Drazin spectrum of operator matrices on the Banach space

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Abstract

When \( A \in B(X) \) and \( B \in B(Y) \) are given, we denote by \( M_C \) the operator acting on the Banach space \( X \oplus Y \) of the form \( M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \). In this paper, it is concluded and proved that for a given pair \( (A, B) \) of operators, \( \sigma_D(A) \cup \sigma_D(B) = \sigma_D(M_C) \cup W \) holds for every \( C \in B(Y, X) \), where \( W \) is the union of certain holes in \( \sigma_D(M_C) \), which happen to be subsets of \( \sigma_D(A) \cap \sigma_D(B) \). Moreover, the set \( \bigcap_{C \in B(Y, X)} \sigma_D(M_C) \) is investigated and an example for it is considered.

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1. Introduction

The concept of the Drazin inverse is based on the associative ring and the semigroup [8]. There are many papers dealing with the Drazin inverse of complex matrices, or of linear bounded operators on Banach and Hilbert spaces (see [3] and references cited there).

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Throughout this paper, let $X$ and $Y$ be infinite dimensional Banach spaces, $X \oplus Y$ is their product space and let $B(X, Y)$ be the set of all bounded linear operators from $X$ into $Y$. For simplicity, we also write $B(X, X)$ as $B(X)$.

For $A \in B(X)$, $B \in B(Y)$ and $C \in B(Y, X)$, we say $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in B(X \oplus Y)$.

For $T \in B(X, Y)$, we use $R(T)$ and $N(T)$ to denote the range and kernel of $T$, respectively. Let $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim Y / R(T)$. The ascent of $T \in B(X)$, denoted by $\text{asc}(T)$, is defined as the smallest nonnegative integer $k$ (if it exists) such that $N(T^k) = N(T^{k+1})$. If such $k$ does not exist, then we say that the ascent of $T$ is equal to infinity. The descent of $T$, denoted by $\text{des}(T)$, is defined as the smallest nonnegative integer $k$ (if it exists) for which $R(T^k) = R(T^{k+1})$ holds. If such $k$ does not exist, then we say that the descent of $T$ is equal to infinity. If the ascent and the descent of $T$ are finite, then they are equal [7]. The sets of Fredholm operators, Weyl operators and Browder operators on $X$ are defined by

\[ \Phi(X) := \{ T \in B(X) : \alpha(T) < \infty \text{ and } \beta(T) < \infty \}, \]
\[ \Phi_0(X) := \{ T \in B(X) : \alpha(T) = \beta(T) < \infty \}, \]
\[ \Phi_b(X) := \{ T \in \Phi(X) : \text{asc}(T) = \text{des}(T) < \infty \}. \]

The spectrum, the approximate point spectrum, the defect spectrum, the point spectrum, the Weyl spectrum, the Browder spectrum, the Browder resolvent of $T \in B(X)$ are, respectively, defined by

\[ \sigma(T) := \{ \lambda \in \mathbb{C} : \lambda - T \text{ is not invertible} \}, \]
\[ \sigma_a(T) := \{ \lambda \in \mathbb{C} : \lambda - T \text{ is not bounded below} \}, \]
\[ \sigma_{su}(T) := \{ \lambda \in \mathbb{C} : R(\lambda - T) \neq X \}, \]
\[ \sigma_w(T) := \{ \lambda \in \mathbb{C} : \lambda - T \notin \Phi_0(X) \}, \]
\[ \sigma_b(T) := \{ \lambda \in \mathbb{C} : \lambda - T \notin \Phi_b(X) \} \]

and

\[ \rho_b(T) = \mathbb{C} \setminus \sigma_b(T). \]

For a Banach space $X$, we use $X^*$ to denote the dual space of $X$. If $T \in B(X, Y)$, then $T' \in B(Y^*, X^*)$ is the dual operator of $T$. It is well-known that $(X \oplus Y)^* = X^* \oplus Y^*$ holds. Thus, if $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in B(X \oplus Y)$, then $M_C' = \begin{pmatrix} A' & 0 \\ C & B \end{pmatrix} \in B(X^* \oplus Y^*)$.

For a compact subset $M$ of $\mathbb{C}$ we use $\text{acc}M$, $\text{int}M$, $\text{iso}M$, $\partial M$, respectively, to denote the set of all points of accumulation of $M$, the interior of $M$, the isolated points of $M$ and the boundary of $M$.

For $T \in B(X)$, $T$ is called Drazin invertible if there exists an operator $T^D \in B(X)$ such that

\[ T T^D = T^D T, \quad T^D T T^D = T^D, \quad T^{k+1} T^D = T^k \]

for some nonnegative integer $k$. Then $T^D$ is called a Drazin inverse of $T$ and it is well known that $T^D$ is unique [7]. The smallest $k$ in the previous definition is known as the Drazin index of $T$, denoted by $i(T)$. The Drazin spectrum and the Drazin resolvent of $T$ are, respectively, defined by

\[ \sigma_D(T) := \{ \lambda \in \mathbb{C} : \lambda - T \text{ is not Drazin invertible} \} \]

and

\[ \rho_D(T) := \mathbb{C} \setminus \sigma_D(T). \]

We also use $\rho_D(T)$ to denote the Drazin spectral radius of $T$.

This paper is organized as follows.

In Section 2, it is concluded and proved that for a given pair $(A, B)$ of operators, $\sigma_D(A) \cup \sigma_D(B) = \sigma_D(M_C) \cup W$ holds for every $C \in B(Y, X)$, where $W$ is the union of certain holes in
\( \sigma_D(M_C) \), which happen to be subsets of \( \sigma_D(A) \cap \sigma_D(B) \). Our result is the extension of Theorem 2.5 in [5] and it is worth to point out that their ideas are different completely.

In Section 3, the set \( \bigcap_{C \in \mathcal{B}(Y,X)} \sigma_D(M_C) \) is investigated. We prove that for a given pair \((A, B)\) of operators, then

\[
\bigcap_{C \in \mathcal{B}(Y,X)} \sigma_D(M_C) \subseteq \left( \bigcap_{C \in \mathcal{B}(Y,X)} \sigma(M_C) \right) \setminus [\rho_D(A) \cap \rho_D(B)].
\]

In particular, if one of the following conditions holds:

(i) \( \sigma\text{su}(B) = \sigma(B) \);  
(ii) \( \text{int}\sigma_p(B) = \emptyset \);  
(iii) \( \sigma(A) \cap \sigma(B) = \emptyset \);  
(iv) \( \sigma_a(A) = \sigma(A) \),

then we have

\[
\bigcap_{C \in \mathcal{B}(Y,X)} \sigma_D(M_C) = \left( \bigcap_{C \in \mathcal{B}(Y,X)} \sigma(M_C) \right) \setminus [\rho_D(A) \cap \rho_D(B)].
\]

2. Drazin spectrum of 2 \( \times \) 2 upper triangular operator matrices

Han et al. [10] have recently considered the filling in holes problem of 2 \( \times \) 2 upper triangular operator matrices. Their main result can be described as follows: if \( M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \) is an operator acting on the Banach space \( X \oplus Y \), then

\[
\sigma(A) \cup \sigma(B) = \sigma(M_C) \cup W,
\]

where \( W \) is the union of certain of the holes in \( \sigma(M_C) \), which happen to be subsets of \( \sigma(A) \cap \sigma(B) \). Zhang et al. [18] have shown that the passage from \( \sigma_\tau(A) \cup \sigma_\tau(B) \) to \( \sigma_\tau(M_C) \) is the punching of some open sets in \( \sigma_\tau(A) \cap \sigma_\tau(B) \), where \( \sigma_\tau(\cdot) \) can be equal to the Browder spectrum or essential spectrum. Some more results concerning the filling in holes problem can also be found in [5,6,12,13].

In this section, our main result is Theorem 2.9. We begin with some Lemmas.

**Lemma 2.1.** Let \( T \in B(X) \), then the following statements are equivalent:

(i) \( T \) is Drazin invertible.
(ii) \( T = T_1 \oplus T_2 \), where \( T_1 \) is invertible and \( T_2 \) is nilpotent.
(iii) There exists a nonnegative integer \( k \) such that \( \text{des}(T) = \text{asc}(T) = k < \infty \).
(iv) There exists a nonnegative integer \( k \) such that \( X = R(T^k) \oplus N(T^k) \) and corresponding

\[
T = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix} : R(T^k) \oplus N(T^k) \longrightarrow R(T^k) \oplus N(T^k),
\]

where \( T_{11} \) is invertible and \( T_{22} \) is nilpotent.

(v) There exists a nonnegative integer \( k \) such that \( X = R(T^k) \oplus N(T^k) \).

The equivalence of (i), (ii) and (iii) of the above Lemma can be found in [1,2] and the equivalence of (iii), (iv) and (v) is the results of Proposition 38.4 in [11].

**Lemma 2.2.** (i) Suppose that \( \lambda_0 \in \sigma(T) \) and suppose that each neighborhood of \( \lambda_0 \) contains a point that is not an eigenvalue of \( T \). Then \( \lambda_0 \) is a pole of the resolvent operator \( R_\lambda = (\lambda - T)^{-1} \) if and only if \( \lambda_0 - T \) has a finite descent.
(ii) If $0 \in \sigma(T)$, then $0$ is a finite order pole of the resolvent operator $R_\lambda$ if and only if desc$(T) < \infty$ and asc$(T) < \infty$.

The result of (i) can be found from Theorem 10.5 in [16] and the result of (ii) can also be obtained easily from Theorems 10.1 and 10.2 in [16].

**Corollary 2.3.** If each neighborhood of $\lambda$ contains a point that is not an eigenvalue of operator $T$ and $\lambda - T$ has a finite descent, then $\lambda - T$ is Drazin invertible.

**Lemma 2.4.** $T \in B(X)$ is Drazin invertible if and only if $T' \in B(X^*)$ is Drazin invertible.

**Proof.** Notice that $T \in B(X)$ is invertible if and only if $T' \in B(X^*)$ is invertible, and $T \in B(X)$ is nilpotent if and only if $T' \in B(X^*)$ is nilpotent. By these two facts and Lemma 2.1, the result is proved. □

**Lemma 2.5** [7]. If asc$(A) = \text{des}(A) = k < \infty$, asc$(B) = \text{des}(B) = l < \infty$, then the Drazin inverse of $M_C$ exists for any $C \in B(Y, X)$.

**Lemma 2.6.** For a given pair $(A, B)$ of operators, if $M_C$ is Drazin invertible for some $C \in B(Y, X)$, then

(a) desc$(B) < \infty$ and asc$(A) < \infty$.
(b) desc$(A') < \infty$ and asc$(B') < \infty$.

**Proof.** If $M_C$ is Drazin invertible, then by Lemma 2.1, there exists some nonnegative integer $k$ such that asc$(M_C) = \text{desc}(M_C) = k < \infty$. Hence $M_C^k(X \oplus Y) = M_C^{k+1}(X \oplus Y)$. In order to prove desc$(B) < \infty$, we only need to show that $R(B^k) \subseteq R(B^{k+1})$. For any $u \in R(B^k)$, let $u = B^ky$, then $M_C^k(0, y) \in R(M_C^k) = R(M_C^{k+1})$. Thus there exists $(x_0, y_0) \in X \oplus Y$ such that $(A^{k-1}Cy + A^{k-2}CBy + \cdots + CB^{k-1}y, B^ky) = M_C^{k-1}(x_0, y_0) = (A^{k-1}x_0 + A^kCy_0 + A^{k-1}CBy_0 + \cdots + CB^{k-1}y_0, B^{k+1}y_0)$, then $u = B^ky = B^{k+1}y \in R(B^{k+1})$. Thus $R(B^k) \subseteq R(B^{k+1})$. This implies desc$(B) < \infty$. Since $N(A) \oplus \{0\} \subseteq N(M_C)$ and asc$(M_C) < \infty$, then asc$(A) < \infty$. If $M_C$ is Drazin invertible, from Lemma 2.4 we know that $M_C' = \begin{pmatrix} b' & c' \\ 0 & A' \end{pmatrix} : Y^* \oplus X^* \rightarrow Y^* \oplus X^*$ is also Drazin invertible. Similarly, we can prove that desc$(A') < \infty$ and asc$(B') < \infty$. □

**Lemma 2.7.** For a given pair $(A, B)$ of operators, if $M_C$ is Drazin invertible for some $C \in B(Y, X)$, then $A$ is Drazin invertible if and only if $B$ is Drazin invertible.

**Proof.** If both $A$ and $M_C$ are Drazin invertible, then $0 \in \rho(A) \cup \text{iso}(\sigma(A))$ and $0 \in \rho(M_C) \cup \text{iso}(\sigma(M_C))$. Thus there exists $\varepsilon > 0$ such that for any $\lambda$, $0 < |\lambda| < \varepsilon$, we have that $A - \lambda I$ and $M_C - \lambda I$ are invertible. And hence it is easy to show that $B - \lambda I$ is invertible for any $\lambda$, $0 < |\lambda| < \varepsilon$. Since $M_C$ is Drazin invertible, then we have desc$(B) < \infty$ from Lemma 2.6. By Corollary 2.3, $B$ is Drazin invertible.

Conversely if $B$ and $M_C$ are Drazin invertible, from Lemma 2.4, we know that both $B'$ and $M_C' = \begin{pmatrix} b' & c' \\ 0 & A' \end{pmatrix} : Y^* \oplus X^* \rightarrow Y^* \oplus X^*$ are also Drazin invertible. In the same way, we have $A'$ is Drazin invertible and hence $A$ is Drazin invertible. □
From Lemma 2.5 and Lemma 2.7 we can get the following.

Corollary 2.8. If any two of operators $A$, $B$ and $M_C$ are Drazin invertible, then the third is Drazin invertible.

The main result of this section follows.

Theorem 2.9. For a given pair $(A, B)$ of operators, and for every $C \in B(Y, X)$, the following holds:

$$\sigma_D(A) \cup \sigma_D(B) = \sigma_D(M_C) \cup W,$$

where $W$ is the union of certain holes in $\sigma_D(M_C)$, which happen to be subsets of $\sigma_D(A) \cap \sigma_D(B)$.

Proof. First, we can claim that, for every $C \in B(Y, X)$,

$$(\sigma_D(A) \cup \sigma_D(B)) \setminus (\sigma_D(A) \cap \sigma_D(B)) \subseteq \sigma_D(M_C) \subseteq \sigma_D(A) \cup \sigma_D(B). \tag{1}$$

Indeed the second inclusion in (1) follows from Corollary 2.8. For the first inclusion, let $\lambda \in (\sigma_D(A) \cup \sigma_D(B)) \setminus (\sigma_D(A) \cap \sigma_D(B))$, then $\lambda \in \sigma_D(B) \setminus \sigma_D(A)$, or $\lambda \in \sigma_D(A) \setminus \sigma_D(B)$. By Lemma 2.7, we have $\lambda \in \sigma_D(M_C)$ for every $C \in B(Y, X)$.

Next we claim that, for every $C \in B(Y, X)$, we have

$$\eta(\sigma_D(A) \cap \sigma_D(B)) = \eta(\sigma_D(M_C)), \tag{2}$$

where $\eta M$ denotes the polynomially convex hull of the compact set $M \subseteq \mathbb{C}$. By Lemma 2.5, $\sigma_D(M_C) \subseteq \sigma_D(A) \cup \sigma_D(B)$ for every $C \in B(Y, X)$. We only need to show that $\partial(\sigma_D(A) \cup \sigma_D(B)) \subseteq \partial \sigma_D(M_C)$ from the Maximum Modulus Theorem. Moreover, since $\text{int} \sigma_D(M_C) \subseteq \text{int}(\sigma_D(A) \cup \sigma_D(B))$, it suffices to show that $\partial(\sigma_D(A) \cup \sigma_D(B)) \subseteq \sigma_D(M_C)$.

In fact, we can claim that $\partial(\sigma_D(A) \cup \sigma_D(B)) \subseteq \partial \sigma_D(A) \cup \partial \sigma_D(B) \subseteq \{\lambda | \text{des}(A' - \lambda) = \infty \text{ or } \text{des}(B - \lambda) = \infty \} \subseteq \sigma_D(M_C)$. The first inclusion is obvious and the last inclusion follows from Lemma 2.6. For the second inclusion, if there exists $\lambda_0 \in (\partial \sigma_D(A) \cup \partial \sigma_D(B)) \setminus \{\lambda | \text{des}(A' - \lambda) = \infty \text{ or } \text{des}(B - \lambda) = \infty \}$, then there are two cases to consider.

Case 1. If $\lambda_0 \in \partial \sigma_D(A)$, then $\lambda_0 \in \sigma(A)$ and for any neighborhood of $\lambda_0$, there exists $\lambda$ such that $A - \lambda$ is Drazin invertible. Thus for any neighborhood of $\lambda_0$, there exists $\lambda$ such that $A - \lambda$ is invertible, and thus $A' - \lambda$ is invertible. Since $\text{des}(A' - \lambda_0) < \infty$, from Corollary 2.3, we know that $A' - \lambda_0$ is Drazin invertible. Moreover, by Lemma 2.4, $A - \lambda_0$ is Drazin invertible. It is in contradiction to the fact that $\lambda_0 \in \partial \sigma_D(A) \subseteq \sigma_D(A)$.

Case 2. If $\lambda_0 \in \partial \sigma_D(B)$, similar to the case 1, we also can induce a contradiction.

Then the second inclusion is true. Consequently, $\partial(\sigma_D(A) \cup \sigma_D(B)) \subseteq \partial \sigma_D(M_C)$ and $\sigma_D(M_C) \subseteq \sigma_D(A) \cup \sigma_D(B)$ tell us that the passage from $\sigma_D(M_C)$ to $\sigma_D(A) \cup \sigma_D(B)$ is the filling in certain holes of $\sigma_D(M_C)$. By (1), we know that $(\sigma_D(A) \cup \sigma_D(B)) \setminus \sigma_D(M_C)$ is contained in $\sigma_D(A) \cap \sigma_D(B)$. It follows that the filling in certain holes in $\sigma_D(M_C)$ should occur in $\sigma_D(A) \cap \sigma_D(B)$. □

From Theorem 2.9, we obtain immediately the following two results.

Corollary 2.10. If $\sigma_D(A) \cap \sigma_D(B)$ has no interior points, then for every $C \in B(Y, X)$,

$$\sigma_D(M_C) = \sigma_D(A) \cup \sigma_D(B). \tag{3}$$

In particular, if either $A \in B(X)$ or $B \in B(Y)$ is a Riesz operator, then equality (3) holds.

Corollary 2.11. For a given pair $(A, B)$ of operators, $r_D(M_C)$ is a constant.
Corollary 2.12. If $\sigma_b(M_C) = \sigma_b(A) \cup \sigma_b(B)$ or $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ or $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, then $\sigma_D(M_C) = \sigma_D(A) \cup \sigma_D(B)$ holds.

Proof. Suppose that $\sigma_b(M_C) = \sigma_b(A) \cup \sigma_b(B)$, we prove that $\sigma_D(M_C) \supseteq \sigma_D(A) \cup \sigma_D(B)$ firstly. Without loss of generality, we suppose that $0 \notin \sigma_D(M_C)$, which implies $0 \in \rho(M_C) \cup iso(\sigma(M_C))$. Thus there exists $\varepsilon > 0$ such that for any $\lambda$, $0 < |\lambda| < \varepsilon$, $M_C - \lambda$ is invertible and it is easy to prove that $B - \lambda$ is right invertible. Therefore we have $\beta(B - \lambda) = 0$. Moreover, since $M_C - \lambda$ is invertible for every $\lambda$, $0 < |\lambda| < \varepsilon$, then $\lambda \notin \sigma_b(M_C) = \sigma_b(A) \cup \sigma_b(B)$. Thus $B - \lambda \in \Phi_b(Y)$. Therefore $\alpha(B - \lambda) = \beta(B - \lambda) = 0$, that is, $B - \lambda$ is invertible for every $\lambda$, $0 < |\lambda| < \varepsilon$. Since $0 \notin \sigma_D(M_C)$, we have $des(B) < \infty$ from Lemma 2.6. By Corollary 2.3 we know that $B$ is Drazin invertible. And hence $A$ is also Drazin invertible from Lemma 2.7. So $0 \notin \sigma_D(A) \cup \sigma_D(B)$. It shows that $\sigma_D(M_C) \supseteq \sigma_D(A) \cup \sigma_D(B)$. From Lemma 2.7 we have $\sigma_D(M_C) \subseteq \sigma_D(A) \cup \sigma_D(B)$ for every $B \in (Y, X)$. It is clear that $\sigma_D(M_C) = \sigma_D(A) \cup \sigma_D(B)$. If $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ or $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, similarly, we can prove the result. □

3. Perturbation of the Drazin spectrums

For given $A \in B(H)$ and $B \in B(K)$, the sets $\bigcap_{C \in B(K,H)} \sigma_{\tau}(MC)$ were studied in some works (see [4,7,9,10,14,15,17]), where $H, K$ are the infinite dimensional separable Hilbert spaces, $\sigma_{\tau}(\cdot)$ can be qual to the spectrum, the essential spectrum, the left (right) spectrum, the essential approximate point spectrum, the Browder spectrum and so on.

In this section we prove the following result, which happens to the main result of the paper.

Theorem 3.1. For a given pair $(A, B)$ of operators, then

(a) $\bigcap_{C \in B(Y,X)} \sigma_D(M_C) \subseteq (\bigcap_{C \in B(Y,X)} \sigma(M_C)) \setminus [\rho_D(A) \cap \rho_D(B)]$.
(b) In particular, if one of the following conditions holds:

(i) $\sigma_{su}(B) = \sigma(B)$; (ii) $\sigma(A) \cap \sigma(B) = \emptyset$;
(ii) $\text{into}_p(B) = \emptyset$; (iv) $\sigma_{iso}(A) = \sigma(A)$,

then we have $\bigcap_{C \in B(Y,X)} \sigma_D(M_C) = (\bigcap_{C \in B(Y,X)} \sigma(M_C)) \setminus [\rho_D(A) \cap \rho_D(B)]$.

Proof. (a) The inclusion is obvious.

(b) In order to prove $\bigcap_{C \in B(Y,X)} \sigma_D(M_C) = (\bigcap_{C \in B(Y,X)} \sigma(M_C)) \setminus [\rho_D(A) \cap \rho_D(B)]$ under certain conditions, we only need to prove $\bigcap_{C \in B(Y,X)} \sigma_D(M_C) \supseteq (\bigcap_{C \in B(Y,X)} \sigma(M_C)) \setminus [\rho_D(A) \cap \rho_D(B)]$ from (a). That is, it suffices to show that $\lambda \notin (\bigcap_{C \in B(Y,X)} \sigma(M_C)) \setminus [\rho_D(A) \cap \rho_D(B)]$ for any $\lambda \notin \bigcap_{C \in B(Y,X)} \sigma_D(M_C)$.

(i) If $\sigma_{su}(B) = \sigma(B)$ and $\lambda \notin \bigcap_{C \in B(Y,X)} \sigma_D(M_C)$, then there exists $C \in B(Y, X)$ such that $M_C - \lambda$ is Drazin invertible. This implies $\lambda \notin \rho(M_C) \cup iso(\sigma(M_C))$. Thus there exists $\varepsilon > 0$ such that for every $\lambda_0$, $0 < |\lambda_0| < \varepsilon$, $M_C - \lambda - \lambda_0$ is invertible, and it is easy to prove that $B - \lambda - \lambda_0$ is right invertible. It follows $\lambda \notin acc_{su}(B) = acc\sigma(B)$. Since $M_C - \lambda$ is Drazin invertible, from Lemma 2.6 we have $des(B - \lambda) < \infty$. By Corollary 2.3, $B - \lambda$ is Drazin invertible. Moreover, using Lemma 2.7, we get that $A - \lambda$ is Drazin invertible. Thus, $\lambda \notin [\rho_D(A) \cap \rho_D(B)]$. This implies $\lambda \notin (\bigcap_{C \in B(Y,X)} \sigma(M_C)) \setminus [\rho_D(A) \cap \rho_D(B)]$ for any $\lambda \notin \bigcap_{C \in B(Y,X)} \sigma_D(M_C)$.

(ii) If $\text{into}_p(B) = \emptyset$ and $\lambda \notin \bigcap_{C \in B(Y,X)} \sigma_D(M_C)$, then there exists $C \in B(Y, X)$ such that $M_C - \lambda$ is Drazin invertible. From Lemma 2.6, we get that $des(B - \lambda) < \infty$. Since into$_p(B) = \emptyset$, which implies $\lambda \notin \text{into}_p(B)$, we have that $B - \lambda$ is Drazin invertible from Corollary 2.3. More-
over, from Lemma 2.7 we can also obtain that \( A - \lambda \) is Drazin invertible. Thus, we induce that \( \lambda \notin [\rho_D(A) \cap \rho_D(B)] \).

Therefore, \( \lambda \notin (\bigcap_{C \in B(Y,X)} \sigma(MC) \setminus [\rho_D(A) \cap \rho_D(B)]) \) for any \( \lambda \notin (\bigcap_{C \in B(Y,X)} \sigma(MC)) \).

(iii) If \( \sigma(A) \cap \sigma(B) = \emptyset \) and \( \lambda \notin (\bigcap_{C \in B(Y,X)} \sigma(MC)) \), then there exists \( C \in B(Y,X) \) such that \( M_C - \lambda \) is Drazin invertible. Since \( \sigma(A) \cap \sigma(B) = \emptyset \), then \( \lambda \notin \rho(A) \) or \( \lambda \notin \rho(B) \) for any \( \lambda \in \mathbb{C} \).

If \( \lambda \notin \rho(A) \), then \( A - \lambda \) is Drazin invertible. By Lemma 2.7, we have that \( B - \lambda \) is Drazin invertible. Thus \( \lambda \in [\rho_D(A) \cap \rho_D(B)] \). If \( \lambda \notin \rho(B) \), similarly, we can get that \( \lambda \notin [\rho_D(A) \cap \rho_D(B)] \).

It proves that \( \lambda \notin (\bigcap_{C \in B(Y,X)} \sigma(MC) \setminus [\rho_D(A) \cap \rho_D(B)]) \) for any \( \lambda \notin (\bigcap_{C \in B(Y,X)} \sigma(MC)) \).

(iv) If \( \sigma_a(A) \neq \sigma(A) \) and \( \lambda \notin (\bigcap_{C \in B(Y,X)} \sigma(MC)) \), then there exists \( C \in B(Y,X) \) such that \( M_C - \lambda \) is Drazin invertible. Thus there exists \( \epsilon > 0 \) such that for any \( \lambda_0 \), \( 0 < |\lambda_0| < \epsilon \), \( M_C - \lambda - \lambda_0 \) is invertible, and it is easy to prove that \( A - \lambda - \lambda_0 \) is left invertible. It follows \( \lambda \notin \text{acc}(\sigma_a(A) = \sigma(A)) \). By Lemma 2.4, we have that \( M_C' - \lambda = \begin{pmatrix} b' & -\lambda \\ 0 & A' - \lambda \end{pmatrix} \): \( Y^* \oplus X^* \rightarrow Y^* \oplus X^* \) is Drazin invertible. From Lemma 2.6, it follows that \( \text{des}(A' - \lambda) < \infty \).

Furthermore, by Corollary 2.3, we obtain that \( A' - \lambda \) is Drazin invertible. Similarly, we can get \( \lambda \notin [\rho_D(A) \cap \rho_D(B)] \). This implies \( \lambda \notin (\bigcap_{C \in B(Y,X)} \sigma(MC) \setminus [\rho_D(A) \cap \rho_D(B)]) \) for any \( \lambda \notin (\bigcap_{C \in B(Y,X)} \sigma(MC)) \).

Notice that if either \( \text{acc}(\sigma_a(B) = \phi) \) or \( \text{int}(\sigma_a(B) = \phi) \) holds, then \( \text{int}(\sigma(B)) = \phi \) and if either \( \text{acc}(\sigma_a(A) = \phi) \) or \( \text{int}(\sigma_a(A) = \phi) \), then \( \sigma_a(A) = \sigma(A) \). Using these two facts and Theorem 3.1, we obtain that

**Corollary 3.2.** For a given pair \( (A, B) \) of operators, if any one of the following conditions is satisfied:

(i) \( \text{acc}(\sigma(A) = \phi) \);
(ii) \( \text{int}(\sigma(A) = \phi) \);

then we have

\[
\bigcap_{C \in B(Y,X)} \sigma(D(M_C)) = \left( \bigcap_{C \in B(Y,X)} \sigma(MC) \right) \setminus [\rho_D(A) \cap \rho_D(B)].
\]

From the following example, we can claim that the above equality does not always true even on complex infinite separable dimensional Hilbert spaces.

**Example 3.3.** Define operators \( T, S, C \in B(l_2) \) by

\[
T(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots),
\]

\[
S(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots), \quad C(x_1, x_2, x_3, \ldots) = (x_1, 0, 0, \ldots)
\]

for any \( (x_1, x_2, x_3, \ldots) \in l_2 \).

Let

\[
M = \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} : l_2 \oplus (l_2 \oplus l_2) \rightarrow l_2 \oplus (l_2 \oplus l_2)
\]

\[
= \begin{pmatrix} T & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{pmatrix} : l_2 \oplus l_2 \oplus l_2 \rightarrow l_2 \oplus l_2 \oplus l_2,
\]

where \( S_1 = \begin{pmatrix} S \\ 0 \end{pmatrix} : l_2 \oplus l_2 \rightarrow l_2 \oplus l_2. \)
Then we can claim that

\[ 0 \in \bigcap_{Q \in B(l_2 \oplus l_2, l_2)} \sigma(M_Q) \setminus \left( \rho_D(T) \bigcap \rho_D(S_1) \right), \quad \text{but } 0 \notin \bigcap_{Q \in B(l_2 \oplus l_2, l_2)} \sigma(M_Q). \]

Thus,

\[ \bigcap_{Q \in B(l_2 \oplus l_2, l_2)} \sigma(D(M_Q)) \neq \left( \bigcap_{Q \in B(l_2 \oplus l_2, l_2)} \sigma(M_Q) \right) \setminus \left( \rho_D(T) \bigcap \rho_D(S_1) \right). \]

In fact,

(i) Since \( M_Q \) cannot be surjective for any \( Q \in B(l_2 \oplus l_2, l_2) \), then \( M_Q = \begin{pmatrix} T & 0 \\ 0 & S_1 \end{pmatrix} \) is not invertible. This implies \( 0 \in \bigcap_{Q \in B(l_2 \oplus l_2, l_2)} \sigma(M_Q) \).

(ii) Since \( \text{des}(T) = \text{asc}(S_1) = \infty \), then neither \( S_1 \) nor \( T \) is Drazin invertible, that is, \( 0 \notin (\rho_D(T) \bigcup \rho_D(S_1)) \). Hence \( 0 \notin (\rho_D(T) \bigcap \rho_D(S_1)) \).

(iii) Let \( M_{C_1} = \begin{pmatrix} T & 0 \\ 0 & S_1 \end{pmatrix} : l_2 \oplus (l_2 \oplus l_2) \to l_2 \oplus (l_2 \oplus l_2) = \begin{pmatrix} T & 0 \\ 0 & S_1 \end{pmatrix} : l_2 \oplus l_2 \oplus l_2 \to l_2 \oplus l_2 \oplus l_2 \), where \( C_1 = \begin{pmatrix} T & 0 \\ 0 & S_1 \end{pmatrix} : l_2 \oplus (l_2 \oplus l_2) \to l_2 \). Then \( M_{C_1} \) is Drazin invertible. Indeed, it is well known that \( \begin{pmatrix} T & 0 \\ 0 & S_1 \end{pmatrix} \) is unitary and hence \( \begin{pmatrix} T & 0 \\ 0 & S_1 \end{pmatrix} \) is invertible. Thus, it follows that \( M_{C_1} \) is Drazin invertible from part (iv) of Lemma 2.1. This implies that \( 0 \notin \bigcap_{Q \in B(l_2 \oplus l_2, l_2)} \sigma_D(M_Q) \).

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References

