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Fuzzy hypergroups based on fuzzy relations

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ABSTRACT

Based on fuzzy reasoning in fuzzy logic, this paper studies a fuzzy hyperoperation and a fuzzy hypergroupoid associated with a fuzzy relation. A sufficient and necessary condition for such a fuzzy hypergroupoid being a fuzzy hypergroup is given, and the properties of the fuzzy hypergroups associated with fuzzy relations are investigated. Furthermore, the definition of normal fuzzy hypergroups is put forward and it is shown that the category **NFHG** of normal fuzzy hypergroups satisfies all the axioms of topos except for the subobject classifier axiom.

1. Introduction

The first step of the development of hyperstructure theory, in particular hypergroup theory, can be traced back to the 8th Congress of Scandinavian Mathematicians in 1934, when Marty [1] introduced the concept of hypergroup, analyzed its properties and applied them to groups, rational fractions and algebraic functions. Since then, the study of hyperstructure theory has made a great achievement both in the theories and applications. Nowadays hundreds of papers and several books on this topic have been published (see [2–5]). The principal notions of hypergroup theory can be found in [2]. Applications of hypergroup theory have mainly appeared in the areas of pure and applied mathematics and computer science [4].

After Zadeh [6] first proposed the concept of fuzzy sets in 1965, Rosenfeld [7] introduced fuzzy sets into group theory to formulate fuzzy subgroups of a group in 1971. Since then, many researchers are engaged in extending the concepts of abstract algebra to the framework of the fuzzy setting [8–10].

The study of fuzzy hyperstructures is an interesting research topic of fuzzy sets theory. There is a considerable amount of work on the connections between fuzzy sets and hyperstructures, which can be divided into three groups of papers. The first group of papers studies the crisp hyperoperations determined through fuzzy sets [11–18]. The second group of papers concerns the fuzzy hyperalgebras, which are a direct extension of the concepts of fuzzy algebras such as fuzzy subgroups, fuzzy rings, etc. For example, given a crisp hypergroup $\langle X, \circ\rangle$ and a fuzzy set μ, we say that μ is a fuzzy subhypergroup of $\langle X, \circ\rangle$ if every cut of μ is a crisp subhypergroup of $\langle X, \circ\rangle$ [19–21]. The third group of papers involves the fuzzy hyperoperations which are defined by assigning to every pair of elements a fuzzy set [22–29].

The last kind of fuzzy hyperstructure seems to have a more practical background. For example, Zadeh put forward the idea of fuzzy inference for complex systems and gave the famous CRI (compositional rule of inference) algorithm [30]. Given a double-input single-output system, let X and Y represent the input universes, and Z represent the output universe. The basic idea of CRI algorithm is as follows: Establish n rules of fuzzy reasoning according to the practical experience of experts, i.e.,

$$\text{IF } x \text{ IS } A_i \text{ AND } y \text{ IS } B_i, \text{ THEN } z \text{ IS } C_i, \quad i = 1, \ldots, n, \tag{1}$$

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where $x \in X$, $y \in Y$ and $z \in Z$ are basic variables, and $A_i \in \mathcal{F}(X)$, $B_i \in \mathcal{F}(Y)$ and $C_i \in \mathcal{F}(Z)$ $(i = 1, \dots, n)$ are linguistic values. The ith rule of fuzzy reasoning in (1) forms a fuzzy relation $R_i \in \mathcal{F}(X \times Y \times Z)$. These n rules are joined by "OR" naturally so that the total fuzzy reasoning relation $R = \bigcup_{i=1}^{n} R_i \in \mathcal{F}(X \times Y \times Z)$ is formed. In particular, when $X = Y = Z = H$, given an input $(x, y) \in H^2$, using the total fuzzy reasoning relation R we can obtain a fuzzy set $C(x, y) = (\{x\} \times \{y\}) \circ R \in \mathcal{F}(H)$. If we denote $x\tilde{\circ}y = C(x, y) = (\{x\} \times \{y\}) \circ R$, then $\tilde{\circ}$ is a fuzzy hyperoperation on H indeed.

Starting with such an analysis, the present paper studies the fuzzy hyperstructures associated with fuzzy relations and also belongs to the third group of papers above.

The rest of the paper is organized as follows. In Section 2 we recall some basic notions on fuzzy hypergroups and topos theory. In Section 3 we study the properties of fuzzy hypergroups based on fuzzy relations. In Section 4 we introduce the normal fuzzy hypergroups. In Section 5 we build up the category of normal fuzzy hypergroups.

2. Preliminaries

First of all, we recall some notions and results that we shall use in the following sections. Let H be a nonempty set. Let $\mathcal{F}(H) = [0, 1]^H$ be the set of all fuzzy subsets of H and $\mathcal{F}^*(H) = \mathcal{F}(H) \setminus \{\emptyset\}$. A fuzzy hyperoperation on H is a mapping $\tilde{\circ}\colon H^2 \to \mathcal{F}(H)$ and the couple $\langle H, \tilde{\circ}\rangle$ is called a partial fuzzy hypergroupoid. If the fuzzy hyperoperation $\tilde{\circ}$ maps H^2 into $\mathcal{F}^*(H)$, then $\langle H, \tilde{\circ}\rangle$ is called a fuzzy hypergroupoid.

If A and B are nonempty fuzzy subsets of H, then $A\tilde{\circ}B \in \mathcal{F}(H)$ is defined by

$$(A\tilde{\circ}B)(y) = \bigvee_{a,\ b\in H} (A(a) \wedge B(b) \wedge (a\tilde{\circ}b)(y)), \quad \forall y \in H.$$

Definition 2.1 (*[22]*).

(i) A fuzzy semihypergroup is a fuzzy hypergroupoid $\langle H, \tilde{\circ}\rangle$ which satisfies the associative law, i.e., $\forall x, y, z \in H,\ (x\tilde{\circ}y)\tilde{\circ}z = x\tilde{\circ}(y\tilde{\circ}z)$.
(ii) A fuzzy quasihypergroup is a fuzzy hypergroupoid $\langle H, \tilde{\circ}\rangle$ which satisfies the reproductive law, i.e., $\forall x \in H,\ x\tilde{\circ}H = H = H\tilde{\circ}x$.
(iii) A fuzzy hypergroup is a fuzzy semihypergroup which is also a fuzzy quasihypergroup.

A *fuzzy subhypergroup* $\langle K, \tilde{\circ}\rangle$ of a fuzzy hypergroup $\langle H, \tilde{\circ}\rangle$ is a nonempty subset $K \subseteq H$ such that for any $k \in K,\ k\tilde{\circ}K = K\tilde{\circ}k = K$.

Let $\langle H_1, \tilde{\circ}_1\rangle$ and $\langle H_2, \tilde{\circ}_2\rangle$ be two fuzzy hypergroups. A mapping $f\colon H_1 \to H_2$ is called a fuzzy hypergroup homomorphism if $\forall x, y \in H_1,\ f(x\tilde{\circ}_1y) \subseteq f(x)\tilde{\circ}_2f(y)$.

Definition 2.2 (*[31]*). A topos is a category $\mathscr{C}$ which satisfies the following five conditions:

(1) Finite products exist in $\mathscr{C}$. That is, for any objects $A, B \in \mathscr{C}$, there exist an object $C \in \mathscr{C}$ and morphisms $p_1 : C \to A$, $p_2 : C \to B$ such that for any morphisms $f : D \to A$ and $g : D \to B$, there exists a unique morphism $h : D \to C$ satisfying $p_1 \circ h = f$ and $p_2 \circ h = g$. Here C is denoted as $C = A \times B$.
(2) Equalizers exist in $\mathscr{C}$. That is, for any morphisms $f, g : A \to B$, there exist an object $E \in \mathscr{C}$ and a morphism $e : E \to A$ such that (i) $f \circ e = g \circ e$; (ii) for any morphism $e' : E' \to A$ satisfying $f \circ e' = g \circ e'$, there exists a unique morphism $\bar{e} : E' \to E$ with $e \circ \bar{e} = e'$.
(3) There is a terminal object U in $\mathscr{C}$. That is, for each object $A \in \mathscr{C}$, there exists exactly one morphism $! : A \to U$.
(4) Exponentials exist in $\mathscr{C}$. That is, for any objects $A,\ B \in \mathscr{C}$, there exist an object $B^A \in \mathscr{C}$ and a morphism $ev : B^A \times A \to B$ such that for any morphism $F : D \times A \to B$, there exists a unique morphism $\bar{F} : D \to B^A$ satisfying $ev \circ (\bar{F} \times \mathrm{Id}_A) = F$.
(5) There is a subobject classifier in $\mathscr{C}$. That is, there are an object $\Omega \in \mathscr{C}$ and a morphism $\top : U \to \Omega$ such that for each monomorphism $f : A' \to A$, there exists a unique morphism $\chi_f : A \to \Omega$ such that
  (i) $\chi_f \circ f = \top\circ !$;
  (ii) for each object B and morphism $g : B \to A$ with $\chi_f \circ g = \top\circ !$, there exists a unique morphism $\bar{g} : B \to A'$ with $g = f \circ \bar{g}$.

Definition 2.3 (*[32]*). Let R be a fuzzy relation on H. Then

(i) R is reflexive if for any $x \in H, R(x, x) = 1$;
(ii) R is symmetric if for any $x, y \in H, R(x, y) = R(y, x)$;
(iii) R is transitive if $R^2 \subseteq R$, i.e., for any $(x, y) \in H^2$,

$$\bigvee_{z\in H}(R(x, z) \wedge R(z, y)) \leq R(x, y).$$

Given two fuzzy relations ρ and σ on H, $\rho\sigma$ is a fuzzy relation on H determined by

$$(\rho\sigma)(x, z) = \bigvee_{y\in H}(\rho(x, y) \wedge \sigma(y, z)), \quad \forall x, z \in H.$$

3. Fuzzy hypergroups based on fuzzy relations

For a fuzzy relation ρ on H, we denote

$$\mathbb{D}(\rho) = \{x \in H \mid \exists y \in H, \rho(x, y) > 0\} \subseteq H$$

and

$$\mathbb{R}(\rho) = \{x \in H \mid \vee_{a \in H} \rho(a, x) = 1\} \subseteq H.$$

Example 3.1. Let $H = \{x_1, x_2, x_3, x_4\}$ and the fuzzy relations ρ_1 and ρ_2 defined on H be

ρ_1	x_1	x_2	x_3	x_4
x_1	0	0	0	0
x_2	0.5	1	0.7	0.9
x_3	0.6	0.8	1	0.6
x_4	0.4	0.8	0.6	0.9

ρ_2	x_1	x_2	x_3	x_4
x_1	1	0.6	0.4	0.2
x_2	0.7	1	0.5	0.3
x_3	0.5	0.8	1	0
x_4	0.3	0	0.6	1

Then, there are

$$\mathbb{D}(\rho_1) = \{x_2, x_3, x_4\} \subset H, \qquad \mathbb{D}(\rho_2) = \{x_1, x_2, x_3, x_4\} = H,$$
$$\mathbb{R}(\rho_1) = \{x_2, x_3\} \subset H, \qquad \mathbb{R}(\rho_2) = \{x_1, x_2, x_3, x_4\} = H.$$

For any $x \in H$, we denote

$$\tilde{L}_x^\rho : H \to [0, 1], \qquad \tilde{L}_x^\rho(z) = \rho(x, z), \quad \forall z \in H$$

and

$$\tilde{R}_x^\rho : H \to [0, 1], \qquad \tilde{R}_x^\rho(z) = \rho(z, x), \quad \forall z \in H.$$

To each fuzzy relation ρ on H, a partial fuzzy hypergroupoid $\mathbb{H}_\rho = \langle H, \tilde{\circ}_\rho \rangle$ is associated as follows:

$$\forall x, y \in H, \quad x \tilde{\circ}_\rho y = \tilde{L}_x^\rho \cup \tilde{L}_y^\rho.$$

Especially, $x \tilde{\circ}_\rho x = \tilde{L}_x^\rho$ for all $x \in H$.

Theorem 3.1. *If ρ is a reflexive and transitive fuzzy relation on H, then $\mathbb{H}_\rho = \langle H, \tilde{\circ}_\rho \rangle$ is a fuzzy hypergroup.*

Proof. Denote $\tilde{\circ} = \tilde{\circ}_\rho$. Since ρ is reflexive and transitive, for any $x, y, v \in H$,

(1) $(x \tilde{\circ} x)(x) = \rho(x, x) = 1$;
(2) $x \tilde{\circ} y = x \tilde{\circ} x \cup y \tilde{\circ} y$;
(3) $(x \tilde{\circ} x)(y) \geq (x \tilde{\circ} x)(v) \wedge (v \tilde{\circ} v)(y)$.

By (1) and (3), we have $(x \tilde{\circ} x)(y) \geq \bigvee_{v \in H}((x \tilde{\circ} x)(v) \wedge (v \tilde{\circ} v)(y)) \geq (x \tilde{\circ} x)(x) \wedge (x \tilde{\circ} x)(y) = (x \tilde{\circ} x)(y)$, i.e., $(x \tilde{\circ} x)(y) = \bigvee_{v \in H}((x \tilde{\circ} x)(v) \wedge (v \tilde{\circ} v)(y))$.

Now we show that $\mathbb{H}_\rho$ is a fuzzy hypergroup. For any $x, y, z \in H$ and $u \in H$, we have

$$\begin{aligned}
((x \tilde{\circ} y) \tilde{\circ} z)(u) &= \bigvee_{v \in H}((x \tilde{\circ} y)(v) \wedge (v \tilde{\circ} z)(u)) \\
&= \bigvee_{v \in H}((x \tilde{\circ} y)(v) \wedge ((v \tilde{\circ} v)(u) \vee (z \tilde{\circ} z)(u))) \\
&= \bigvee_{v \in H}(((x \tilde{\circ} y)(v) \wedge (v \tilde{\circ} v)(u)) \vee ((x \tilde{\circ} y)(v) \wedge (z \tilde{\circ} z)(u))) \\
&= \left(\bigvee_{v \in H}((x \tilde{\circ} y)(v) \wedge (v \tilde{\circ} v)(u))\right) \vee \left(\bigvee_{v \in H}((x \tilde{\circ} y)(v) \wedge (z \tilde{\circ} z)(u))\right) \\
&= \bigvee_{v \in H}(((x \tilde{\circ} x)(v) \wedge (v \tilde{\circ} v)(u)) \vee ((y \tilde{\circ} y)(v) \wedge (v \tilde{\circ} v)(u))) \vee (z \tilde{\circ} z)(u) \\
&= ((x \tilde{\circ} x)(u) \vee (y \tilde{\circ} y)(u)) \vee (z \tilde{\circ} z)(u).
\end{aligned}$$

Similarly, we obtain $(x \tilde{\circ} (y \tilde{\circ} z))(u) = (x \tilde{\circ} x)(u) \vee ((y \tilde{\circ} y)(u) \vee (z \tilde{\circ} z)(u))$. Thus $(x \tilde{\circ} y) \tilde{\circ} z = x \tilde{\circ} (y \tilde{\circ} z)$, and $\mathbb{H}_\rho$ is a fuzzy semihypergroup. For any $x \in H$ and $u \in H$, $(x \tilde{\circ} H)(u) = \bigvee_{v \in H}((x \tilde{\circ} v)(u) \wedge H(v)) = \bigvee_{v \in H}((x \tilde{\circ} x)(u) \vee (v \tilde{\circ} v)(u)) = (u \tilde{\circ} u)(u) = 1$ and $(H \tilde{\circ} x)(u) = \bigvee_{v \in H}(H(v) \wedge (v \tilde{\circ} x)(u)) = \bigvee_{v \in H}((v \tilde{\circ} v)(u) \vee (x \tilde{\circ} x)(u)) = (u \tilde{\circ} u)(u) = 1$. So $x \tilde{\circ} H = H = H \tilde{\circ} x$. Thus $\mathbb{H}_\rho$ is a fuzzy quasihypergroup. Therefore, $\mathbb{H}_\rho$ is a fuzzy hypergroup. □

Definition 3.1. Let $\lambda \in (0, 1]$. An element $x \in H$ is called a λ-outer element of ρ if there exists an $h_\lambda \in H$ such that $\rho^2(h_\lambda, x) < \lambda$. An element $x \in H$ is called an outer element of ρ if x is a λ-outer element of ρ for any $\lambda \in (0, 1]$.

Remark 3.1. In Example 3.1, for $x_1 \in H$, there are

$$\rho_2^2(x_1,x_1)=\bigvee_{y\in H}\rho_2(x_1,y)\wedge\rho_2(y,x_1)=\max\{1,0.6,0.4,0.2\}=1,$$

$$\rho_2^2(x_2,x_1)=\bigvee_{y\in H}\rho_2(x_2,y)\wedge\rho_2(y,x_1)=\max\{0.7,0.7,0.5,0.3\}=0.7,$$

$$\rho_2^2(x_3,x_1)=\bigvee_{y\in H}\rho_2(x_3,y)\wedge\rho_2(y,x_1)=\max\{0.5,0.7,0.5,0\}=0.7,$$

and

$$\rho_2^2(x_4,x_1)=\bigvee_{y\in H}\rho_2(x_4,y)\wedge\rho_2(y,x_1)=\max\{0.3,0,0.5,0.3\}=0.5.$$

Thus, for any $\lambda \in (0.5, 1]$, x_1 is a λ-outer element of ρ_2. Similarly, for any $\lambda \in (0.6, 1]$, x_2 is a λ-outer element of ρ_2; for any $\lambda \in (0.5, 1]$, x_3 is a λ-outer element of ρ_2; and for any $\lambda \in (0.3, 1]$, x_3 is a λ-outer element of ρ_2.

But for the fuzzy relation ρ_1, there are $\rho_1^2(x_1,x_1)=0$, $\rho_1^2(x_1,x_2)=0$, $\rho_1^2(x_1,x_3)=0$ and $\rho_1^2(x_1,x_4)=0$. Thus, for any $\lambda \in (0, 1]$, x_i $(i = 1, 2, 3, 4)$ is a λ-outer element of ρ_1, hence x_i $(i = 1, 2, 3, 4)$ is an outer element of ρ_1.

Remark 3.2. For $x \in H$, if there exists $h \in H$ such that $\rho^2(h,x)=0$, then x is an outer element of ρ.

Theorem 3.2. *$\mathbb{H}_\rho$ is a fuzzy hypergroup if and only if*

(a) $H=\mathbb{D}(\rho)$;
(b) $H=\mathbb{R}(\rho)$;
(c) $\rho\subseteq\rho^2$;
(d) *For any $\lambda\in(0,1]$, if $x\in H$ is a λ-outer element of ρ, then for $a\in H$, $\rho^2(a,x)\ge\lambda\Rightarrow\rho(a,x)\ge\rho^2(a,x)$.*

Proof. Denote $\tilde{\circ}=\tilde{\circ}_\rho$. "*Necessity*" Suppose $\mathbb{H}_\rho$ is a fuzzy hypergroup.

(a) For any $x \in H$, we have $x\tilde{\circ}x\in\mathcal{F}^*(H)$, and there exists a $y \in H$ such that $(x\tilde{\circ}x)(y)>0$, so $x\in\mathbb{D}(\rho)$. Hence $H\subseteq\mathbb{D}(\rho)$, and we have $H=\mathbb{D}(\rho)$.
(b) For any $y\in H$, $1=\chi_H(y)=(y\tilde{\circ}H)(y)=\bigvee_{a\in H}(y\tilde{\circ}a)(y)=\bigvee_{a\in H}(a\tilde{\circ}a)(y)$, so $y\in\mathbb{R}(\rho)$. Hence $H\subseteq\mathbb{R}(\rho)$, and we have $H=\mathbb{R}(\rho)$.
(c) For any $x,z\in H$, let $\lambda=\rho(x,z)$, i.e., $(x\tilde{\circ}x)(z)=\lambda$. Since $(x\tilde{\circ}(x\tilde{\circ}z))(z)=\bigvee_{y\in H}((x\tilde{\circ}y)(z)\wedge(x\tilde{\circ}z)(y))\ge\bigvee_{y\in H}((x\tilde{\circ}x)(z)\wedge(x\tilde{\circ}z)(y))=(x\tilde{\circ}x)(z)=\lambda$, we have

$$\begin{aligned}((x\tilde{\circ}x)\tilde{\circ}z)(z)&=\bigvee_{y\in H}((y\tilde{\circ}z)(z)\wedge(x\tilde{\circ}x)(y))\\&=\bigvee_{y\in H}(((y\tilde{\circ}y)(z)\wedge(x\tilde{\circ}x)(y))\vee((z\tilde{\circ}z)(z)\wedge(x\tilde{\circ}x)(y)))\\&=\left(\bigvee_{y\in H}(y\tilde{\circ}y)(z)\wedge(x\tilde{\circ}x)(y)\right)\vee\left(\bigvee_{y\in H}(z\tilde{\circ}z)(z)\wedge(x\tilde{\circ}x)(y)\right)\\&\ge\lambda.\end{aligned}$$

If $\bigvee_{y\in H}((y\tilde{\circ}y)(z)\wedge(x\tilde{\circ}x)(y))\ge\lambda$, then it implies $\rho^2(x,z)\ge\lambda$. If $\bigvee_{y\in H}((z\tilde{\circ}z)(z)\wedge(x\tilde{\circ}x)(y))\ge\lambda$, then we have $(z\tilde{\circ}z)(z)\ge\lambda$, thus $\rho^2(x,z)\ge\rho(x,z)\wedge\rho(z,z)=\lambda$. Hence $\rho^2(x,z)\ge\rho(x,z)$. Therefore, $\rho\subseteq\rho^2$.
(d) Let $\lambda\in(0,1]$ and $x\in H$ be a λ-outer element of ρ. For $a\in H$, let $\theta=\rho^2(a,x)\ge\lambda$. Suppose $\rho(a,x)<\theta$. Since x is a λ-outer element of ρ, there exists an $h_\lambda\in H$ such that $\rho^2(h_\lambda,x)=\bigvee_{b\in H}(\rho(h_\lambda,b)\wedge\rho(b,x))<\lambda\le\theta$. Then

$$\begin{aligned}\theta>\bigvee_{b\in H}((a\tilde{\circ}b)(x)\wedge(h_\lambda\tilde{\circ}h_\lambda)(b))&=(a\tilde{\circ}(h_\lambda\tilde{\circ}h_\lambda))(x)\\&=((a\tilde{\circ}h_\lambda)\tilde{\circ}h_\lambda)(x)\\&=\bigvee_{y\in H}((y\tilde{\circ}h_\lambda)(x)\wedge(a\tilde{\circ}h_\lambda)(y))\\&\ge\bigvee_{y\in H}((a\tilde{\circ}a)(y)\wedge(y\tilde{\circ}h_\lambda)(x))\\&\ge\bigvee_{y\in H}((a\tilde{\circ}a)(y)\wedge(y\tilde{\circ}y)(x))\\&=\rho^2(a,x),\end{aligned}$$

which leads to a contradiction to $\rho^2(a,x)=\theta$. Hence $\rho(a,x)\ge\theta$, i.e., $\rho(a,x)\ge\rho^2(a,x)$.

"*Sufficiency*" By condition (a), $\mathbb{H}_\rho$ is a fuzzy hypergroupoid.

For any $x, y, \in H$ and $u \in H$, we have $((x\tilde{\circ}y)\circ z)(u) = \bigvee_{a\in H}((x\tilde{\circ}x)(a) \wedge (a\tilde{\circ}a)(u)) \vee \bigvee_{b\in H}((y\tilde{\circ}y)(b) \wedge (b\tilde{\circ}b)(u)) \vee (z\tilde{\circ}z)(u)$ and $(x\tilde{\circ}(y\circ z))(u) = (x\tilde{\circ}x)(u) \vee \bigvee_{b\in H}((y\tilde{\circ}y)(b) \wedge (b\tilde{\circ}b)(u)) \vee \bigvee_{c\in H}((z\tilde{\circ}z)(c) \wedge (c\tilde{\circ}c)(u))$. By condition (c),

$$(x\tilde{\circ}x)(u) = \rho(x, u) \leq \rho^2(x, u) = \bigvee_{a\in H}((x\tilde{\circ}x)(a) \wedge (a\tilde{\circ}a)(u)).$$

First, we show that $((x\tilde{\circ}y)\circ z)(u) \leq (x\tilde{\circ}(y\circ z))(u)$. Let $\lambda = ((x\tilde{\circ}y)\circ z)(u)$. If $(z\tilde{\circ}z)(u) = \lambda$ or $\bigvee_{b\in H}((y\tilde{\circ}y)(b) \wedge (b\tilde{\circ}b)(u)) = \lambda$, then we have $(x\tilde{\circ}(y\circ z))(u) \geq \lambda$. Otherwise, $\bigvee_{a\in H}((x\tilde{\circ}x)(a) \wedge (a\tilde{\circ}a)(u)) = \rho^2(x, u) = \lambda$and $\bigvee_{b\in H}((y\tilde{\circ}y)(b) \wedge (b\tilde{\circ}b)(u)) \vee (z\tilde{\circ}z)(u) < \lambda$. Next, we will show that $(x\tilde{\circ}x)(u) = \lambda$. Since $\bigvee_{b\in H}((y\tilde{\circ}y)(b) \wedge (b\tilde{\circ}b)(u)) < \lambda$, u is a λ-outer element of ρ. By conditions (c), (d) and $\rho^2(x, u) = \lambda$, we have $\rho(x, u) = \rho^2(x, u)$, i.e., $(x\tilde{\circ}x)(u) = \lambda$. So $((x\tilde{\circ}y)\circ z)(u) \leq (x\tilde{\circ}(y\circ z))(u)$. Similarly, we can show $(x\tilde{\circ}(y\circ z))(u) \leq ((x\tilde{\circ}y)\circ z)(u)$. Hence $((x\tilde{\circ}y)\circ z)(u) = (x\tilde{\circ}(y\circ z))(u)$. Thus $(x\tilde{\circ}y)\circ z = x\tilde{\circ}(y\circ z)$.

For any $x, u \in H$, $(x\tilde{\circ}H)(u) = \bigvee_{a\in H}(x\tilde{\circ}a)(u) = \bigvee_{a\in H}(a\tilde{\circ}a)(u)$. By condition (b), for $u \in H = \mathbb{R}(\rho)$, we have $\bigvee_{a\in H}(a\tilde{\circ}a)(u) = 1$, so $(x\tilde{\circ}H)(u) = 1$. Thus $x\tilde{\circ}H = H$. Similarly we have $H\tilde{\circ}x = H$.

Therefore, $\mathbb{H}_\rho$ is a fuzzy hypergroup. □

Next, we will illustrate Theorem 3.2 step by step by an example.

Example 3.2. Let $H = \{x, y, z\}$ and ρ be a fuzzy relation on H defined by

ρ	x	y	z
x	1	0.8	0.7
y	0.6	1	0.6
z	0.5	0.5	1

Then the partial fuzzy hypergroupoid $\mathbb{H}_\rho = \langle H, \tilde{\circ}_\rho\rangle$ is determined by

$$\begin{cases} x\tilde{\circ}_\rho x = \tilde{L}_x^\rho = (1, 0.8, 0.7) \\ y\tilde{\circ}_\rho y = \tilde{L}_y^\rho = (0.6, 1, 0.6) \\ z\tilde{\circ}_\rho z = \tilde{L}_z^\rho = (0.5, 0.5, 1) \\ x\tilde{\circ}_\rho y = y\tilde{\circ}_\rho x = x\tilde{\circ}_\rho x \cup y\tilde{\circ}_\rho y = (1, 1, 0.7) \\ x\tilde{\circ}_\rho z = z\tilde{\circ}_\rho x = x\tilde{\circ}_\rho x \cup z\tilde{\circ}_\rho z = (1, 0.8, 1) \\ y\tilde{\circ}_\rho z = z\tilde{\circ}_\rho y = y\tilde{\circ}_\rho y \cup z\tilde{\circ}_\rho z = (0.6, 1, 1). \end{cases}$$

Since ρ is reflexive and transitive, it is only necessary to check the "*Necessity*" since by Theorem 3.1 that $\mathbb{H}_\rho = \langle H, \tilde{\circ}_\rho\rangle$ is a fuzzy hypergroup.

Firstly, by the definitions of $\mathbb{D}(\rho)$ and $\mathbb{R}(\rho)$ and the given fuzzy relation ρ that $\mathbb{D}(\rho) = \mathbb{R}(\rho) = H$. Thus the conditions (a) and (b) in Theorem 3.2 hold.

Secondly, there is

$$\rho^2 = \begin{pmatrix} 1 & 0.8 & 0.7 \\ 0.6 & 1 & 0.6 \\ 0.5 & 0.5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0.8 & 0.7 \\ 0.6 & 1 & 0.6 \\ 0.5 & 0.5 & 1 \end{pmatrix} \supseteq \begin{pmatrix} 1 & 0.8 & 0.7 \\ 0.6 & 1 & 0.6 \\ 0.5 & 0.5 & 1 \end{pmatrix} = \rho. \tag{2}$$

Thirdly, for any $\lambda \in (0.5, 1]$, x_1 and x_2 are λ-outer elements of ρ and for any $\lambda \in (0.6, 1]$, x_3 is a λ-outer element of ρ. For any $\lambda \in (0.5, 0.6]$, there is $\rho^2(x_1, x_1) > \rho^2(x_2, x_1) \geq \lambda$, it follows from Eq. (2) that $\rho(x_1, x_1) \geq \rho^2(x_1, x_1)$ and $\rho(x_2, x_1) \geq \rho^2(x_2, x_1)$. Similarly, it can be shown that condition (d) in Theorem 3.2 holds.

Till then, the necessary conditions of Theorem 3.2 hold, thus the partial fuzzy hypergroupoid $\mathbb{H}_\rho = \langle H, \tilde{\circ}_\rho\rangle$ is a fuzzy hypergroup. This can also be deduced directly from Theorem 3.1 for that ρ is reflexive and transitive.

Proposition 3.1. *Let $\mathbb{H}_\rho$ be a fuzzy hypergroup. Then*

(1) *ρ^2 is a transitive fuzzy relation on H.*
(2) *If ρ is symmetric, then ρ^2 is a fuzzy equivalence relation on H.*

Proof. (1) Suppose that there exist $x, y, z \in H$ such that $\rho^2(x, z) < \rho^2(x, y) \wedge \rho^2(y, z)$. Let $\lambda = \rho^2(x, y) \wedge \rho^2(y, z)$. Then $\rho^2(x, z) < \lambda$, which means that z is a λ-outer element of ρ. By Theorem 3.2(c) and (d), $\rho^2(y, z) \geq \lambda \Rightarrow \rho(y, z) = \rho^2(y, z) \geq \lambda$. If $\rho^2(y, z) = \lambda$, then $\rho^2(x, y) \geq \lambda$. Thus

$$\begin{aligned} \lambda &= \rho^2(x, y) \wedge \rho(y, z) \\ &= \left(\bigvee_{a\in H} \rho(x, a) \wedge \rho(a, y)\right) \wedge \rho(y, z) \\ &= \bigvee_{a\in H}(\rho(x, a) \wedge \rho(a, y) \wedge \rho(y, z)) \end{aligned}$$

$$\leq \bigvee_{a\in H}(\rho(x,a)\wedge\rho^2(a,z))$$
$$= \bigvee_{a\in H,\ \rho^2(a,z)\geq\lambda}(\rho(x,a)\wedge\rho(a,z))\leq\rho^2(x,z),$$

which is a contradiction to $\rho^2(x,z)<\lambda$. If $\rho^2(y,z)>\lambda$, then $\rho^2(x,y)=\lambda$. Since $\rho(y,z)=\rho^2(y,z)>\lambda$, we have $\lambda=\rho^2(x,y)\wedge\rho(y,z)\leq\rho^2(x,z)$, which also leads to a contradiction. Thus $\rho^2(x,z)\geq\bigvee_{y\in H}\rho^2(x,y)\wedge\rho^2(y,z)$ for all $x,z\in H$, i.e., ρ^2 is transitive.

(2) Let ρ be symmetric. For any $x\in H$, $x\in\mathbb{R}(\rho)$ by Theorem 3.2(b), and so $\bigvee_{a\in H}\rho(a,x)=1$. Hence $\rho^2(x,x)=\bigvee_{a\in H}(\rho(x,a)\wedge\rho(a,x))=\bigvee_{a\in H}\rho(a,x)=1$, proving the reflexivity of ρ^2. Clearly, ρ^2 is symmetric. By (1), ρ^2 is a fuzzy equivalence relation on H. □

Proposition 3.2. *If ρ and σ are fuzzy relations on H, then for any $a\in H$,*

(1) $a\tilde{\circ}_\rho a\tilde{\circ}_\rho a=(a\tilde{\circ}_{\rho^2}a)\cup(a\tilde{\circ}_\rho a)$;
(2) $(a\tilde{\circ}_\rho a)\tilde{\circ}_\sigma(a\tilde{\circ}_\rho a)=a\tilde{\circ}_{\rho\sigma}a$;
(3) $a\tilde{\circ}_{\rho\cup\sigma}a=a\tilde{\circ}_\rho a\cup a\tilde{\circ}_\sigma a$; $a\tilde{\circ}_{\rho\cap\sigma}a=a\tilde{\circ}_\rho a\cap a\tilde{\circ}_\sigma a$.

Proof. (1) For any $u\in H$,

$$\begin{aligned}(a\tilde{\circ}_\rho a\tilde{\circ}_\rho a)(u) &= \bigvee_{t\in H}((t\tilde{\circ}_\rho a)(u)\wedge(a\tilde{\circ}_\rho a)(t))\\ &= \bigvee_{t\in H}(((t\tilde{\circ}_\rho t)(u)\vee(a\tilde{\circ}_\rho a)(u))\wedge(a\tilde{\circ}_\rho a)(t))\\ &= \left(\bigvee_{t\in H}((t\tilde{\circ}_\rho t)(u)\wedge(a\tilde{\circ}_\rho a)(t))\right)\vee\left(\bigvee_{t\in H}((a\tilde{\circ}_\rho a)(u)\wedge(a\tilde{\circ}_\rho a)(t))\right)\\ &= \left(\bigvee_{t\in H}((t\tilde{\circ}_\rho t)(u)\wedge(a\tilde{\circ}_\rho a)(t))\right)\vee(a\tilde{\circ}_\rho a)(u)\\ &= (a\tilde{\circ}_{\rho^2}a)(u)\vee(a\tilde{\circ}_\rho a)(u)\\ &= (a\tilde{\circ}_{\rho^2}a\cup a\tilde{\circ}_\rho a)(u).\end{aligned}$$

(2) For any $u\in H$,

$$\begin{aligned}((a\tilde{\circ}_\rho a)\tilde{\circ}_\sigma(a\tilde{\circ}_\rho a))(u) &= \bigvee_{x,\ y\in H}((x\tilde{\circ}_\sigma y)(u)\wedge(a\tilde{\circ}_\rho a)(x)\wedge(a\tilde{\circ}_\rho a)(y))\\ &= \left(\bigvee_{x,\ y\in H}((x\tilde{\circ}_\sigma x)(u)\wedge(a\tilde{\circ}_\rho a)(x)\wedge(a\tilde{\circ}_\rho a)(y))\right)\\ &\quad\vee\left(\bigvee_{x,\ y\in H}((y\tilde{\circ}_\sigma y)(u)\wedge(a\tilde{\circ}_\rho a)(x)\wedge(a\tilde{\circ}_\rho a)(y))\right)\\ &= \left(\bigvee_{t\in H}((a\tilde{\circ}_\rho a)(t)\wedge(t\tilde{\circ}_\sigma t)(u))\right)\wedge\left(\bigvee_{t\in H}(a\tilde{\circ}_\rho a)(t)\right)\\ &= (a\tilde{\circ}_{\rho\sigma}a)(u).\end{aligned}$$

(3) is straightforward. □

Corollary 3.1. *Let ρ be a fuzzy relation on H. If $\rho\subseteq\rho^2$, then*

(1) $\rho^2(a,x)=(a\tilde{\circ}_\rho a\tilde{\circ}_\rho a)(x)$ *for any $a,x\in H$;*
(2) *$x\in H$ is a λ-outer element of ρ if and only if there exists an $a_\lambda\in H$ such that $(a_\lambda\tilde{\circ}_\rho a_\lambda\tilde{\circ}_\rho a_\lambda)(x)<\lambda$;*
(3) *H has no 1-outer element of ρ if and only if $a\tilde{\circ}_\rho a\tilde{\circ}_\rho a=H$ for any $a\in H$.*

Remark 3.3. Let ρ be a fuzzy relation on H. Then

(1) ρ is transitive if and only if $a\tilde{\circ}_\rho a\tilde{\circ}_\rho a=a\tilde{\circ}_\rho a$ for any $a\in H$;
(2) $\rho\subseteq\rho^2$ if and only if $a\tilde{\circ}_\rho a\tilde{\circ}_\rho a=(a\tilde{\circ}_\rho a)\tilde{\circ}_\rho(a\tilde{\circ}_\rho a)$ for any $a\in H$.

Theorem 3.3. *If ρ is a reflexive and non-transitive fuzzy relation on H, then the following are equivalent to each other:*

(1) *$x\tilde{\circ}_\rho x\tilde{\circ}_\rho x=H$ for any $x\in H$;*
(2) *H has no 1-outer element of ρ;*
(3) *$\rho^2=H\times H$.*

On a fuzzy hypergroup $\langle H, \tilde{\circ}\rangle$, the following fuzzy equivalence relations, called *the fuzzy operational equivalence*, *the fuzzy inseparability* and *the fuzzy essential indistinguishability*, respectively, may be defined:

- $x \sim_{fo} y \Leftrightarrow x\tilde{\circ}a = y\tilde{\circ}a$ and $a\tilde{\circ}x = a\tilde{\circ}y$ for any $a \in H$;
- $x \sim_{fi} y \Leftrightarrow (a\tilde{\circ}b)(x) = (a\tilde{\circ}b)(y)$ for any $a, b \in H$;
- $x \sim_{fe} y \Leftrightarrow x \sim_{fo} y$ and $x \sim_{fi} y$.

Example 3.3. Let $H = \{x_1, x_2, x_3\}$. The fuzzy relation defined on H is given by

ρ	x_1	x_2	x_3
x_1	1	1	0.8
x_2	1	1	0.8
x_3	0.8	0.8	1

Then $\langle H, \tilde{\circ}_\rho\rangle$ is a fuzzy hypergroup. Since

$$x\tilde{\circ}_\rho y = x\tilde{\circ}_\rho x = y\tilde{\circ}_\rho x = (1, 1, 0.5), \qquad x\tilde{\circ}_\rho y = y\tilde{\circ}_\rho y = y\tilde{\circ}_\rho x = (1, 1, 0.5)$$

and

$$x\tilde{\circ}_\rho z = y\tilde{\circ}_\rho z = z\tilde{\circ}_\rho x = z\tilde{\circ}_\rho y = (1, 1, 1),$$

then there is $x \sim_{fo} y$. Similarly, there is $x \sim_{fi} y$, therefore, $x \sim_{fe} y$.

Proposition 3.3. *If $\mathbb{H}_\rho$ is a fuzzy hypergroup, then for any $x, y \in H$,*

(1) $x \sim_{fo} y \Leftrightarrow \tilde{L}_x^\rho = \tilde{L}_y^\rho$;
(2) $x \sim_{fi} y \Leftrightarrow \tilde{R}_x^\rho = \tilde{R}_y^\rho$.

Proof. (1) For any $a \in H$, $x\tilde{\circ}_\rho a = y\tilde{\circ}_\rho a$ and $a\tilde{\circ}_\rho x = a\tilde{\circ}_\rho y$ are equivalent to $\tilde{L}_x^\rho \cup \tilde{L}_a^\rho = \tilde{L}_y^\rho \cup \tilde{L}_a^\rho$. So if $\tilde{L}_x^\rho = \tilde{L}_y^\rho$, then $x \sim_{fo} y$. Suppose $x \sim_{fo} y$. Then for any $a \in H$, $\tilde{L}_x^\rho \cup \tilde{L}_a^\rho = \tilde{L}_y^\rho \cup \tilde{L}_a^\rho$. Putting $a = x$ gives $\tilde{L}_x^\rho = \tilde{L}_y^\rho \cup \tilde{L}_x^\rho$, so $\tilde{L}_y^\rho \subseteq \tilde{L}_x^\rho$. Putting $a = y$ gives $\tilde{L}_y^\rho = \tilde{L}_x^\rho \cup \tilde{L}_y^\rho$, so $\tilde{L}_x^\rho \subseteq \tilde{L}_x^\rho$. Thus $\tilde{L}_x^\rho = \tilde{L}_y^\rho$.

(2) For any $a, b \in H$, $(a\tilde{\circ}_\rho b)(x) = (a\tilde{\circ}_\rho b)(y)$ is equivalent to $(a\tilde{\circ}_\rho a)(x) \vee (b\tilde{\circ}_\rho b)(x) = (a\tilde{\circ}_\rho a)(y) \vee (b\tilde{\circ}_\rho b)(y)$, i.e., $\tilde{R}_x^\rho(a) \vee \tilde{R}_x^\rho(b) = \tilde{R}_y^\rho(a) \vee \tilde{R}_y^\rho(b)$. So if $\tilde{R}_x^\rho = \tilde{R}_y^\rho$, then $x \sim_{fi} y$. Suppose $x \sim_{fi} y$. Then for any $a, b \in H$, $\tilde{R}_x^\rho(a) \vee \tilde{R}_x^\rho(b) = \tilde{R}_y^\rho(a) \vee \tilde{R}_y^\rho(b)$. Putting $a = b$ gives $\tilde{R}_x^\rho(a) = \tilde{R}_y^\rho(a)$. Hence $\tilde{R}_x^\rho = \tilde{R}_y^\rho$. □

Proposition 3.4. *Let ρ and σ be two reflexive and transitive fuzzy relations on H and $x, y \in H$. Then $x \sim_{fe} y$ in $\mathbb{H}_{\rho\cap\sigma}$ if and only if $x \sim_{fe} y$ in $\mathbb{H}_\rho$ and $x \sim_{fe} y$ in $\mathbb{H}_\sigma$.*

Proof. Since $(\rho \cap \sigma)^2 \subseteq \rho^2 \subseteq \rho$ and $(\rho \cap \sigma)^2 \subseteq \sigma^2 \subseteq \sigma$, $(\rho \cap \sigma)^2 \subseteq \rho \cap \sigma$ and so $\rho \cap \sigma$ is also reflexive and transitive. By Theorem 3.1, the fuzzy hypergroupoids $\mathbb{H}_\rho$, $\mathbb{H}_\sigma$ and $\mathbb{H}_{\rho\cap\sigma}$ are fuzzy hypergroups.

Suppose that $x \sim_{fe} y$ in H_ρ and $x \sim_{fe} y$ in $\mathbb{H}_\sigma$. Then $x\tilde{\circ}_\rho x = y\tilde{\circ}_\rho x = y\tilde{\circ}_\rho y$ and $x\tilde{\circ}_\sigma x = y\tilde{\circ}_\sigma x = y\tilde{\circ}_\sigma y$. So $x\tilde{\circ}_\rho x \cap x\tilde{\circ}_\sigma x = y\tilde{\circ}_\rho y \cap y\tilde{\circ}_\sigma y$. By Proposition 3.2(3), $x\tilde{\circ}_{\rho\cap\sigma} x = y\tilde{\circ}_{\rho\cap\sigma} y$, so $x \sim_{fo} y$ in $\mathbb{H}_{\rho\cap\sigma}$. For any $z \in H$, we have $(z\tilde{\circ}_\rho z)(x) = (z\tilde{\circ}_\rho z)(y)$ and $(z\tilde{\circ}_\sigma z)(x) = (z\tilde{\circ}_\sigma z)(y)$. By Proposition 3.2(3), $(z\tilde{\circ}_{\rho\cap\sigma} z)(x) = (z\tilde{\circ}_{\rho\cap\sigma} z)(y)$, so $x \sim_{fi} y$ in $\mathbb{H}_{\rho\cap\sigma}$. Therefore, $x \sim_{fe} y$ in $\mathbb{H}_{\rho\cap\sigma}$.

Conversely, suppose $x \sim_{fe} y$ in $\mathbb{H}_{\rho\cap\sigma}$. It suffices to show that $x\tilde{\circ}_{\rho\cap\sigma} x = y\tilde{\circ}_{\rho\cap\sigma} y$ implies $x\tilde{\circ}_\rho x = y\tilde{\circ}_\rho y$ and $x\tilde{\circ}_\sigma x = y\tilde{\circ}_\sigma y$, and that $(z\tilde{\circ}_{\rho\cap\sigma} z)(x) = (z\tilde{\circ}_{\rho\cap\sigma} z)(y)$ implies $(z\tilde{\circ}_\rho z)(x) = (z\tilde{\circ}_\rho z)(y)$ and $(z\tilde{\circ}_\sigma z)(x) = (z\tilde{\circ}_\sigma z)(y)$.

In fact, since ρ and σ are reflexive, $(x\tilde{\circ}_{\rho\cap\sigma} x)(x) = 1$, and so $(y\tilde{\circ}_{\rho\cap\sigma} y)(x) = 1$. For any $z \in H$, $(y\tilde{\circ}_\rho y)(z) \geq (y\tilde{\circ}_\rho y)(x) \wedge (x\tilde{\circ}_\rho x)(z) = (x\tilde{\circ}_\rho x)(z)$, so $x\tilde{\circ}_\rho x \subseteq y\tilde{\circ}_\rho y$. Similarly, we have $y\tilde{\circ}_\rho y \subseteq x\tilde{\circ}_\rho x$. Hence $x\tilde{\circ}_\rho x = y\tilde{\circ}_\rho y$. In the same way, we obtain $x\tilde{\circ}_\sigma x = y\tilde{\circ}_\sigma y$.

For any $z \in H$, $(z\tilde{\circ}_\rho z)(y) \geq (z\tilde{\circ}_\rho z)(x) \wedge (x\tilde{\circ}_\rho x)(y) = (z\tilde{\circ}_\rho z)(x)$. Similarly, we have $(z\tilde{\circ}_\rho z)(x) \geq (z\tilde{\circ}_\rho z)(y)$. Hence $(z\tilde{\circ}_\rho z)(y) = (z\tilde{\circ}_\rho z)(x)$. In the same way, we get $(z\tilde{\circ}_\sigma z)(y) = (z\tilde{\circ}_\sigma z)(x)$. □

Proposition 3.5. *Let ρ and σ be two reflexive and transitive fuzzy relations on H such that $\rho\sigma = \sigma\rho$. If for $x, y \in H$, $x \sim_{fo} y$ in $\mathbb{H}_\rho$ and $x \sim_{fi} y$ in $\mathbb{H}_\sigma$, then $x \sim_{fe} y$ in $\mathbb{H}_{\rho\sigma}$. Moreover, $x \sim_{fe} y$ in $\mathbb{H}_\rho$ and $x \sim_{fe} y$ in $\mathbb{H}_\sigma$ lead to $x \sim_{fe} y$ in $\mathbb{H}_{\rho\sigma}$.*

Proof. Since $(\rho\sigma)^2 = \rho\sigma\rho\sigma = \rho\rho\sigma\sigma \subseteq \rho\sigma$, $\rho\sigma$ is also reflexive and transitive. By Theorem 3.1, the fuzzy hypergroupoids $\mathbb{H}_\rho$, $\mathbb{H}_\sigma$ and $\mathbb{H}_{\rho\sigma}$ are fuzzy hypergroups.

Suppose $x \sim_{fo} y$ in $\mathbb{H}_\rho$ and $x \sim_{fi} y$ in $\mathbb{H}_\sigma$. Then $\tilde{L}_x^\rho = \tilde{L}_y^\rho$ and $\tilde{R}_x^\sigma = \tilde{R}_y^\sigma$. For any $z \in H$, $\tilde{L}_x^{\rho\sigma}(z) = \bigvee_{t\in H}(\tilde{L}_x^\rho(t) \wedge \tilde{L}_t^\sigma(z)) = \bigvee_{t\in H}(\tilde{L}_y^\rho(t) \wedge \tilde{L}_t^\sigma(z)) = \tilde{L}_y^{\rho\sigma}(z)$ and $\tilde{R}_x^{\rho\sigma}(z) = \bigvee_{t\in H}(\tilde{R}_x^\rho(t) \wedge \tilde{R}_t^\sigma(z)) = \bigvee_{t\in H}(\tilde{R}_y^\rho(t) \wedge \tilde{R}_t^\sigma(z)) = \tilde{R}_y^{\rho\sigma}(z)$. Hence $\tilde{L}_x^{\rho\sigma} = \tilde{L}_y^{\rho\sigma}$ and $\tilde{R}_x^{\rho\sigma} = \tilde{R}_y^{\rho\sigma}$. Thus $x \sim_{fe} y$ in $\mathbb{H}_{\rho\sigma}$. The last assertion is straightforward. □

4. Normal fuzzy hypergroups

In this section, we introduce the normal fuzzy hypergroups.

Definition 4.1. A fuzzy hypergroup $\langle H, \tilde{\circ}\rangle$ is said to be normal if it satisfies the following three conditions:

(1) $(x\tilde{\circ}x)(x) = 1$ for all $x \in H$;
(2) $x\tilde{\circ}y = x\tilde{\circ}x \cup y\tilde{\circ}y$ for all $x, y \in H$;
(3) $(x\tilde{\circ}x)(z) \geq (x\tilde{\circ}x)(y) \wedge (y\tilde{\circ}y)(z)$ for all $x, y, z \in H$.

Theorem 4.1. *Let $\tilde{\circ}$ be a fuzzy hyperoperation on H. If*

(i) $(x\tilde{\circ}x)(x) = 1$ *for all* $x \in H$*;*
(ii) $x\tilde{\circ}y = x\tilde{\circ}x \cup y\tilde{\circ}y$ *for all* $x, y \in H$*;*
(iii) $(x\tilde{\circ}x)(z) \geq (x\tilde{\circ}x)(y) \wedge (y\tilde{\circ}y)(z)$ *for all* $x, y, z \in H$*,*

then the fuzzy hypergroupoid $\langle H, \tilde{\circ}\rangle$ is a normal fuzzy hypergroup.

Proof. As in the proof of Theorem 3.1 we can verify that $\langle H, \tilde{\circ}\rangle$ is a fuzzy hypergroup. Then $\langle H, \tilde{\circ}\rangle$ is a normal fuzzy hypergroup. □

Corollary 4.1. *Let ρ be a fuzzy relation on H. If ρ is reflexive and transitive, then $\mathbb{H}_\rho$ is a normal fuzzy hypergroup.*

Theorem 4.2. *Let $\langle H, \tilde{\circ}\rangle$ be a normal fuzzy hypergroup. Then the fuzzy relation defined by $\rho(x, y) = (x\tilde{\circ}x)(y) \wedge (y\tilde{\circ}y)(x)$, $\forall x,\ y \in H$, is a fuzzy equivalence relation on H.*

Proof. By the definition of ρ, there are $\rho(x, x) = (x\tilde{\circ}x)(x) = 1$, $\rho(x, y) = (x\tilde{\circ}x)(y) \wedge (y\tilde{\circ}y)(x) = \rho(y, x)$, and

$$\begin{aligned}\rho(x, y) \wedge \rho(y, z) &= ((x\tilde{\circ}x)(y) \wedge (y\tilde{\circ}y)(x)) \wedge ((y\tilde{\circ}y)(z) \wedge (z\tilde{\circ}z)(y)) \\ &= ((x\tilde{\circ}x)(y) \wedge (y\tilde{\circ}y)(z)) \wedge ((z\tilde{\circ}z)(y) \wedge (y\tilde{\circ}y)(x)) \\ &\leq (x\tilde{\circ}x)(z) \wedge (z\tilde{\circ}z)(x) \\ &= \rho(x, z).\end{aligned}$$

Thus, ρ is a fuzzy equivalence relation on H. □

Example 4.1. Let $H = \{x_1, x_2, x_3, x_4, x_5\}$. Define a fuzzy hyperoperation $\tilde{\circ}$ on H by

$$\begin{cases} x_1\tilde{\circ}x_1 = (1, 0.4, 0.8, 0.5, 0.5) \\ x_2\tilde{\circ}x_2 = (0.4, 1, 0.4, 0.4, 0.4) \\ x_3\tilde{\circ}x_3 = (0.8, 0.4, 1, 0.5, 0.5) \\ x_4\tilde{\circ}x_4 = (0.5, 0.4, 0.5, 1, 0.6) \\ x_5\tilde{\circ}x_5 = (0.5, 0.4, 0.5, 0.6, 1) \\ x_i\tilde{\circ}x_j = x_i\tilde{\circ}x_i \cup x_j\tilde{\circ}x_j, \quad \forall 1 \leq i,\ j \leq 5. \end{cases}$$

Obviously, conditions (i)–(iii) in Theorem 4.1 are satisfied. Thus $\langle H, \tilde{\circ}\rangle$ is a normal fuzzy hypergroup.

5. The category of normal fuzzy hypergroups

Let **NFHG** be a category, where (a) objects are normal fuzzy hypergroups; (b) a morphism from a normal fuzzy hypergroup $\langle H_1, \tilde{\circ}_1\rangle$ to a normal fuzzy hypergroup $\langle H_2, \tilde{\circ}_2\rangle$ is a mapping $f : H_1 \to H_2$ such that $f(x\tilde{\circ}_1 y) \subset f(x)\tilde{\circ}_2 f(y)$; (c) an identity $\mathrm{Id}_H : \langle H, \tilde{\circ}\rangle \to \langle H, \tilde{\circ}\rangle$ is an identity mapping; (d) the composition of morphisms f and g is the composition of mappings f and g.

For the category of normal fuzzy hypergroup, the following theorem holds.

Theorem 5.1. *The category* NFHG *satisfies all the axioms of topos except for the subobject classifier axiom.*

Proof. (1) The category **NFHG** has equalizers property. In fact, let $\langle H_1, \tilde{\circ}_1\rangle$ and $\langle H_2, \tilde{\circ}_2\rangle$ be two normal fuzzy hypergroups and $f, g : \langle H_1, \tilde{\circ}_1\rangle \to \langle H_2, \tilde{\circ}_2\rangle$ be two morphisms.

Let $E = \{x \in H_1 | f(x) = g(x)\}$. For any $x_1, x_2 \in E$, set $x_1 \diamond x_2 = x_1\tilde{\circ}_1 x_2 \cap E$. Then $\langle E, \diamond\rangle$ is a normal fuzzy hypergroup. In fact, $\diamond$ is a fuzzy hyperoperation from E^2 to $\mathcal{F}^*(E)$, which satisfies (1) for any $x \in E$, $(x \diamond x)(x) = (x\tilde{\circ}_1 x)(x) \wedge \chi_E(x) = 1$; (2) for any $x, y \in E$, $x \diamond y = x\tilde{\circ}_1 y \cap E = (x\tilde{\circ}_1 x \cap E) \cup (y\tilde{\circ}_1 y \cap E) = x \diamond x \cup y \diamond y$; (3) for any $x, y, z \in E$, $(x \diamond x)(y) \wedge (y \diamond y)(z) = (x\tilde{\circ}_1 x)(y) \wedge \chi_E(y) \wedge (y\tilde{\circ}_1 y)(z) \wedge \chi_E(z) = (x\tilde{\circ}_1 x)(y) \wedge (y\tilde{\circ}_1 y)(z) \leq (x\tilde{\circ}_1 x)(z) = (x\tilde{\circ}_1 x)(z) \wedge \chi_E(z) = (x \diamond x)(z)$. Let $e : E \to H_1$, $x \mapsto x$. Then for any $x, y \in E$ and $z \in H_1$,

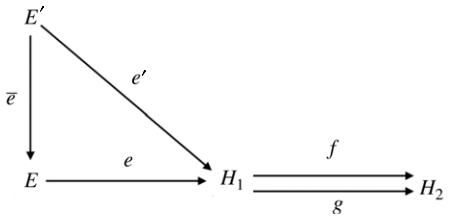

Fig. 1. Illustration of the existence of equalizers in **NFHG**.

$$\begin{aligned} e(x \diamond y)(z) &= \bigvee_{e(z')=z} (x \diamond y)(z') \\ &= \begin{cases} (x \diamond y)(z), & z \in E \\ 0, & z \notin E \end{cases} \\ &\leq (x\tilde{\circ}_1 y)(z) \\ &= (e(x)\tilde{\circ}_1 e(y))(z). \end{aligned}$$

Thus e is a morphism from $\langle E, \diamond\rangle$ to $\langle H_1, \tilde{\circ}_1\rangle$ and $f \circ e = g \circ e$, where $\circ$ represents the composition of two morphisms.

In Fig. 1, let $e' : \langle E', *\rangle \to \langle H_1, \tilde{\circ}_1\rangle$ be a morphism and $f \circ e' = g \circ e'$. For any $x' \in E'$, $(f \circ e')(x') = (g \circ e')(x')$, and so $e'(x') \in E$. Thus a mapping $\bar{e} : E' \to E, x' \mapsto \bar{e}(x') = e'(x')$ is defined. For any $z \in E$,

$$\begin{aligned} \bar{e}(x' * y')(z) &= \bigvee_{\bar{e}(z')=z} (x' * y')(z') \\ &= \bigvee_{e'(z')=z} (x' * y')(z') \\ &= e'(x' * y')(z) \\ &\leq (e'(x')\tilde{\circ}_1 e'(y'))(z) \\ &= (e'(x') \diamond e'(y'))(z) \\ &= (\bar{e}(x') \diamond \bar{e}(y'))(z), \end{aligned}$$

and so $\bar{e}(x' * y') \subseteq \bar{e}(x') \diamond \bar{e}(y')$. Thus $\bar{e}$ is a morphism. Clearly, $e \circ \bar{e} = e'$ and such a morphism $\bar{e}$ is unique. Therefore, $\{\langle E, \diamond\rangle, e\}$ is an equalizer of f and g.

(2) The category **NFHG** has finite products property.

In fact, let $\langle H_1, \tilde{\circ}_1\rangle$ and $\langle H_2, \tilde{\circ}_2\rangle$ be two normal fuzzy hypergroups, and let $Z = H_1 \times H_2$. For $(x_i, y_i) \in Z (i = 1, 2)$, define

$$((x_1, y_1) \odot (x_2, y_2))(z_1, z_2) = ((x_1\tilde{\circ}_1 x_1)(z_1) \wedge (y_1\tilde{\circ}_2 y_1)(z_2)) \vee ((x_2\tilde{\circ}_1 x_2)(z_1) \wedge (y_2\tilde{\circ}_2 y_2)(z_2)).$$

Then $((x, y) \odot (x, y))(x, y) = (x\tilde{\circ}_1 x)(x) \wedge (y\tilde{\circ}_2 y)(y) = 1$, $(x_1, y_1) \odot (x_2, y_2) = (x_1, y_1) \odot (x_1, y_1) \cup (x_2, y_2) \odot (x_2, y_2)$, and

$$\begin{aligned} &((x, y) \odot (x, y))(x', y') \wedge ((x', y') \odot (x', y'))(x'', y'') \\ &\quad = (x\tilde{\circ}_1 x)(x') \wedge (y\tilde{\circ}_2 y)(y') \wedge (x'\tilde{\circ}_1 x')(x'') \wedge (y'\tilde{\circ}_2 y')(y'') \\ &\quad = ((x\tilde{\circ}_1 x)(x') \wedge (x'\tilde{\circ}_1 x')(x'')) \wedge ((y\tilde{\circ}_2 y)(y') \wedge (y'\tilde{\circ}_2 y')(y'')) \\ &\quad \leq (x\tilde{\circ}_1 x)(x'') \wedge (y\tilde{\circ}_2 y)(y'') \\ &\quad = ((x, y) \odot (x, y))(x'', y''). \end{aligned}$$

Thus $\langle Z, \odot\rangle$ is a normal fuzzy hypergroup. Let $p_1 : Z \to H_1, (x, y) \mapsto x$ and $p_2 : Z \to H_2, (x, y) \mapsto y$. Then for any $x \in H_1$ and $y \in H_2$,

$$\begin{aligned} p_1((x, y) \odot (x, y))(x) &= \bigvee_{p_1(x',y')=x} (x\tilde{\circ}_1 x)(x') \wedge (y\tilde{\circ}_2 y)(y') \\ &= \bigvee_{y' \in H_2} (x\tilde{\circ}_1 x)(x) \wedge (y\tilde{\circ}_2 y)(y') \\ &= (x\tilde{\circ}_1 x)(x) \wedge \left(\bigvee_{y' \in H_2} (y\tilde{\circ}_2 y)(y')\right) \\ &= (x\tilde{\circ}_1 x)(x) \\ &= (p_1(x, y)\tilde{\circ}_1 p_1(x, y))(x). \end{aligned}$$

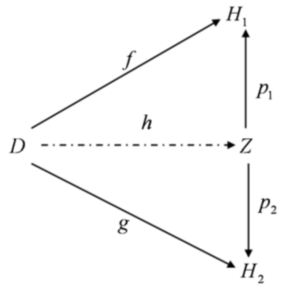

Fig. 2. Illustration of the existence of finite products in **NFHG**.

Hence $p_1((x,y)\odot(x,y)) = x\tilde{\circ}_1 x$ and $p_1((x_1,y_1)\odot(x_2,y_2)) = p_1((x_1,y_1)\odot(x_1,y_1))\cup p_1((x_2,y_2)\odot(x_2,y_2)) = x_1\tilde{\circ}_1 x_1\cup x_2\tilde{\circ}_1 x_2 = x_1\tilde{\circ}_1 x_2$. Thus p_1 is a morphism. Similarly, p_2 is also a morphism.

Let $\langle D,\otimes\rangle$ be a normal fuzzy hypergroup, $f: D\to H_1$ and $g: D\to H_2$ be two morphisms (see Fig. 2).

Let $h: D\to Z, d\mapsto (f(d), g(d))$. For $(x,y)\in Z$, if $(x,y)=h(d)$ for some $d\in D$, then

$$\begin{aligned}
h(d_1\otimes d_1)(x,y) &= \bigvee_{h(d)=(x,y)} (d_1\otimes d_1)(d)\\
&= \bigvee_{f(d)=x, g(d)=y} (d_1\otimes d_1)(d)\\
&\le \left(\bigvee_{f(d)=x}(d_1\otimes d_1)(d)\right)\wedge\left(\bigvee_{g(d')=y}(d_1\otimes d_1)(d')\right)\\
&= f(d_1\otimes d_1)(x)\wedge g(d_1\otimes d_1)(y)\\
&\le (f(d_1)\tilde{\circ}_1 f(d_1))(x)\wedge (g(d_1)\tilde{\circ}_2 g(d_1))(y)\\
&= ((f(d_1),g(d_1))\odot(f(d_1),g(d_1)))(x,y)\\
&= (h(d_1)\odot h(d_1))(x,y).
\end{aligned}$$

If $(x,y)\neq h(d)$ for any $d\in D$, then $h(d_1\otimes d_1)(x,y)=\bigvee_{h(d)=(x,y)}(d_1\otimes d_1)(d)=0$. Hence $h(d_1\otimes d_1)\subseteq h(d_1)\odot h(d_1)$. Thus $h(d_1\otimes d_2)=h(d_1\otimes d_1)\cup h(d_2\otimes d_2)\subseteq h(d_1)\odot h(d_1)\cup h(d_2)\odot h(d_2)=h(d_1)\odot h(d_2)$, which means that h is a morphism. Clearly, $p_1\circ h=f$ and $p_2\circ h=g$, and such an h is unique. Therefore, $(\langle Z,\odot\rangle, p_1, p_2)$ is a finite product of $\langle H_1,\tilde{\circ}_1\rangle$ and $\langle H_2,\tilde{\circ}_2\rangle$.

(3) There is a terminal object in **NFHG**.

In fact, let $M=\{0\}$ and $0\tilde{\circ}0=\{0\}$. Then $\langle M,\tilde{\circ}\rangle$ is a fuzzy normal hypergroup. For any normal fuzzy hypergroup $\langle H,\tilde{\circ}\rangle$, there exists exactly one morphism $f: H\to M, x\mapsto 0$. Therefore, $\langle M,\tilde{\circ}\rangle$ is a terminal object of **NFHG**.

(4) The category **NFHG** has exponentials property.

In fact, let $\langle H_1,\tilde{\circ}_1\rangle$ and $\langle H_2,\tilde{\circ}_2\rangle$ be two normal fuzzy hypergroups. Let Γ be the set of mappings $f: H_1\to H_2$ such that $f(x\tilde{\circ}_1 x)\subseteq f(x)\tilde{\circ}_2 f(x)$ for all $x\in H_1$. For $f, h\in\Gamma$, let

$$\begin{cases}(f\Diamond f)(h)=\bigwedge_{x\in H_1}(f(x)\tilde{\circ}_2 f(x))(h(x))\\ f\Diamond g=f\Diamond f\cup g\Diamond g.\end{cases}$$

Then $(f\Diamond f)(f)=\bigwedge_{x\in H_1}(f(x)\tilde{\circ}_2 f(x))(f(x))=1$, $f\Diamond g=f\Diamond f\cup g\Diamond g$ and

$$\begin{aligned}
(f\Diamond f)(h)\wedge(h\Diamond h)(g) &= \left(\bigwedge_{x\in H_1}(h(x)\tilde{\circ}_2 h(x))(g(x))\right)\wedge\left(\bigwedge_{x\in H_1}(f(x)\tilde{\circ}_2 f(x))(h(x))\right)\\
&\le \bigwedge_{x\in H_1}((h(x)\tilde{\circ}_2 h(x))(g(x))\wedge(f(x)\tilde{\circ}_2 f(x))(h(x)))\\
&\le \bigwedge_{x\in H_1}(f(x)\tilde{\circ}_2 f(x))(g(x))\\
&= (f\Diamond f)(g).
\end{aligned}$$

Thus $\langle\Gamma,\Diamond\rangle$ is a normal fuzzy hypergroup. Let $ev:\Gamma\times H_1\to H_2, (f,x)\mapsto f(x)$. Then for any $(f,x)\in\Gamma\times H_1$ and $y\in H_2$,

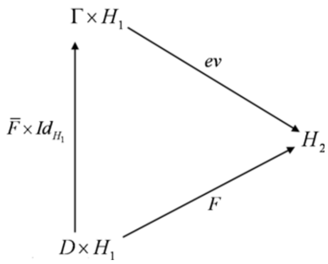

Fig. 3. Illustration of the existence of exponentials in **NFHG**.

$$\begin{aligned} ev((f,x)\odot(f,x))(y) &= \bigvee_{g\in\Gamma,\ g(x')=y} ((f\diamondsuit f)(g)\wedge(x\tilde{\circ}_1 x)(x')) \\ &\le \bigvee_{g\in\Gamma}((f\diamondsuit f)(g)\wedge g(x\tilde{\circ}_1 x)(y)). \end{aligned}$$

Since $g(x\tilde{\circ}_1 x)(y)\le(g(x)\tilde{\circ}_2 g(x))(y)$,

$$\begin{aligned} (f\diamondsuit f)(g)\wedge g(x\tilde{\circ}_1 x)(y) &\le \left(\bigwedge_{u\in H_1}(f(u)\tilde{\circ}_2 f(u))(g(u))\right)\wedge(g(x)\tilde{\circ}_2 g(x))(y) \\ &\le (f(x)\tilde{\circ}_2 f(x))(g(x))\wedge(g(x)\tilde{\circ}_2 g(x))(y) \\ &\le (f(x)\tilde{\circ}_2 f(x))(y). \end{aligned}$$

Thus $ev((f,x)\odot(f,x))(y)\le(f(x)\tilde{\circ}_2 f(x))(y)=(ev(f,x)\tilde{\circ}_2 ev(f,x))(y)$, i.e., $ev((f,x)\odot(f,x))\subseteq ev(f,x)\tilde{\circ}_2 ev(f,x)$. Hence, ev is a morphism.

In Fig. 3, let $\langle D,\otimes\rangle$ be a normal fuzzy hypergroup and $F:D\times H_1\to H_2$ be a morphism. Let $\bar{F}:D\to\Gamma$, $d\mapsto\bar{F}(d)$, where $\bar{F}(d)(x)=F(d,x)$ for all $x\in H_1$.

For any $y\in H_2$,

$$\begin{aligned} \bar{F}(d)(x\tilde{\circ}_1 x)(y) &= \bigvee_{\bar{F}(d)(a)=y}(x\tilde{\circ}_1 x)(a) \\ &= \bigvee_{F(d,a)=y}(x\tilde{\circ}_1 x)(a) \\ &\le \bigvee_{F(d_1,a)=y}((d\otimes d)(d_1)\wedge(x\tilde{\circ}_1 x)(a)) \\ &= \bigvee_{F(d_1,a)=y}((d,x)\odot(d,x))(d_1,a) \\ &= F((d,x)\odot(d,x))(y) \\ &\le (F(d,x)\tilde{\circ}_2 F(d,x))(y) \\ &= (\bar{F}(d)(x)\tilde{\circ}_2\bar{F}(d)(x))(y). \end{aligned}$$

So $\bar{F}(d)(x\tilde{\circ}x)\subseteq\bar{F}(d)(x)\tilde{\circ}_2\bar{F}(d)(x)$, which implies that $\bar{F}(d)\in\Gamma$. Since for any $x\in H_1$,

$$\begin{aligned} \bar{F}(d_1\otimes d_1)(g) &= \bigvee_{\bar{F}(d)=g}(d_1\otimes d_1)(d) \\ &\le \bigvee_{\bar{F}(d)(x)=g(x)}(d_1\otimes d_1)(d)\wedge(x\tilde{\circ}_1 x)(x) \\ &= \bigvee_{F(d,x)=g(x)}(d_1,x)\odot(d_1,x)(d,x) \\ &\le \bigvee_{F(d,x')=g(x)}(d_1,x)\odot(d_1,x)(d,x') \\ &= F((d_1,x)\odot(d_1,x))(g(x)) \\ &\le (\bar{F}(d_1)(x)\tilde{\circ}_2\bar{F}(d_1)(x))(g(x)), \end{aligned}$$

we have

$$\bar{F}(d_1\otimes d_1)(g)\le\bigwedge_{x\in H_1}(\bar{F}(d_1)(x)\tilde{\circ}_2\bar{F}(d_1)(x))(g(x))=(\bar{F}(d_1)\diamondsuit\bar{F}(d_1))(g).$$

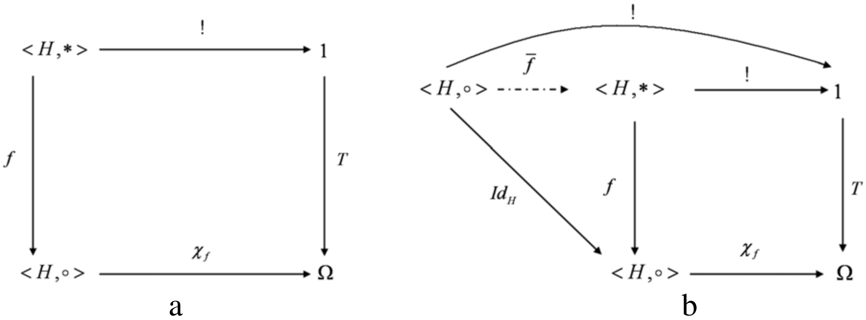

Fig. 4. Illustration of the existence of a subobject classifier in **NFHG**.

Thus $\bar{F}(d_1 \otimes d_1) \subseteq \bar{F}(d_1)\Diamond\bar{F}(d_1)$. Hence $\bar{F}(d_1 \otimes d_2) \subseteq \bar{F}(d_1)\Diamond\bar{F}(d_2)$, and so $\bar{F}$ is a morphism. Clearly, $ev \circ (\bar{F} \times \mathrm{Id}_{H_1}) = F$ and such an $\bar{F}$ is unique. Therefore, $\{\langle \Gamma, \Diamond\rangle, ev\}$ is an exponential.

Remark 5.1. Γ can be an empty set in the theory of category. In fact, as shown in Fig. 3, if the morphism F exists, then for any $d \in D$, there is $\bar{F}(d) \in \Gamma$, where $\bar{F}(d) : H_1 \to H_2, x \to \bar{F}(d)(x) = F(d, x)$. In this case, Γ is nonempty. If such F does not exist, Γ can be an empty set, but this item still holds.

(5) There is no subobject classifier in the category **NFHG**.

Assume that there exists a normal fuzzy hypergroup and a morphism $T : 1 \to \Omega$ satisfying the subobject classifiers axiom, where 1 is a terminal object in **NFHG**.

Let $H = \{a, b, c\}$. Define a hyperoperation $\tilde{\circ}$ on H by

$\tilde{\circ}$	a	b	c
a	$\{a, b\}$	$\{a, b\}$	$\{a, b, c\}$
b	$\{a, b\}$	$\{b\}$	$\{b, c\}$
c	$\{a, b, c\}$	$\{b, c\}$	$\{c\}$

Then $\langle H, \tilde{\circ}\rangle$ is a normal fuzzy hypergroup. We define another hyperoperation $*$ on H by

$*$	a	b	c
a	$\{a\}$	$\{a, b\}$	$\{a, c\}$
b	$\{a, b\}$	$\{b\}$	$\{b, c\}$
c	$\{a, c\}$	$\{b, c\}$	$\{c\}$

Then $\langle H, *\rangle$ is also a normal fuzzy hypergroup.

Let $f : \langle H, *\rangle \to \langle H, \tilde{\circ}\rangle, x \mapsto x$. Then f is a monomorphism in **NFHG**, and there exists a unique morphism $\chi_f : \langle H, \tilde{\circ}\rangle \to \Omega$ such that the Fig. 4(a) is a pullback.

In Fig. 4(b), we have $\chi_f \circ \mathrm{Id}_H = T \circ !$. Then there exists a unique morphism $\bar{f} : \langle H, \tilde{\circ}\rangle \to \langle H, *\rangle$ such that $f \circ \bar{f} = \mathrm{Id}_H$. Hence $x = \mathrm{Id}_H(x) = (f \circ \bar{f})(x) = f(\bar{f}(x)) = \bar{f}(x)$, and $\{a, b\} = a\tilde{\circ}a = \bar{f}(a\tilde{\circ}a) \subseteq \bar{f}(a) * \bar{f}(a) = a * a = \{a\}$, i.e., $\{a, b\} \subset \{a\}$, which leads to a contradiction. Therefore, **NFHG** has no subobject classifier. $\square$

6. Conclusion

In this paper, motivated by a fuzzy hyperoperation induced from the fuzzy inference, we studied a fuzzy hyperstructure associated with a fuzzy relation. We gave a sufficient and necessary condition for such a fuzzy hypergroupoid being a fuzzy hypergroup and investigated the properties of such fuzzy hypergroups. Furthermore, we presented the definition of normal fuzzy hypergroups and showed that the category **NFHG** of normal fuzzy hypergroups satisfies all the axioms of topos except for the subobject classifier axiom.

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