Strongly hopfian manifolds as codimension-2 fibrators

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Abstract

If a closed $n$-manifold $N$ has a $2$--$1$ covering, we consider the covering space $\tilde{N}$ of $N$ corresponding to $H$, where $H$ is the intersection of all subgroups $H_i$ of index 2 in $\pi_1(N)$, i.e., $H = \bigcap_{i \in I} H_i$ with $[\pi_1(N) : H_i] = 2$ for $i \in I$. We will show that if $\pi_1(N)$ is residually finite, $\chi(N) \neq 0$, and $\tilde{N}$ is hopfian, then $N$ is a codimension 2 fibration. And then, we will also get several results about codimension-2 fibrators as its corollaries. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Daverman [5] introduced the following definition: A closed $n$-manifold $N^n$ is a codimension-2 fibration (respectively, a codimension-2 orientable fibration) if, whenever $p : M \to B$ is a proper map from an arbitrary (respectively, orientable) $(n + 2)$-manifold $M$ to a 2-manifold $B$ such that each $p^{-1}(b)$ is shape equivalent to $N$, then $p : M \to B$ is an approximate fibration. Then we have the following natural question:

Main question. Which manifolds $N$ are codimension-2 fibrators?

In [5], Daverman showed that all simply connected manifolds, closed surfaces with nonzero Euler characteristic, and real projective $n$-spaces ($n > 1$) are codimension-2 fibrators. And he asked whether every closed $n$-manifold with a finite fundamental

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group is a codimension-2 fibrator. The answer for the case of $N$ having a nonzero Euler characteristic is yes [1]. But, in general, surprisingly, the answer turned out to be no [7].

A group $\Gamma$ is said to be hopfian if every epimorphism $f : \Gamma \to \Gamma$ is necessarily an isomorphism. A group $\Gamma$ is said to be residually finite if for any nontrivial element $x$ of $\Gamma$ there is a homomorphism $f$ from $\Gamma$ onto a finite group $K$ such that $f(x) \neq 1_K$. It is well known that every finitely generated residually finite group is hopfian. Call a closed manifold $N$ hopfian if it is orientable and every degree one map $N \to N$ is a homotopy equivalence. In [6], Daverman showed the following theorem:

Every hopfian $n$-manifold $N$ with hopfian $\pi_1(N)$ and a nonzero Euler characteristic is a codimension-2 orientable fibrator.

Whether it is a codimension-2 fibrator is still an open question. But, Chinen [1] showed that the answer is yes if $N$ has no 2–1 covering. So it is natural to ask the following:

**Question.** What if $N$ has a 2–1 covering?

In this paper, we will get a partial answer of that question, in a sense. If a closed $n$-manifold $N$ has a 2–1 covering, we consider the covering space $\tilde{N}$ of $N$ corresponding to $H$, where $H$ is the intersection of all subgroups $H_i$ of index 2 in $\pi_1(N)$, i.e., $H = \bigcap_{i \in I} H_i$ with $[\pi_1(N) : H_i] = 2$ for $i \in I$. Then we see that, by Hall’s Theorem (for any finitely generated group $G$, the number of subgroups of $G$ having any fixed finite index is finite), the index set $I$ is finite, and $\tilde{N}$ is an $n$-dimensional orientable manifold, which follows from the facts that a (finite) covering space of an $n$-dimensional orientable manifold is an $n$-dimensional orientable manifold and any nonorientable manifold has a 2–1 orientable covering. We will show that if $\pi_1(N)$ is residually finite, $\chi(N) \neq 0$, and $\tilde{N}$ is hopfian, then $N$ is a codimension-2 fibrator. And then, we will also get several results about codimension-2 fibrators as its corollaries.

2. Preliminaries

Throughout this paper, the symbols $\sim$, $\cong$, and $\cong$ denote homotopy equivalence, homeomorphism, and isomorphism in that order. The symbol $\chi$ is used to denote Euler characteristic. All manifolds are understood to be finite dimensional, connected, metric, and boundaryless. Whenever the presence of boundary is tolerated, the object will be called a manifold with boundary.

Approximate fibrations were introduced by Coram and Duvall [2] as a generalization of Hurewicz fibrations and cell-like maps. A proper map $p : M \to B$ between locally compact ANRs is called an approximate fibration if it has the following approximate homotopy lifting property: Given an open cover $\varepsilon$ of $B$, an arbitrary space $X$, and two maps $g : X \to M$ and $F : X \times I \to B$ such that $p \circ g = F_0$, there exists a map $G : X \times I \to M$ such that $G_0 = g$ and $p \circ G$ is $\varepsilon$-close to $F$. The latter means: for each $z \in X \times I$ there exists an $U_z \in \varepsilon$ such that $\{F(z), p \circ G(z)\} \subset U_z$. Much of the theory of Hurewicz fibrations carries over to the set of approximate fibrations. For example, if a
A proper map \( p : M \to B \) is an approximate fibration, there is a homotopy exact sequence between \( M, B \) and fibers of \( p \) as follows:

\[
\cdots \to \pi_{i+1}(B) \to \pi_i(p^{-1}b) \to \pi_i(M) \to \pi_i(B) \to \cdots.
\]

Furthermore, the set of approximate fibrations is a closed subset of the space of maps between two compact ANRs with the sup-norm metric, while the set of Hurewicz fibrations may not be closed [3].

Let \( N^n \) be a closed manifold. A proper map \( p : M \to B \) is \( N^n \)-like if each fiber \( p^{-1}(b) \) is shape-equivalent to \( N \). For simplicity or familiarity, we shall assume that each fiber \( p^{-1}(b) \) in an \( N^n \)-like map to be an ANR having the homotopy type of \( N^n \). Let \( N \) and \( N' \) be closed \( n \)-manifolds and \( f : N \to N' \) be a map. If both \( N \) and \( N' \) are orientable, then the degree of \( f \) is the nonnegative integer \( d \) such that the induced endomorphism of \( f_* : H_n(N; Z) \cong Z \to H_n(N'; Z) \cong Z \) amounts to multiplication by \( d \), up to sign.

In general, the degree mod 2 of \( f \) is the nonnegative integer \( d \) such that the induced endomorphism of

\[
f_* : H_n(N; Z_2) \cong Z_2 \to H_n(N'; Z_2) \cong Z_2
\]

amounts to multiplication by \( d \). Any degree one mod 2 map \( f : N \to N \) with \( \chi(N) \neq 0 \) induces a \( \pi_1 \)-epimorphism \( f_* : \pi_1(N) \to \pi_1(N) \) (see [1, Lemma 3.4])

Suppose that \( N \) is a closed \( n \)-manifold and a proper map \( p : M \to B \) is \( N \)-like. Let \( G \) be the set of all fibers, i.e., \( G = \{ p^{-1}(b) : b \in B \} \). Put \( C = \{ p(g) \in B : g \in G \) and there exist a neighborhood \( U_g \) of \( g \) in \( M \) and a retraction \( R_g : U_g \to g \) such that \( R_g|g' : g' \to g \) is a degree one map for all \( g' \in G \) with \( g' \in G \) in \( U_g \} \), and \( C' = \{ p(g) \in B : g \in G \) and there exist a neighborhood \( U_g \) of \( g \) in \( M \) and a retraction \( R_g : U_g \to g \) such that \( R_g|g' : g' \to g \) is a degree one mod 2 map for all \( g' \in G \) with \( g' \in G \) in \( U_g \} \). Call \( C \) the continuity set of \( p \) and \( C' \) the mod 2 continuity set of \( p \). D. Coram and P. Duvall [4] showed that \( C \) and \( C' \) are dense, open subsets of \( B \).

The following [4, Proposition 2.8] is very useful for investigating codimension-2 fibrators.

**Proposition 2.1.** If \( G \) is a usc decomposition of an orientable \((n + 2)\)-manifold \( M \) into closed, orientable \( n \)-manifolds, then the decomposition space \( B = M/G \) is a 2-manifold and \( D = B \setminus C \) is locally finite in \( B \), where \( C \) represents the continuity set of the decomposition map \( p : M \to B \); if either \( M \) or some elements of \( G \) are nonorientable, \( B \) is a 2-manifold with boundary (possibly empty) and \( D' = (\text{int } B) \setminus C' \) is locally finite in \( B \), where \( C' \) represents the mod 2 continuity set of \( p \).

And, the following ([6, Theorem 2.2] or [9]) gives us useful information connecting hopfian manifolds and hopfian fundamental groups.

**Proposition 2.2.** A closed, orientable \( n \)-manifold \( N \) is a hopfian manifold if any one of the following conditions holds:

1. \( n \leq 4 \) and \( \pi_1(N) \) is hopfian;
2. \( \pi_1(N) \) contains a nilpotent subgroup of finite index.
3. Strongly hopfian manifolds as codimension-2 fibrators

**Definition.** Let \( N \) be a closed \( n \)-manifold. \( N \) is strongly hopfian if \( \widetilde{N} \) is hopfian, where \((\widetilde{N}, \tilde{g})\) is the covering space of \( N \) corresponding to \( H = \bigcap_{i \in I} H_i \) with \( I = \{ i : [\pi_1(N) : H_i] = 2 \} \neq \emptyset \), and \( \widetilde{N} = N \) when \( I = \emptyset \).

From now on, we reserve the symbols \( \widetilde{N} \) and \( H \) for the above meanings.

**Lemma 3.1.** Let \( N \) be a strongly hopfian closed \( n \)-manifold with residually finite fundamental group and nonzero Euler characteristic. If a proper map \( p : M^{n+2} \to B^2 \) from an \((n + 2)\)-manifold \( M \) onto a 2-manifold \( B \) is \( N \)-like, then \( p \) is an approximate fibration over \( C' \), where \( C' \) denotes the mod2 continuity set of \( p \).

**Proof.** If \( I = \emptyset \), then \( N = \widetilde{N} \) is hopfian. Since \( \pi_1(N) \) is residually finite, it is hopfian. Hence \( N \) is a hopfian \( n \)-manifold with \( \chi(N) \neq 0 \) and hopfian fundamental group. By [6, Theorem 5.10], \( N \) is a codimension-2 orientable fibrator. Since \( I = \emptyset \) implies that \( N \) has no 2–1 covering, by [1, Corollary 3.3], \( N \) is a codimension-2 fibrator. Now we assume that \( I \neq \emptyset \). Set \( G = \{ p^{-1}(b) : b \in B \} \). Fix \( g_0 \in G \) with \( p(g_0) \in C' \). Take a neighborhood \( U \) of \( p(g_0) \) such that \( p^{-1}(U) \) retracts to \( g_0 \), and take a smaller connected neighborhood \( V \) of \( p(g_0) \) such that \( p^{-1}(V) \) deformation-retracts to \( g_0 \) in \( p^{-1}(U) \). Call this retraction \( R : p^{-1}(V) \to g_0 \). Then, we have that \( R_g : \pi_1(p^{-1}(V)) \to \pi_1(g_0) \) is an epimorphism. By Coram and Duvall [3], it is enough to show that \( p|p^{-1}(V) : p^{-1}(V) \to V \) is an approximate fibration. Take the covering map \( q : V^* \to p^{-1}(V) \) corresponding to \( R_g^{-1}(H) \). Since \( [\pi_1(p^{-1}(V)) : R_g^{-1}(H)] = [\pi_1(g_0) : H] < \infty \), \( q \) is finite. So, by [6, Lemma 2.5], it suffices to show that \( p \circ q : V^* \to V \) is an approximate fibration.

**Claim.** For all \( g \in G \) with \( g \subset p^{-1}(V) \), \( q^{-1}(g) \equiv g^* \) and \( \widetilde{N} \) have the same homotopy type.

First, appealing to the method of the proof of claim in [1, Proposition 3.5], we see that for all \( g \in G \) with \( g \subset p^{-1}(V) \), \( q^{-1}(g) \equiv g^* \) is connected. Hence \( \pi_0(g^*) = 1 \). Now, let \( i : g \to p^{-1}(V) \), \( i^* : g^* \to V^* \), \( j : (p^{-1}(V)) \to (p^{-1}(V), g) \) and \( j^* : V^* \to (V^*, g^*) \) be the inclusion maps for \( g \in G \) with \( g \subset p^{-1}(V) \). From the homotopy exact sequence of \((V^*, g^*)\) and \((p^{-1}(V), g)\), we have the following diagram:

\[
\begin{array}{ccccccccc}
\pi_2(V^*, g^*) & \to & \pi_1(g^*) & \to & \pi_1(V^*) & \to & \pi_1(p^{-1}(V), g) & \to & 1 \\
\cong & \downarrow & \cong & \phi & \downarrow & \cong & \\
\pi_2(p^{-1}(V), g) & \to & \pi_1(g) & \to & \pi_1(p^{-1}(V), g) & \to & 1 \\
\end{array}
\]

Since \( R|g : g \to g_0 \) has degree one mod 2, by [1, Lemma 3.4], the induced map \((R|g) : \pi_1(g) \to \pi_1(g_0)\) is onto, so it is an isomorphism by the fact that \( \pi_1(g) = \pi_1(g_0) \) is hopfian. Since \( R|g = R \circ i, i_g \) is a monomorphism, and so is \( i^* \). We easily see that
Moreover, by the diagram chasing argument (using the serpent lemma (see [15, p. 141])), we have that \( \pi_1(g)/K \cong \pi_1(p^{-1}(V))/q\#(\pi_1(V^*)) \). Since

\[
\begin{align*}
[\pi_1(g) : K] = [\pi_1(g) : H][H : K] \quad \text{and} \quad q\#(\pi_1(V^*)) = R\#^{-1}(H),
\end{align*}
\]

we have \([H : K] = 1\), i.e., \( K = H \). It follows from the uniqueness of lifting that \( g^* \) and \( \tilde{N} \) have the same homotopy type.

Since \( \chi(g) \neq 0 \) and \( q \) is finite, \( \chi(g^*) \neq 0 \). And since every subgroup of a residually finite group is residually finite, \( q\#(\pi_1(g^*)) \cong \pi_1(g^*) \) is residually finite, and so \( \pi_1(g^*) \) is hopfian. Recall that \( V^* \) is orientable. It follows from [6, Theorem 5.10] that \( p^* : V^* \to B^* = V^*/G^* = V \) is an approximate fibration, where \( G^* = \{ g^* : g \in G \text{ with } g \subset p^{-1}(V) \} \) is the usc decomposition of \( V^* \).

Lemma 3.2. Let \( N \) be a strongly hopfian closed \( n \)-manifold with hopfian \( \pi_1(N) \) and \( \chi(N) \neq 0 \). If an \( N \)-like proper map \( p : M^{n+2} \to B^2 \) from an \( (n + 2) \)-manifold onto a \( 2 \)-manifold with boundary is an approximate fibration over \( \text{int} \ B \), then \( \partial B = \emptyset \).

Proof. Suppose not. Then there exist \( a_0 \in \partial B \), a neighborhood \( U \) of \( a_0 \) in \( B \), and a deformation retract \( H : p^{-1}(U) \to p^{-1}(a_0) \) such that

1. \( U \approx \) the upper half plane \( \{(x, y) \in \mathbb{E}^2 \mid y \geq 0\} \),
2. \( A = (\partial B) \cap U \) is an open arc, and
3. for all \( a \in A \), \( R|p^{-1}(a) : p^{-1}(a) \to p^{-1}(a_0) \) is a homotopy equivalence.

Take the covering map \( q : M^* \to p^{-1}(U) \) corresponding to \( H \). Then by another argument similar to the proof in the Lemma 3.1, we have that for all \( a \in A \), \( q^{-1}(p^{-1}(a)) \) is connected and \( q^{-1}(p^{-1}(a)) \sim q^{-1}(p^{-1}(a_0)) \sim \tilde{N} \). And, since \( p \) is an approximate fibration over \( p^{-1}(\text{int} \ U) \), for all \( b, b' \in \text{int} \ U \), \( q^{-1}(p^{-1}(b))_C \sim q^{-1}(p^{-1}(b'))_C \sim (\text{say} \ N^*)_*, \) where \( q^{-1}(p^{-1}(b))_C \) and \( q^{-1}(p^{-1}(b'))_C \) are components of \( q^{-1}(p^{-1}(b)) \) and \( q^{-1}(p^{-1}(b')) \), respectively. Hence, by the fact of \( M^* \) is orientable and [5, Proposition 2.9], we see that for all \( b \in \text{int} \ U \) and for all \( a \in A \), the components \( q^{-1}(p^{-1}(b))_C \) of \( q^{-1}(p^{-1}(b)) \) and \( q^{-1}(p^{-1}(a)) \) are orientable. Therefore, if \( G^* = \{ q^{-1}(p^{-1}(b))_C, q^{-1}(p^{-1}(a)) \mid b \in \text{int} \ U, a \in A \} \) is the usc decomposition of \( M^* \), then by Proposition 2.1, \( B^* = M^*/G^* \) is a 2-manifold without boundary.

Let \( p^* : M^* \to B^* \) be the decomposition map and \( C^* \) be its continuity set. Since \( p^*(q^{-1}(p^{-1}(A))) \) is homeomorphic to an open arc and \( B^* \setminus C^* \) is locally finite in \( B^* \), there is a point \( a^* \in p^*(q^{-1}(p^{-1}(A))) \cap C^* \). So we have a map \( N^* \sim q^{-1}(p^{-1}(b))_C \to q^{-1}(p^{-1}(a)) \sim \tilde{N} \) with degree one, where \( p^*|(q^{-1}(p^{-1}(a))) = a^* \) and for some \( b \in \text{int} \ U \). Hence we have \( \beta_i(N^*) \geq \beta_i(\tilde{N}) \) for each \( i \).

For \( g \in G \) with \( p(g) = b \), let \( i : g \to p^{-1}(U) \) be the inclusion map. Set

\[
\begin{align*}
\bar{H} &= i_\#^{-1}(q\#(\pi_1(M^*)) \cap i_\#(\pi_1(g))) = i_\#^{-1}(H \cap i_\#(\pi_1(g))) \quad \text{and} \\
K &= (q|g^*_C)_\#(\pi_1(g^*_C)),
\end{align*}
\]
where \( g_C^* = q^{-1}(p^{-1}(b))_C \). Then we can easily see that \( K \subset \overline{H}, \ H \subset \overline{H} \). But since \( g^* \) has two (or more) components, \( H \neq \overline{H} \). By [16, Proposition 11.1], we have \( K = \overline{H} \).

Now we take the covering map \( N_H \to g \) corresponding to \( H \), and take the covering map \( N_H \to g \) corresponding to \( H \). And since \( H \subset \overline{H} \) and \( H \neq \overline{H} \), we have a \( d-1 \) covering map \( N \sim N_H \to N_H \approx N_K \sim N^* \) with \( d \geq 2 \), so we have for each \( i, \beta_i(N) \geq \beta_i(N^*) \) and \( \chi(N) = d(\chi(N^*)) \). As before \( \chi(N^*) = \chi(N) = d(\chi(N^*)) \), which gives a contradiction \( \chi(N) = 0 \).

**Theorem 3.3.** A strongly hopfian \( n \)-manifold \( N \) with residually finite fundamental group and nonzero Euler characteristic is a codimension-2 fibration.

**Proof.** We may assume that \( I \neq \emptyset \), i.e., \( N \) has a 2–1 covering. Let a proper map \( p: M^{n+2} \to B^2 \) from an \((n+2)\)-manifold \( M^{n+2} \) onto a 2-manifold \( B \) with boundary be \( N \)-like, and \( G = \{ p^{-1}(b): b \in B \} \). By Proposition 2.1, Lemmas 3.1 and 3.2, it suffices to show that \( p \) is an approximate fibration over \( \text{int } B \).

Let \( D' = (\text{int } B) \setminus C' \). If \( D' = \emptyset \), by the Lemma 3.1, there is nothing to prove. So assume that \( D' \neq \emptyset \). Let \( b_0 \in D \). We localize the situation so that \( \text{int } B \) is an open disk containing \( b_0 = p(g_0) \) and \( p \) is an approximate fibration over \( (\text{int } B) \setminus b_0 \). Also we may assume that \( R: p^{-1}(\text{int } B) \to g_0 \) is a strong deformation retraction. Take a covering \( q: M^* \to p^{-1}(\text{int } B) \) corresponding to \( R^{-1}(H) = H \). By an argument similar to the proof of claims in the Lemma 3.1, we see that \( g_0^* = q^{-1}(g_0) \) is connected and has the homotopy type of \( \overline{N} \). Since

\[
p|p^{-1}(\text{int } B) \setminus g_0) : p^{-1}(\text{int } B) \setminus g_0) \to (\text{int } B) \setminus b_0
\]

is an approximate fibration, for any \( g, g'(\neq g_0) \in G \) in \( p^{-1}(\text{int } B) \), their components \( q^{-1}(g)_C = g_C^* \) and \( q^{-1}(g')_C = g_C^* \) have the same homotopy type. And since \( M^* \) is orientable, by [6, Proposition 2.9], \( g_C^* \) and \( g_0^* \) are orientable.

Now we follow the method of the proof in [6, Theorem 5.10], then we have

\[
(R^*|g_C^*)_* : H_1(g_C^*) \to H_1(g_0^*)
\]

is an epimorphism, where \( R^* \) is a lifting of \( R \). By [6, Lemma 5.2'1], \( R^*|g_C^* \) has a positive degree. It follows from [15, p. 399] that \( \beta_i(g_C^*) \geq \beta_i(g_0^*) \) for each \( i \).

Now, for \( g(\neq g_0) \in G \), let \( i: g \to p^{-1}(\text{int } B) \) be the inclusion map. Set

\[
\overline{H} = i_{#}^{-1}(g_{#}(\pi_1(M^*))) \cap i_{#}(\pi_1(g)) = i_{#}^{-1}(H \cap i_{#}(\pi_1(g))) \quad \text{and} \quad K = (q|g_C^*)_{#}(\pi_1(g_C^*)).
\]

Then we can easily see that \( K \subset \overline{H} \) and \( H \subset \overline{H} \). And by [16, Proposition 11.1], we have \( \overline{H} \subset K \), i.e., \( K = \overline{H} \).

Now, let us examine the induced map \( i_{#} \) case by case.

**Case 1.** \( i_{#} \) is an epimorphism. Then, since \( \pi_1(g) = \pi_1(p^{-1}(\text{int } B)) \) is hopfian, \( i_{#} \) is an isomorphism. So we have that for all \( g \in G \) with \( p(g) \in \text{int } B \), \( g^* \) is connected and has the homotopy type of \( \overline{N} \). By the same proof as the Lemma 3.1, \( p \) is an approximate fibration over \( \text{int } B \).

**Case 2.** \( i_{#} \) is not onto.
Subcase 1. \( H = \overline{H} = K \). Then for all \( g \in G \) with \( p(g) \in \text{int} \, B \), \( g^* \) is connected and has the homotopy type of \( \tilde{N} \). By the same reason of the proof in the Lemma 3.1, \( p \) is an approximate fibration over \( \text{int} \, B \).

Subcase 2. \( H \subset \overline{H} = K \) but \( H \neq \overline{H} \). We will show that this case cannot happen. Take the covering map \( N_H \to g \) corresponding to \( \overline{H} \), and take the covering map \( N_H \to g \) corresponding to \( H \). Consider

\[
\begin{array}{ccc}
N_H & \to & g \\
\downarrow & & \downarrow \\
N_H & \to & g
\end{array}
\]

Since \( H \subset \overline{H} \) and \( H \neq \overline{H} \), we have a \( d-1 \) covering map \( N_H \to N_{\overline{H}} \) with \( d \geq 2 \). By the facts of \( g_0^* \sim \tilde{N} \sim N_H \) and \( N_{\overline{H}} \sim N_K \sim g_0^* \) with \( d \geq 2 \), we see that \( \beta_i(g_0^*) \geq \beta_i(g_0^*) \) for each \( i \) (from [8, Corollary 1]) and \( \chi(g_0^*) = dx(g_0^*) \) with \( d \geq 2 \). But since we already have that \( \beta_i(g_0^*) \geq \beta_i(g_0^*) \) for each \( i \), \( \chi(g_0^*) = \chi(g_0^*) = dx(g_0^*) \) with \( d \geq 2 \), which gives the contradiction \( \chi(N) = \chi(g_0^*) \neq 0 \). □

Note. A subgroup of a hopfian group may not be hopfian, while every subgroup of a residually finite group is residually finite (see [17]). Call a group \( I \) hereditarily hopfian if every subgroup of \( I \) is hopfian. The preceding argument actually gives the more general result stated below:

Let \( N \) be a strongly hopfian \( n \)-manifold with \( \chi(N) \neq 0 \). If \( \pi_1(N) \) is hereditarily hopfian, then \( N \) is a codimension-2 fibration.

Remark. In the theorem, we cannot omit the condition \( \chi(N) \neq 0 \) (see [7, Theorem 2.1]).

Corollary 3.1. Let \( N^n \) be a closed \( n \)-manifold with \( \chi(N) \neq 0 \). Then \( N \) is a codimension-2 fibration if any one of the following conditions holds:

1. \( \pi_1(N) \) is abelian;
2. \( \pi_1(N) \) is residually finite and \( n_i(N) = 0 \) for \( 1 < i < n - 1 \);
3. \( n = 4 \) and \( \pi_1(N) \) is residually finite;
4. \( [1] \pi_1(N) \) is finite;
5. \([5]\) \( n = 2 \).

Proof. (1) Case 1. \( N \) has no 2–1 covering. Then, \( N \) must be orientable. Since \( \pi_1(N) \) is abelian, it is nilpotent. By Proposition 2.2, \( N \) is hopfian. We have that \( N \) is a hopfian manifold with hopfian fundamental group and nonzero Euler characteristic, so by [6, Theorem 5.10], \( N \) is a codimension-2 orientable fibration. By Chinen [1, Corollary 3.3], \( N \) is a codimension-2 fibration.

Case 2. \( N \) has a 2–1 covering. Since a finitely generated abelian group is residually finite, \( \pi_1(N) \) is residually finite. And, since \( \pi_1(N) \cong H \) is abelian, \( \tilde{N} \) is hopfian, and so \( N \) is a strongly hopfian manifold. Hence, \( N \) is a codimension-2 fibration.

(2) Note that for \( i \geq 2 \), \( 0 \to \pi_i(\tilde{N}) \to \pi_i(N) \to 0 \). Since \( \pi_i(N) = 0 \) for \( 1 < i < n - 1 \), \( \pi_i(\tilde{N}) = 0 \) for \( 1 < i < n - 1 \). By Swarup [18, Lemma 1.1], \( N \) is strongly hopfian.
(3) and (4) By Proposition 2.2, \( N \) is strongly hopfian.

(5) This follows from the facts that any closed surface has a residually finite fundamental group \([11]\) and Proposition 2.2. \( \square \)

Now, let us consider the following question:

**Question.** Is any finite product of codimension-2 fibrators a codimension-2 fibrator?, i.e., if \( N_1, N_2, \ldots, N_k \) are closed manifolds which are codimension-2 fibrators, is \( N_1 \times N_2 \times \cdots \times N_k \) a codimension-2 fibrator?

The answer is not yet settled. But the answer is yes for the case of each \( N_j \) (\( j = 1, 2, \ldots, k \)) a closed orientable surface ([12] and [13]). Here, we have an affirmative answer without assuming orientability for any \( N_j \) as follows:

**Corollary 3.2.** Any finite product of closed surfaces which are codimension-2 fibrators is a codimension-2 fibrator.

**Proof.** Let \( N_1, N_2, \ldots, N_k \) be closed surfaces which are codimension-2 fibrators, and \( N = N_1 \times N_2 \times \cdots \times N_k \). First, note that for all \( j = 1, \ldots, k \), \( \chi(N_j) \neq 0 \), for the torus and Klein bottle are the only examples of noncodimension-2 fibrators (see [5]). So we have \( \chi(N) \neq 0 \). Moreover,

\[
\pi_1(N) \cong \bigoplus_{j=1}^k \pi_1(N_j)
\]

is residually finite. Hence it suffices to show that \( N \) is strongly hopfian. If \( N \) has no 2–1 covering, then \( N \) must be orientable, so that each \( N_j \) is orientable. In [12] and [13], Im took care of this case. Hence we consider the case that \( N \) has a 2–1 covering. Since \( N \) is of the form \( \text{products of } RP^2 \times \text{products of } S^2 \times \text{products of closed surfaces which are neither } RP^2 \text{ nor } S^2 \), \( N \) must be of the form \( \text{products of } S^2 \times \text{products of closed surfaces which are aspherical} \). Hence \( \bar{N} \) is hopfian, which follows from the fact that any finite product of simply connected manifolds and aspherical closed manifolds with hopfian fundamental groups is hopfian (see [14]). \( \square \)

In closing, we mention the following unsettled topics.

**Question 1.** If \( N \) and \( N' \) are closed strongly hopfian manifolds with residually finite fundamental groups and nonzero Euler characteristics, then is \( N \times N' \) a codimension-2 fibrator? Furthermore, is any finite product of such manifolds a codimension-2 fibrator?

**Question 2.** What conditions on a closed manifold are necessary for being a codimension-2 fibrator? What if \( \chi(N) \neq 0 \)?
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References


[13] Young Ho Im, Decompositions into codimension two submanifolds that induce approximate fibrations, Topology Appl. 56 (1) (1994) 1–11.

[14] Young Ho Im, Mee Kwang Kang and Ki Mun Woo, Codimension-2 fibrators that is closed under finite product, Preprint.


