.64) or (6.65) and (6.60) or (6.61), as a function of \( Z_0^1 \) or \( Z_0^2 \) and \( \xi \). Therefore the dependence on \( \xi \eta \) is not needed in (6.59) and \( \xi \eta \) is simply dropped.
The third possibility is that
\[ c_3 = c_7 = 0 \] (6.66)
in (6.60), (6.61). Then \( Z^1_0 \) and \( Z^2_0 \) depend only on \( \xi \), and without loss of generality we set
\[ Z^1_0 = Z^2_0 = -\xi; \] (6.67)
\[ \bar{\xi} = \bar{z}_{2m+3}. \] (6.68)

As the form of the system (2.1) is determined by the relations between \( Z^1_0, Z^2_0, \xi, \eta, \xi \), this exhausts the possibilities. □

In either case (6.64), (6.65), the resulting system (2.1) is conveniently interpreted as two Euler systems, coupled through an “energy equation”. Motivated by (4.8), (4.1), (4.2), (4.3), (5.44), we introduce fluid temperatures \( T_1, T_2 \), and make the identifications
\[ \frac{1}{T_k} \overset{\text{def}}{=} -\frac{Z^k_0}{\xi}, \] (6.69)
\[ G_k \overset{\text{def}}{=} \tau_k, \quad k = 1, 2. \] (6.70)

From (5.32), (6.69)
\[ Z^k_1 = \xi u_k/T_k, \quad k = 1, 2; \] (6.71)
from (6.70), (5.38), (6.69), (5.32)
\[ Z^k_2 = \xi \left( G_k - \frac{1}{2} |u_k|^2 \right)/T_k, \quad k = 1, 2. \] (6.72)

From (6.43), (6.37), (6.38)
\[ \bar{\phi} = \xi (P_1/T_1 + P_2/T_2), \] (6.73)
\[ \bar{\psi} = \xi (u_1 P_1/T_1 + u_2 P_2/T_2). \] (6.74)

Coupling of the two fluids is through the temperatures in each case, using (6.69) and (6.64) or (6.65). For the case (6.64), from (6.69) we have
\[ T \overset{\text{def}}{=} T_1 \]
\[ = T_2, \] (6.75)
and we identify (somewhat arbitrarily) but without loss of generality
\[ \bar{z}_{2m+3} \overset{\text{def}}{=} Z^1_0 \]
\[ = Z^2_0 \]
\[ = -\xi/T. \] (6.76)
Using (6.43), (6.70), (6.75), the equations of state assume the form

$$P_k = P_k(G_k, 1/T), \quad k = 1, 2.$$ \hspace{1cm} (6.77)

This is recognized as the “common temperature” system (4.12), (4.13).

Using (6.76), (6.73), (6.74), we compute the remaining equation in the system (6.2), obtaining (as expected)

$$\left(\frac{1}{2} \rho_1 |u_1|^2 + \rho_1 E_1 + \frac{1}{2} \rho_2 |u_2|^2 + \rho_2 E_2\right)_t + \sum_{i=1}^m \left(u_{1,i} \rho_1 \left(\frac{1}{2} |u_1|^2 + H_1\right) + u_{2,i} \rho_2 \left(\frac{1}{2} |u_2|^2 + H_2\right)\right)_{x_i} = 0.$$ \hspace{1cm} (6.78)

From (6.3), (6.73), (6.74), we obtain the corresponding entropy inequality, also of expected form

$$(-\rho_1 S_1 - \rho_2 S_2)_t + \sum_{i=1}^m (-u_{1,i} \rho_1 S_1 - u_{2,i} \rho_2 S_2)_{x_i} \leq 0.$$ \hspace{1cm} (6.79)

For the case (6.65), from (6.69) the fluid temperatures satisfy

$$\frac{1}{T_1} + \frac{1}{T_2} = 1.$$ \hspace{1cm} (6.80)

In this case we regard the temperatures as artificial, related to a void fraction $\alpha$ by

$$\frac{1}{T_1} = \alpha,$$ \hspace{1cm} (6.81)

$$\frac{1}{T_2} = 1 - \alpha.$$ \hspace{1cm} (6.82)

Then given expressions for the fluid pressures of the form

$$P_1 = f_1(\rho_1, \alpha),$$ \hspace{1cm} (6.83)

$$P_2 = f_2(\rho_2, 1 - \alpha)$$ \hspace{1cm} (6.84)

are understood as equations of state of the form

$$P_k = f_k(\rho_k, 1/T_k), \quad k = 1, 2.$$ \hspace{1cm} (6.85)

As $1/T$ is among the components of $z$ for the Euler system, we include $\alpha$ as a component of $z$, setting

$$\tilde{z}_{2m+3} \overset{\text{def}}{=} \xi \left(\frac{1}{2} - \alpha\right)$$

$$= \xi \left(\frac{1}{2} - \frac{1}{T_1}\right)$$

$$= -\xi \left(\frac{1}{2} - \frac{1}{T_2}\right).$$ \hspace{1cm} (6.86)

using (6.81), (6.82).
Using (6.86), (6.73), (6.74), setting \( j = 2m + 3 \) in (6.2) we obtain the remaining equation in the system (2.1) of the form
\[
\left( \frac{1}{2} \rho_1 |u_1|^2 + \rho_1 E_1 - \frac{1}{2} \rho_2 |u_2|^2 - \rho_2 E_2 \right)_t + \sum_{i=1}^m \left( u_{1,i} \rho_1 \left( \frac{1}{2} |u_1|^2 + H_1 \right) - u_{2,i} \rho_2 \left( \frac{1}{2} |u_2|^2 + H_2 \right) \right) x_i = 0,
\]
(6.87)
and the entropy inequality from (6.3) of the form
\[
\left( -\rho_1 S_1 + \frac{1}{4} \rho_1 |u_1|^2 + \frac{1}{2} \rho_1 E_1 - \rho_2 S_2 + \frac{1}{4} \rho_2 |u_2|^2 + \frac{1}{2} \rho_2 E_2 \right)_t + \sum_{i=1}^m \left( u_{1,i} \left( -\rho_1 S_1 + \frac{1}{4} \rho_1 |u_1|^2 + \frac{1}{2} \rho_1 H_1 \right) ight) \leq 0.
\]
(6.88)

We point out that if the given equations of state (6.85) are such that
\[
E_k(\rho_k, T_k) > 0, \quad k = 1, 2,
\]
and
\[
\frac{\partial E_k}{\partial T_k}(\rho_k, T_k) > 0, \quad k = 1, 2,
\]
(6.89) (6.90)
for all \( \rho_k, T_k > 0, k = 1, 2 \), then at any point \( x, t \) where \( \rho_1(x, t), \rho_2(x, t) > 0 \), a value \( \alpha(x, t) \in (0, 1) \) is uniquely determined from the value of \( (\rho_1 E_1 - \rho_2 E_2)(x, t) \).

For either system, convexity of the corresponding entropy density is obtained from a simple assumption on the equations of state. This is the essential use of the identification as Euler systems.

**Theorem 6.10.** For either the “common temperature” system (6.41), (6.42), \( k = 1, 2, (6.78) \), or the “void fraction” system (6.41), (6.42), \( k = 1, 2, (6.87) \), assume a given convex equation of state for each fluid. Then the entropy density \( W \) is convex in the primitive variables \( w \).

**Remark.** Throughout this theorem, “convex” may be replaced by “strictly convex” or “uniformly convex”.

**Proof of Theorem 6.10.** A given convex equation of state is equivalent to \( P/T \) convex in the pair \( (G/T, -1/T) \). Thus from (6.43), (6.72), (6.69), (5.38), (6.71), a given convex equation of state for each fluid implies each \( \tilde{\phi} \) convex in \( (Z^k_2, Z^k_1, Z^k_0) \).

Then from (6.37) and (6.37) and (6.76) or (6.86), it follows that \( \tilde{\phi} \) is convex in \( (Z_2^1, Z_2^2, Z_1^1, Z_1^2, Z_{2m+3}) \). From (2.14), (2.16), this is the same as \( \phi \) convex in \( z \). And \( \phi \) convex in \( z \) is equivalent to \( W \) convex in \( w \) using (2.7), (2.9), (2.10).

The system obtained in the case (6.67), (6.68) also admits interpretation as two Euler systems with a common fluid temperature. Here we retain the identification (6.70) and from (6.68) we denote
\[ T \overset{\text{def}}{=} T_1 = T_2 = -\eta \]  

(6.91)

so positive fluid temperatures correspond as previously to \( \tilde{z}_{2m+3} \in (-\infty, 0) \).

In this case, from (6.43), (6.67)

\[ \phi_k^\xi = \xi P_k, \quad k = 1, 2. \]  

(6.92)

The form of the two equations of state follows from (6.59), (6.68), (6.70), (6.91)

\[ P_k = P_k(\tau_k, -\eta) = P_k(G_k, T), \quad k = 1, 2. \]  

(6.93)

From (5.32), (6.67)

\[ Z^k_1 = \xi u_k, \quad k = 1, 2, \]  

(6.94)

and from (5.38), (6.70), (6.94), (6.67)

\[ Z^k_2 = \xi \left( G_k - \frac{1}{2} |u_k|^2 \right), \quad k = 1, 2. \]  

(6.95)

From (6.92), (6.93), (6.68), (6.91), (6.94), (6.95)

\[ \frac{\partial \phi_k^\xi}{\partial \tilde{z}_{2m+3}} = - \frac{\partial P_k}{\partial T} \bigg|_{G_k} = -\rho_k S_k, \quad k = 1, 2. \]  

(6.96)

Using (6.37), (6.38), (6.94), (6.96), Eq. (6.2) with \( j = 2m + 3 \) is obtained as

\[ (-\rho_1 S_1 - \rho_2 S_2)_t + \sum_{i=1}^{m} (-\rho_1 u_{1,i} S_1 - \rho_2 u_{2,i} S_2)_x_i = 0. \]  

(6.97)

The corresponding entropy inequality is obtained similarly, of the form

\[ \left( \frac{1}{2} \rho_1 |u_1|^2 + \rho_1 E_1 + \frac{1}{2} \rho_2 |u_2|^2 + \rho_2 E_2 \right)_t \]

\[ + \sum_{i=1}^{m} \left( \rho, u_{1,i} \left( \frac{1}{2} |u_1|^2 + H_1 \right) + \rho_{2u_{2,i}} \left( \frac{1}{2} |u_2|^2 + H_2 \right) \right)_x_i \leq 0. \]  

(6.98)

The system (2.1) so obtained includes (6.41), (6.42), \( k = 1, 2 \); (6.97). The entropy inequality (2.4) is (6.98). This is simply the “common temperature” system above, including (6.78), (6.79), with the energy and entropy exchanged.

Such might be anticipated. It is known [16] that convexity of the entropy density survives exchange of the entropy inequality, as determined by \( (W, Q) \), with one of the primitive equations, as
determined by \((w_j q_{,j})\) provided that the corresponding symmetric variable \(z_j\) remains nonvanishing throughout a specific homotopy in phase space. Such is the case here. Given a convex equation of state for each fluid, the entropy density of the “common temperature” system, as shown in (6.79), is convex in the primitive variables \(w\) by application of Theorem 6.7. Thus the entropy density shown in (6.98) is also convex in the corresponding primitive variables. Thus the system (6.41), (6.42), \(k = 1, 2, (6.97)\) is a member of class \(S\). As a member of this class, this system must appear among the cases given in Theorem 6.6, and so must correspond to (6.67), (6.68).

We observe finally that dropping the assumption of fluid interchangeability in the definition of class \(S\) above seems of little consequence. In the absence of this assumption, the additional possibilities for the relation between \(z_1^0, z_2^0, \bar{z}, \bar{z}_{2m+3}\) are only the following:

\[
Z_1^0 = Z_0^2 - \xi, \quad (6.99)
\]

and

\[
Z_0^1 = \bar{z}_{2m+3}, \quad Z_0^2 = -\xi, \quad (6.100)
\]

and (6.99), (6.100) with \(Z_1^0, Z_2^0\) reversed.

For the case (6.99), we use (6.69) and obtain

\[
\frac{1}{T_1} = \frac{1}{T_2} + 1 \quad (6.101)
\]

leading to the “common temperature” system containing (6.78), (6.79), but with the fluid temperatures related by (6.101). Such a relation among the temperatures is equivalent, however, to a change in the equation of state of one of the fluids. In particular convexity of the entropy density is unaffected.

For the case (6.100), we obtain \(T_1\) from (6.69), (6.100), and \(T_2\) from (6.91), (6.68), (6.100), obtaining

\[
T_2 = -\bar{z}_{2m+3}/\xi. \quad (6.102)
\]

Proceeding similarly, we obtain a system (2.1) conserving the sum of the energy of fluid 1 and the (thermodynamic) entropy of fluid 2. Again convexity of the system entropy density is unaffected. Further details are omitted.

References