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Correct initial boundary value problems for dispersive equations

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ABSTRACT

This paper deals with correctness of initial boundary value problems for general dispersive equations of finite odd orders. For the Kawahara and KdV equations we prove existence, uniqueness and stability of strong global solutions in a bounded domain for different signs of a coefficient of the highest derivative as well as their asymptotics when the coefficient of the higher-order derivative in the Kawahara equation approaches zero.

1. Introduction

In $Q = (0,1) \times (0,T)$ we consider initial boundary value problems for the Kawahara equation

$$Lu = u_t + uDu + aD^3u - bD^5u = 0, \tag{1.1}$$

where $D^i = \frac{\partial^i}{\partial x_i}$; a, b are real constants.

This dispersive equation describes one-dimensional evolutions of small amplitude long waves in various problems of fluid dynamics and physics [19,25]. If $b = 1$, $a = 1$, we have the classical Kawahara equation which was derived as a perturbation of the KdV equation

$$Lu = u_t + uDu + aD^3u = 0 \tag{1.2}$$

when the coefficient a is small and frequently is called the perturbed KdV equation or the special version of the Benney–Lin equation, see [3]. A sign of the coefficient b depends on a nature of physical processes modeled by the Kawahara equation and may be positive or negative.

Historically, interest in dispersive-type evolution equations dates from the 19th century when Russel [24], Airy [1], Boussinesq [9] and later Korteweg and de Vries [20] studied propagation of waves in dispersive media. Due to physical reasons, these and posterior studies mostly dealt with one-dimensional problems posed on the entire real line, see [2,3,5,6,11–13,18,19,25] and references therein. Moreover, the emphasis in these works was mainly focused on the existence and qualitative structure of the solitary, cnoidal and other specific types of waves, whereas correctness of corresponding mathematical problems attracted minor interest. Known mathematical results for (1.1) concerned the Cauchy problem, see [5,13], but correctness of initial boundary value problems was not studied.

On the other hand, if one is interested in calculating solutions to the Cauchy problem or stationary problems on the whole real line using some numerical schemes, there appears a problem of approximating the real line by finite intervals and solving mixed problems in bounded domains. Therefore precise mathematical analysis of initial boundary value problems in bounded domains for dispersive equations attracts more attention in last years, see [7,10,14,16,17,22,23].

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Because (1.1) and (1.2) are of odd orders, boundary conditions for them in points $x = 0$ and $x = 1$ are not symmetric and study of these equations in bounded domains is interesting from the purely mathematical point of view: a type of boundary conditions needed to ensure the correctness of a problem depends on a sign of the higher derivative coefficient; whereas this sign is of no importance in the case of the Cauchy problem. Therefore, methods for studying of these two kinds of problems should be different. Solvability of boundary value problems for linear third-order dispersive equations in bounded multidimensional domains was studied in [21] (see also the references). As concerns nonlinear odd-order equations considered in bounded regions, there is no satisfactory general theory to answer questions of existence, uniqueness and asymptotics, nevertheless initial–boundary value problems for the KdV equation on bounded intervals were studied by various authors, see [7,10,14,22,23]. Usually, there are one condition on the left end and two conditions on the right end of an interval. This is explained by physical arguments and it must be noted that this situation is common for all dispersive equations, nevertheless we did not find a general mathematical approach to this question. Because of that, we tried to treat the questions of ill and well-posed problems for general dispersive equations.

Our paper has the following structure: Section 1 is Introduction, in Section 2 we present some auxiliary mathematical results including Lopatinskii condition for ODE systems. In Section 3 we treat the question of ill and well-posedness for general dispersive equations and give an example of an ill-posed problem for the third-order ordinary differential equation while the set of boundary conditions is different from what we propose. This is actually an example of an ill-posed problem for the linear KdV equation. In Section 4 we prove our main results: Theorem 1 on solvability of the initial boundary value problem for the Kawahara equation on a bounded interval when $b > 0$ and Theorem 2 on the exponential decay of the energy as $t \to \infty$. To prove Theorem 1, we exploit a regularization of the original problem by an initial boundary value problem for a higher-order parabolic operator in a similar manner as we have done in [22] for the KdV equation. In turn, we use the Faedo–Galerkin method with a special basis to solve the parabolic problem. We prove the existence, uniqueness and the exponential decay of the L^2-norms of solutions for the Kawahara equation. In Section 5 we show that its solutions converge to solutions of the KdV equation as the coefficient b tends to zero and a is positive. Finally, in Section 6 we consider the initial boundary value problem for the Kawahara equation when $b < 0$.

2. Notations and auxiliary results

We will use the following notations:

$$(u, v)(t) = \int_0^1 u(x,t)v(x,t)\,dx, \qquad \left|u(t)\right|^2 = (u, u)(t), \qquad \|v\| = \left|v(0)\right|$$

and usual notations for Sobolev spaces $H^l(0,1)$. We will need two inequalities of calculus: the Ehrling inequality which in our case can be formulated as follows, see [4].

Lemma 1. *Let $u \in H^s(0,1)$ and $0 \leqslant l < s$. Then for an arbitrary $\varepsilon > 0$ there exists $K(\varepsilon) > 0$ such that*

$$\|u\|_{H^l(0,1)} \leqslant \varepsilon \|u\|_{H^s(0,1)} + K(\varepsilon)\|u\|_{L^2(0,1)}.$$

The other is the Gagliardo–Nirenberg inequality [4], which in our case reads.

Lemma 2. *Let $n = 1$, $1 \leqslant p_1 \leqslant \infty$, $1 \leqslant p_2 \leqslant \infty$, $0 \leqslant r < l$ and*

$$\frac{1}{p} - r = (1-\theta)\frac{1}{p_1} + \theta\left(\frac{1}{p_2} - l\right), \quad r < l, \ \frac{r}{l} \leqslant \theta \leqslant 1.$$

Then there exists a constant C which depends on $r, l, p, p_1, p_2, \theta$ such that

$$\left\|D^r f\right\|_{L^p(0,1)} \leqslant C\|f\|^{1-\theta}_{L^{p_1}(0,1)}\left\|D^l f\right\|^{\theta}_{L^{p_2}(0,1)}.$$

We will need two lemmas from the theory of ODE. For details one can see [15].

Boundary value problem on a segment

Let us study the problem of finding a solution $y(x)$ of the nonhomogeneous equation

$$\frac{d}{dt}y(x) = Ay(x) + f(x), \quad x_L \leqslant x \leqslant x_R; \qquad x_l < x_R, \tag{2.1}$$

with the boundary conditions

$$Ly(x_L) = l, \qquad Ry(x_R) = r. \tag{2.2}$$

Here A is a quadratic $N \times N$ matrix; each of the matrices L, R has N columns, where N is the number of components of the vector $y(x)$. The number of rows of the matrix L can differ from that of the matrix R. We denote the number of the rows of the matrix L by k_L and the number of the rows of the matrix R by k_R. Let $Y(x)$ be a fundamental matrix of solutions to the homogeneous equation

$$\frac{d}{dt} y(x) = Ay(x).$$

The following assertion is true [15].

Lemma 3. *The boundary value problem* (2.1)–(2.2) *has a unique solution $y(x)$ for any $l, r, f(x)$ if and only if the following conditions hold*:

$$k_L + k_R = N$$

and the determinant of the matrix

$$\begin{Vmatrix} LY(x_L) \\ RY(x_R) \end{Vmatrix}$$

is different of zero.

The Lopatinskii condition

Proposition 1. *The boundary value problem for the homogeneous vector equation*

$$\frac{d}{dt} v(x) = Av(x)$$

with the boundary condition

$$Mv(0) = \phi$$

has a unique bounded solution $v(x)$, $0 \leqslant x < \infty$ ($\|v(x)\| < \infty$) for any vector ϕ such that the number of its components coincides with the number of rows of the matrix M which in turn coincides with the number of those eigenvalues of the matrix A that have the negative real parts.

3. Initial boundary value problems for odd-order evolution equations

We consider odd-order evolution equations of dispersive type:

$$u_t + uDu + (-1)^{l+1} D^{2l+1} u = 0 \tag{3.1}$$

in $Q = (0,1) \times (0,T)$, $Q^- = (-\infty, 0) \times (0,T)$, $Q^+ = (0, \infty,) \times (0,T)$, where l is a nonnegative entire number, $T > 0$, $D^k = \frac{d^k}{dx^k}$.

To set a correct initial boundary value problem for (3.1), we prescribe initial data

$$u(x,0) = u_0(x)$$

for $x \in (0,1)$ or $x \in (-\infty, 0) = R^-$ or $x \in (0,\infty) = R^+$.

Besides the initial data we must set boundary conditions at $x = 0$ and $x = 1$ for $x \in (0,1)$ or at $x = 0$ for $x \in R^+$ and $x \in R^-$ which are defined by the principal part of (3.1):

$$u_t + (-1)^{l+1} D^{2l+1} u = 0. \tag{3.2}$$

We will show that when $x \in (0,1)$, one has to set l conditions at $x = 0$ and $l + 1$ conditions at $x = 1$. In the case $x \in R^+$, one must set l conditions at $x = 0$ and when $x \in R^-$ one must set $l + 1$ conditions at $x = 0$. A correct set of boundary conditions guarantees that a corresponding initial boundary value problem for (3.1) is well posed (it has a unique regular bounded solution) and that a different choice of numbers of boundary conditions leads to ill-posed problems (nonexistence of solutions or non-uniqueness).

To construct solutions of (3.2), we use discretization of it with respect to time: let N be a natural number, then we define

$$h = \frac{|0,T|}{N}, \qquad u^n(x) = u(x, nh), \qquad u^0(x) = u_0(x).$$

Substitution in (3.2) $u(x,t)$ by $u^n(x)$ and $u_t(x,t)$ by $\frac{u^n(x) - u^{n-1}(x)}{h}$ gives

$$\frac{u^n}{h} + (-1)^{l+1} D^{2l+1} u^n = \frac{u^{n-1}}{h} \equiv f(x), \quad n = 1, \dots, N.$$

Since $u^0 = u_0(x)$, finding $u^1(x)$, we can find $u^2(x)$, etc. It is clear that we must set well-posed boundary value problems for the stationary equation

$$sv + (-1)^{l+1} D^{2l+1} v = f(x), \quad s > 0. \tag{3.3}$$

Because our goal here is not to write explicitly solutions, but to find out which boundary value problems are well-posed and which are ill-posed in $(0,1)$, R^+ or R^-, it is sufficient to construct a fundamental system of solutions for the linear homogeneous equation

$$sv + (-1)^{l+1} D^{2l+1} v = 0 \tag{3.4}$$

which is defined by the roots of the characteristic equation

$$s + (-1)^{l+1} \lambda^{2l+1} = 0.$$

Without loss of generality they can be written as

$$\lambda_k = d_0 \exp\left(i\pi \frac{\beta + 2k}{2l+1}\right), \quad k = 0, \dots, 2l,$$

where

$$d_0 = |s|^{\frac{1}{2l+1}}, \qquad \beta = 0 \quad \text{for } l = 2s \quad \text{and} \quad \beta = 1 \quad \text{for } l = 2s+1.$$

It is easy to see that there is always one real root

$$\lambda_0 = (-1)^{\beta} d_0 \tag{3.5}$$

and $2l$ complex roots

$$\lambda_j = d_0 \exp\left(i\pi \frac{\beta + 2j}{2l+1}\right), \quad j = 1, \dots, 2l. \tag{3.6}$$

If $l = 2s$, among them are l roots with positive real parts

$$\lambda_j^+ = d_0 \exp\left(\pm i \frac{2\pi j}{2l+1}\right), \quad j = 1, \dots, \frac{l}{2},$$

and l roots with negative real parts

$$\lambda_j^- = d_0 \exp\left\{\pm i \frac{2\pi j}{2l+1}\right\}, \quad j = \frac{l}{2} + 1, \dots, l.$$

If $l = 2s+1$, then there are $l+1$ roots with positive real parts

$$\lambda_j^+ = d_0 \exp\left\{\pm i\pi \frac{1+2j}{2l+1}\right\}, \quad j = 0, \dots, \frac{l-1}{2},$$

and $l-1$ roots with negative real parts

$$\lambda_j^- = d_0 \exp\left\{\pm i\pi \frac{1+2j}{2l+1}\right\}, \quad j = \frac{l-1}{2} + 1, \dots, l-1.$$

3.1. Problem I: $x \in R^+$

We seek bounded solutions

$$\sup_{x \in R^+} |u(x)| < \infty. \tag{3.7}$$

It is easy to see that there are l roots with real parts negative. Rewriting (3.4) as a first-order ODE system and using Proposition 1, we pose l conditions at $x = 0$. This can be resumed as

Lemma 4. *Well-posed Problem I for* (3.1) *in R^+ has exactly l conditions at $x = 0$.*

3.2. Problem II: $x \in R^-$

Changing $-x = y$, we get

$$sv - (-1)^{l+1} D_y^{2l+1} v = 0, \quad y > 0. \tag{3.8}$$

In this case there are $l+1$ roots with negative real parts. By Proposition 1, we need to pose $l+1$ conditions at $y = 0$. This can be resumed as

Lemma 5. *Well-posed Problem II for* (3.1) *in R^- has exactly $l+1$ conditions at $x = 0$.*

3.3. Problem III: $x \in (0, 1)$

To pose a correct boundary value problem on the segment $(0, 1)$, we rewrite (3.3) as a first-order ODE system and use Lemma 3 setting l conditions at $x = 0$ and $l+1$ conditions at $x = 1$. This gives $2l+1$ conditions on both ends of the segment and corresponds to a correct boundary value problem on R^+.

From the physical point of view it means one-way propagation of waves. This conception was used while simplifying Boussinesq system to the KdV and BBM equations. Details can be found in [8]. This setting of boundary conditions is usual for the KdV equation [7,10,14,22]. On the other hand, setting $l+1$ conditions at $x = 0$ and l conditions at $x = 1$ leads to nonexistence of solutions which can be proved for the linear KdV equation. We resume all above as

Lemma 6. *Well-posed initial boundary value problems for* (3.1) *on R^+, R^- and $(0, 1)$ have the following set of boundary conditions*:

1. *l conditions at $x = 0$, while $x \in R^+$, $t > 0$.*
2. *$l+1$ conditions at $x = 0$, while $x \in R^-$, $t > 0$.*
3. *l conditions at $x = 0$ and $l+1$ conditions at $x = 1$ while $x \in (0, 1)$, $t > 0$.*

Ill-posed problem for the linear KdV equation

In the case of the KdV equation we have $l = 1$ and (3.4) becomes

$$sv + D^3 v = 0 \tag{3.9}$$

for which we pose $l+1 = 2$ boundary conditions at $x = 0$ and $l = 1$ boundary condition at $x = 1$:

$$v(0) = v_x(0) = v(1) = 0. \tag{3.10}$$

This contradicts the choice of boundary conditions proposed in Lemma 6. The characteristic equation for (3.9)

$$s + \lambda^3 = 0$$

has three roots

$$\lambda_k = |s|^{1/3} \exp\left(i\left[\frac{\pi + 2k\pi}{3}\right]\right), \quad k = 1, 2, 3,$$

where $\lambda_1 = -|s|^{1/3}$ has the negative real parts. These roots define the following fundamental system of solutions to (3.9):

$$u_1(x) = \exp\left(-|s|^{1/3} x\right), \qquad u_2(x) = \exp\left(\frac{1}{2}|s|^{1/3} x\right) \cos\left(\frac{\sqrt{3}}{2}|s|^{1/3} x\right),$$

$$u_3(x) = \exp\left(\frac{1}{2}|s|^{1/3} x\right) \sin\left(\frac{\sqrt{3}}{2}|s|^{1/3} x\right).$$

Due to Lemma 3, applied to the equivalent first-order ODE system, a correct setting of boundary conditions on $(0, 1)$ guarantees that a corresponding determinant is different from zero for all $s > 0$ and vice versa. We calculate

$$\left\| \begin{matrix} u_1(0) & u_2(0) & u_3(0) \\ u_1(1) & u_2(1) & u_3(1) \\ u_{1x}(0) & u_{2x}(0) & u_{3x}(0) \end{matrix} \right\| = \frac{\sqrt{3}t}{2} \exp\left(\frac{t}{2}\right)\left\{2\cos\left(\frac{\sqrt{3}t}{2} + \frac{\pi}{3}\right) - \exp\left(-\frac{3t}{2}\right)\right\},$$

where $t = |s|^{1/3} > 0$. Consider on R^+ the continuous function

$$G(t) = 2\cos\left(\frac{\sqrt{3}t}{2} + \frac{\pi}{3}\right) - \exp\left(-\frac{3t}{2}\right).$$

At points t_k, $k = 0, 1, 2, 3, \dots$ such that $t_0 = \frac{4\pi}{3\sqrt{3}}$, $t_{k+1} - t_k = \frac{2\pi}{\sqrt{3}}$, we have

$$\cos\left(\frac{\sqrt{3}t_{2p}}{2} + \frac{\pi}{3}\right) = -1, \quad p = 0, 1, 2, 3, \dots,$$

$$\cos\left(\frac{\sqrt{3}t_{2p+1}}{2} + \frac{\pi}{3}\right) = +1, \quad p = 0, 1, 2, 3, \dots.$$

It is easy to see that $G(t_0) < 0$ and $G(t_1) > 0$. By the Intermediate Value Theorem, there exists at least one point $t^* \in (t_0, t_1)$ such that $G(t^*) = 0$. Proceeding with this argument for all intervals (t_k, t_{k+1}), we find that $G(t)$ is equal to zero on a countable set of points on R^+. It means that problem (3.9)–(3.10) does not have solutions for all $s > 0$. Hence the problem

$$u_t + D^3 u = 0, \quad x \in (0, 1),$$

$$u(x, 0) = u_0(x),$$

$$u(0, t) = u_x(0, t) = u(1, t) = 0$$

is ill posed.

On the other hand, the problem

$$u_t + D^3 u = 0, \quad x \in (0, 1), \tag{3.11}$$

$$u(x, 0) = u_0(x), \tag{3.12}$$

$$u(0, t) = u(1, t) = u_x(1, t) = 0 \tag{3.13}$$

is well posed. Indeed, it is easy to see that a corresponding determinant is strictly positive for all $s > 0$:

$$\begin{Vmatrix} u_1(0) & u_2(0) & u_3(0) \\ u_1(1) & u_2(1) & u_3(1) \\ u_{1x}(1) & u_{2x}(1) & u_{3x}(1) \end{Vmatrix} = te^t\left[\frac{\sqrt{3}}{2} - \exp\left(-\frac{3t}{2}\right)\cos\left(\frac{\sqrt{3}t}{2} + \frac{\pi}{6}\right)\right] > 0, \quad \forall t > 0.$$

By Lemma 3, problem (3.11)–(3.13) is well posed.

4. Kawahara equation

In this section we will prove our main results: Theorems 1 and 2.

In $Q_T = (0, 1) \times (0, T)$, $T > 0$, consider the following mixed problem:

$$Lu = u_t + uDu + aD^3 u - bD^5 u = 0 \quad \text{in } Q_T, \tag{4.1}$$

$$u(0, t) = Du(0, t) = 0, \quad t > 0, \tag{4.2}$$

$$u(1, t) = Du(1, t) = D^2 u(1, t) = 0, \quad t > 0, \tag{4.3}$$

$$u(x, 0) = u_0(x), \quad x \in (0, 1), \tag{4.4}$$

where $a, b \in R$; $b > 0$.

Theorem 1. *Let $u_0 \in H^5(0, 1)$ satisfy boundary conditions* (4.2), (4.3). *Then for all finite $T > 0$ there exists a unique regular solution to* (4.1)–(4.4) $u(x, t)$:

$$u \in C\big(0, T; H^5(0, 1)\big) \cap L^2\big(0, T; H^7(0, 1)\big),$$

$$u_t \in L^\infty\big(0, T; L^2(0, 1)\big) \cap L^2\big(0, T; H^2(0, 1)\big).$$

Proof. We will prove that this initial boundary value problem is well posed which corresponds to the choice of boundary conditions proposed in Lemma 6. As was shown in the example of the ill-posed problem in the case $l = 1$, a different choice of boundary conditions may lead to the non-existence of solutions even in a linear case. To solve it we will use regularization by a mixed problem for a higher-order parabolic equation:

Regularized problem

let ε be an arbitrary positive real number, then we define

$$L_\varepsilon u_\varepsilon = Lu_\varepsilon - \varepsilon D^{10} u_\varepsilon = 0, \tag{4.5}$$

$$u_\varepsilon(0,t) = Du_\varepsilon(0,t) = D^5 u_\varepsilon(0,t) = D^6 u_\varepsilon(0,t) = D^7 u_\varepsilon(0,t) = 0, \tag{4.6}$$

$$u_\varepsilon(1,t) = Du_\varepsilon(1,t) = D^2 u_\varepsilon(1,t) = D^5 u_\varepsilon(1,t) = D^6 u_\varepsilon(1,t) = 0, \tag{4.7}$$

$$u_\varepsilon(x,0) = u_{0m}(x), \tag{4.8}$$

where $u_{0m} \in H^{15}(0,1)$.

Lemma 7. *Let $u_{0m} \in H^{15}(0,1)$ satisfy boundary conditions* (4.6), (4.7) *and*

$$D^{10} u_{0m}(0) = D^{11} u_{0m}(0) = D^{11} u_{0m}(1) = D^{12} u_{0m}(1) = 0.$$

Then for all $\varepsilon > 0$ there exist unique solutions to (4.5)–(4.8), $u_\varepsilon(x,t)$:

$$u_\varepsilon \in C\big(0,T; H^{10}(0,1)\big), \qquad u_{\varepsilon t} \in C\big(0,T; H^5(0,1)\big) \cap L^2\big(0,T; H^{10}(0,1)\big), \qquad u_{\varepsilon tt} \in L^2(Q_T).$$

Proof. To solve (4.5)–(4.8), we will use the Faedo–Galerkin method with a special basis provided by the following proposition.

Proposition 2. *There exist eigenfunctions of the following eigenvalue problem*:

$$L_\lambda w_j = -D^{10} w_j + \lambda w_j = 0, \quad x \in (0,1),\ j = 1,2,\dots, \tag{4.9}$$

satisfying boundary conditions (4.6), (4.7), *which create a basis in $H^{10}(0,1)$ orthonormal in $L^2(0,1)$.*

Proof. It is easy to show that the operator (4.9), (4.6), (4.7) is self-adjoint and positive in $H^{10}(0,1)$. Hence, assertions of the last proposition follow from the well-known facts of the functional analysis. □

For a fixed $\varepsilon > 0$ we construct approximate solutions to (4.5)–(4.8) in the form

$$u_\varepsilon^N(x,t) = \sum_{j=1}^{N} g_j^N(t) w_j(x),$$

where $g_j^N(t)$ are to be found from the following system of ODE:

$$\big(L_\varepsilon u_\varepsilon^N, w_j\big)(t) = 0, \tag{4.10}$$

$$g_j^N(0) = (u_{0m}, w_j), \quad j = 1,\dots,N. \tag{4.11}$$

Obviously, solutions of (4.10), (4.11) exist on some interval $(0,T_N)$. To extend them to any finite $T > 0$ and to pass to the limit as $N \to \infty$, we need a priori estimates.

Estimate I

From now on we will drop in calculations indices ε, N. Replacing in (4.10) w_j by u_ε^N and integrating by parts, we get

$$\frac{d}{dt}\big|u(t)\big|^2 + b\big|D^2(0,t)\big|^2 + 2\varepsilon\big|D^5 u(t)\big|^2 = 0,$$

whence,

$$\big|u_\varepsilon^N(t)\big|^2 + 2\varepsilon \int_0^t \big|D^5 u_\varepsilon^N(s)\big|^2\, ds \leqslant |u_{0m}|^2. \tag{4.12}$$

Estimate II

Due to Proposition 2, we replace in (4.10) w_j by $-D^{10}u_\varepsilon^N$ and obtain

$$\frac{d}{dt}\left|D^5u(t)\right|^2+\varepsilon\left|D^{10}u(t)\right|^2\leqslant C(\varepsilon)\big(\big(\left|u(t)\right|^2+1\big)\left|D^5u(t)\right|^2\big).$$

Taking into account (4.12), we get

$$\left|D^5u^N(t)\right|^2+\varepsilon\int_0^t\left|D^{10}u^N(s)\right|^2ds\leqslant C(\varepsilon)\|u_{0m}\|^2_{H^5(0,1)}, \tag{4.13}$$

where the constant C depends on $\varepsilon>0$, $\|u_{0m}\|$, but does not depend on N.

Estimate III

Differentiating (4.10) with respect to t and replacing w_j by $-\frac{D^{10}w_jg_{jt}^N}{\lambda_j}$, we come to the inequality

$$\frac{d}{dt}\left|D^5u_t(t)\right|^2+\varepsilon\left|D^{10}u_t(t)\right|^2\leqslant C(\varepsilon)\big[\left|D^5u_t(t)\right|^2\big(1+\left|u(t)\right|^2\big)+\left|D^5u(t)\right|^2\big]$$

which implies

$$\left|D^5u^N_{\varepsilon t}(t)\right|^2+\varepsilon\int_0^t\left|D^{10}u^N_{\varepsilon s}(s)\right|^2ds\leqslant C(\varepsilon)\big\{1+\left\|u^N_{\varepsilon t}(0)\right\|^2_{H^5(0,1)}\big\}. \tag{4.14}$$

We calculate from (4.10)

$$\left\|u^N_{\varepsilon t}(0)\right\|_{H^5(0,1)}\leqslant C\|u_{0m}\|_{H^{15}(0,1)},$$

hence

$$\left|D^5u^N_{\varepsilon t}(t)\right|^2+\varepsilon\int_0^t\left|D^{10}u^N_{\varepsilon s}(s)\right|^2ds\leqslant C(\varepsilon)\|u_{0m}\|^2_{H^{15}(0,1)} \tag{4.15}$$

with the constant C independent of N, but dependent of $\varepsilon>0$. In turn, (4.15) and (4.10) imply

$$\int_0^t\left|u^N_{\varepsilon tt}(s)\right|^2ds\leqslant C(\varepsilon), \tag{4.16}$$

where C does not depend on N. Estimates (4.12)–(4.16) allow us to pass to the limit in (4.10) as $N\to\infty$ and to prove the existence part of Lemma 7. Uniqueness can be proved by standards arguments.

Solvability of (4.1)–(4.4)

To prove the existence part of Theorem 1, we need a priori estimates uniform in $\varepsilon\in(0,1]$ which will allow us to pass to the limit as $\varepsilon\to0$.

Estimate (4.12) is uniform in $\varepsilon>0$, N. In the limit case, when $N\to\infty$, it reads

$$\left|u_\varepsilon(t)\right|^2+2\varepsilon\int_0^t\left|D^5u_\varepsilon(s)\right|^2ds\leqslant\|u_{0m}\|^2. \tag{4.17}$$

Estimate IV

Multiplying (4.5) by $(1+x)u_\varepsilon$, integrating by parts and dropping the index ε, we get

$$\frac{1}{2}\frac{d}{dt}\big((1+x),u^2\big)(t)+\big(uDu,(1+x)u\big)(t)+\frac{a}{2}\left|Du(t)\right|^2+\frac{5b}{2}\left|D^2u(t)\right|^2+\frac{b}{2}\left|D^2u(0,t)\right|^2$$
$$+\varepsilon\big((1+x),\left|D^5u\right|^2\big)(t)+5\varepsilon\big(D^5u,D^4u\big)(t)=0. \tag{4.18}$$

Here a can be positive or negative. Estimating sign-indefinite terms, we have

$$I = \left|\left(uDu, (1+x)u\right)(t)\right| \leqslant \frac{1}{3}\max_{[0,1]}\left|u(x,t)\right|\left(u^2,1\right)(t) \leqslant \eta\left|Du(t)\right|^2 + \frac{1}{36\eta}\left|u(t)\right|^4$$

for all $\eta > 0$.

Substituting I into (4.18) and using the Ehrling inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\left((1+x),u^2\right)(t) + \frac{a}{2}\left|Du(t)\right|^2 + \frac{5b}{2}\left|D^2u(t)\right|^2 - \eta\left|Du(t)\right|^2 - \frac{1}{36\eta}\left|u(t)\right|^4 - C\varepsilon\left(\left|D^5u(t)\right|^2 + \left|u(t)\right|^2\right) \leqslant 0, \tag{4.19}$$

where the constant C does not depend on $\varepsilon > 0$. There are two cases:

(i) If $a > 0$, then setting $\eta = \frac{a}{4}$ and taking into account (4.12), we get

$$a\int_0^t \left|Du(s)\right|^2 ds + b\int_0^t \left|D^2u(s)\right|^2 ds \leqslant C\left(\|u_{0m}\|\right)\|u_{0m}\|^2, \tag{4.20}$$

where the constant C depends on a, $\|u_{0m}\|$, but does not depend on $\varepsilon > 0$.

(ii) If $a \leqslant 0$, using the obvious inequality

$$|a||Du|^2 \leqslant \eta\left|D^2u\right| + \frac{a^2}{4\eta}|u|^2$$

with $2\eta = b$, we transform (4.18) into the form

$$\frac{d}{dt}\left((1+x),u^2\right)(t) + 3b\left|D^2u(t)\right|^2 \leqslant \frac{1}{9b}\left|u(t)\right|^4 + \frac{|a|^2}{2b^2}\left|u(t)\right|^2 + \varepsilon C\left(\left|D^5u(t)\right|^2 + \left|u(t)\right|^2\right)$$

which after integration and making use of (4.17) reads

$$b\int_0^t \left|D^2u(s)\right|^2 ds \leqslant C\|u_{0m}\|^2, \tag{4.21}$$

where the constant C does not depend on ε.

Estimate V

Differentiating (4.5), which is possible due to (4.13), (4.16), multiplying the result by $(1+x)u_t$, after standard calculations we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left((1+x),u_t^2\right)(t) + \left((uDu)_t,(1+x)u_t\right)(t) + \frac{a}{2}\left|Du_t(t)\right|^2 + \frac{5b}{2}\left((1+x),\left|D^2u_t\right|^2\right)(t)\\ &\quad + \frac{b}{2}\left|D^2u_t(0,t)\right|^2 + 5\varepsilon\left(D^5u_t,D^4u_t\right)(t) + \varepsilon\left((1+x),\left|D^5u_t\right|^2\right)(t) = 0.\end{aligned} \tag{4.22}$$

Estimating the second term in the left-hand side, we find

$$\begin{aligned}I &= \left((uDu)_t,(1+x)u_t\right)(t) = -\left(u,u_t^2\right)(t) - \left(uu_t,(1+x)Du_t\right)(t) \geqslant -\max_{[0,1]}\left|u(x,t)\right|\left|u_t(t)\right|^2 - \eta\left|Du_t(t)\right|^2 - \frac{1}{\eta}\left(u^2,u_t^2\right)(t)\\ &\geqslant -\eta\left|Du_t(t)\right|^2 - \left(\left[1 + \left(\frac{1}{4}+\frac{1}{\eta}\right)\left|Du(t)\right|^2\right],u_t^2\right)(t), \quad \forall\eta > 0.\end{aligned}$$

Substituting I into (4.22) and using the Ehrling inequality, we obtain

$$\frac{d}{dt}\left((1+x),u_t^2\right)(t) + a\left|Du_t(t)\right|^2 + 5b\left|D^2u_t(t)\right|^2 + \varepsilon\left|D^5u_t(t)\right|^2 \leqslant \eta\left|Du_t(t)\right|^2 + \left(\left[1 + \varepsilon C + \left(\frac{1}{4}+\frac{1}{\eta}\right)\left|Du(t)\right|^2\right],u_t^2\right)(t). \tag{4.23}$$

We consider two cases:

(i) If $a > 0$, we take $\eta = \frac{1}{2}a$ in (4.23) and after integration and making use of (4.20), we get

$$\left|u_t(t)\right|^2 + \varepsilon\int_0^t \left|D^5u_s(s)\right|^2 ds + a\int_0^t \left|Du_s(s)\right|^2 ds + b\int_0^t \left|D^2u_s(s)\right|^2 ds \leqslant C\left(\|u_{0m}\|^2_{H^5(0,1)} + \varepsilon^2\|u_{0m}\|^2_{H^{10}(0,1)}\right) \tag{4.24}$$

with the constant C independent of ε.

(ii) If $a \leqslant 0$, then the same arguments that we have used to prove (4.21), show

$$\left|u_t(t)\right|^2 + \varepsilon \int_0^t \left|D^5 u_s(s)\right|^2 ds + b \int_0^t \left|Du_s(s)\right|^2 ds + b \int_0^t \left|D^2 u_s(s)\right|^2 ds \leqslant C\left(\|u_{0m}\|^2_{H^5(0,1)} + \varepsilon^2 \|u_{0m}\|^2_{H^{10}(0,1)}\right). \tag{4.25}$$

Estimate VI

Multiplying (4.5) by $-D^5 u$, we obtain

$$b\left|D^5 u(t)\right|^2 + \frac{1}{2}\varepsilon\left|D^7 u(1,t)\right|^2 \leqslant \left|\left(u_t, D^5 u\right)(t) + \left(uDu, D^5 u\right)(t) + a\left(D^3 u, D^5 u\right)(t)\right|. \tag{4.26}$$

Exploiting the Ehrling inequality with an arbitrary $\eta > 0$, we get

$$I_1 = a\left(D^3 u, D^5 u\right)(t) \leqslant |a|\left|D^5 u(t)\right|\left|D^3 u(t)\right| \leqslant \eta\left|D^5 u(t)\right|^2 + C(\eta)a^2\left|u(t)\right|^2,$$
$$I_2 = \left(uDu, D^5 u\right)(t) \leqslant \left|D^5 u(t)\right|\left\|u(t)\right\|_{L^4(0,1)}\left\|Du(t)\right\|_{L^4(0,1)},$$
$$I_3 = \left(u_t, D^5 u\right)(t) \leqslant \eta\left|D^5 u(t)\right|^2 + C(\eta)\left|u_t(t)\right|^2.$$

Using the Gagliardo–Nirenberg inequality, we find

$$\left\|u(t)\right\|_{L^4(0,1)} \leqslant C\left|D^5 u(t)\right|^{\frac{1}{20}}\left|u(t)\right|^{\frac{19}{20}}, \qquad \left\|Du(t)\right\|_{L^4(0,1)} \leqslant C\left|D^5 u(t)\right|^{\frac{1}{4}}\left|u(t)\right|^{\frac{3}{4}}.$$

With this I_2 for all $\eta > 0$ becomes

$$I_2 \leqslant \eta\left|D^5 u(t)\right|^2 + C(\eta)\left|u(t)\right|^{\frac{34}{7}}. \tag{4.27}$$

Substituting $I_1 - I_3$ into (4.26) and setting $\eta = \frac{b}{6}$, we come to the inequality:

$$b\left|D^5 u(t)\right|^2 \leqslant C\left\{\left|u_t(t)\right|^2 + \left|u(t)\right|^2 + \left|u(t)\right|^{\frac{34}{7}}\right\}.$$

Making use of (4.25), (4.26), we find

$$b\left|D^5 u_\varepsilon(t)\right|^2 \leqslant C\left\{1 + \|u_{0m}\|^2_{H^5(0,1)} + \varepsilon^2\|u_{0m}\|^2_{H^{10}(0,1)}\right\}. \tag{4.28}$$

The constant in (4.28) does not depend on $\varepsilon > 0$.

For every $\varepsilon \in (0,1]$ we have solvability of (4.5)–(4.8), moreover, the sequence of solutions $u_\varepsilon(x,t)$ satisfies estimates (4.17), (4.20), (4.21), (4.25), (4.26), (4.28) which are uniform in $\varepsilon > 0$. Whence, there exists a function $u(x,t)$ and a subsequence $\{u_\varepsilon(x,t)\}$ such that for all fixed m

$$u_\varepsilon \to u \quad \text{weak star in } L^\infty\left(0,T;H^5(0,1)\right),$$
$$u_{\varepsilon t} \to u_t \quad \text{weak star in } L^\infty\left(0,T;L^2(0,1)\right) \cap L^2\left(0,T;H^2(0,1)\right),$$
$$\varepsilon D^5 u_\varepsilon \to 0 \quad \text{weak star in } L^\infty\left(0,T;L^2(0,1)\right).$$

It is clear that $u(x,t)$ is a solution to (4.1)–(4.4).

Now directly from (4.1) $u \in L^2(0,T;H^7(0,1))$. □

Remark 1. To prove solvability of (4.1)–(4.4), from the technical reasons we needed an excessive regularity of initial data $u_{0m} \in H^{15}(0,1)$. In reality, it is sufficient $u_0 \in H^5(0,1)$. We approximate a function $u_(x)$: $u(0) = Du(0) = u(1) = Du(1) = D^2u(1) = 0$ by a sequence of functions $u_{0m} \in H^{15}(0,1)$ satisfying boundary conditions (4.6), (4.7) and also

$$D^{10}u_{0m}(0) = D^{10}u_{0m}(1) = D^{11}u_{0m}(0) = D^{11}u_{0m}(1) = D^{12}u_{0m}(1) = 0.$$

Of course,

$$\lim_{m\to\infty}\|u_0 - u_{0m}\|_{H^5(0,1)} = 0.$$

Such functions can be developed in series of eigenfunctions of problem (4.9) given by Proposition 2, see [4]. Passing to the limit as $m \to \infty$ in estimates (4.26), (4.28), we prove solvability of (4.1)–(4.4) for $u_0 \in H^5(0,1)$ satisfying boundary conditions (4.2), (4.3) as was claimed in Theorem 1. This completes the proof of the existence part of Theorem 1.

4.1. Uniqueness

Let $u_1(x,t)$, $u_2(x,t)$ be two regular solutions to (4.1)–(4.4), then $z = u_1 - u_2$ is a solution to the following problem:

$$z_t - bD^5 z + aD^3 z + \frac{1}{2}D\left(u_1^2 - u_2^2\right) = 0, \tag{4.29}$$

$$z(0,t) = Dz(0,t) = z(1,t) = Dz(1,t) = D^2 z(1,t) = 0, \tag{4.30}$$

$$z(x,0) = 0. \tag{4.31}$$

Multiplying (4.29) by $(1+x)z$ and integrating over $(0,1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left((1+x), z^2\right)(t) + \frac{5b}{2}\left|D^2 z(t)\right|^2 + \frac{b}{2}\left|D^2 z(0,t)\right|^2 + \frac{3a}{2}\left|Dz(t)\right|^2 + \frac{1}{2}\left((1+x)z, D\left[u_1^2 - u_2^2\right]\right)(t) = 0. \tag{4.32}$$

For all $\eta > 0$, we have

$$I_1 = \frac{3a}{2}\left|Dz(t)\right|^2 \geqslant -\eta\left|D^2 z(t)\right|^2 - \frac{9a^2}{16\eta}\left|z(t)\right|^2,$$

$$I_2 = \frac{1}{2}\left((1+x)z, D\left[u_1^2 - u_2^2\right]\right)(t) = \frac{1}{4}\left((1+x)D(u_1+u_2) - (u_1+u_2), z^2\right)(t) \geqslant -\frac{3}{8}\left(\left[2 + \left|D^2 u_1(t)\right|^2 + \left|D^2 u_2(t)\right|^2\right], z^2\right)(t).$$

Taking $\eta = \frac{1}{2}b$ and substituting I_1, I_2 into (4.32), we get

$$\frac{d}{dt}\left((1+x), z^2\right)(t) \leqslant C\left\{1 + \left|Du_1(t)\right|^2 + \left|Du_2(t)\right|^2\right\}\left((1+x), z^2\right)(t).$$

Because $u_i \in L^2(0,T; H_0^2(0,1))$, $i = 1,2$, then by the Gronwall's lemma,

$$\left((1+x), z^2\right)(t) = 0 \quad \text{a.e. in } (0,T),$$

hence

$$z(x,t) = 0 \quad \text{in } Q.$$

This proves uniqueness of a regular solution to (4.1)–(4.4) and completes the proof of Theorem 1. □

The exponential decay of the energy

We prove the exponential decay of the energy without restriction on a sign of the coefficient a.

Theorem 2. *Let $b > 0$ and $35b + 9a - \frac{8}{3}\|u_0\| = b_0 > 0$. Then for regular solutions of* (4.1)–(4.4) *the following inequality is valid*:

$$\left|u(t)\right|^2 \leqslant 4\|u_0\|^2 \exp(-\kappa t),$$

where $\kappa = \frac{b_0}{4}$.

Proof. First we multiply (4.1) by u to get

$$\frac{d}{dt}\left|u(t)\right|^2 + b\left|D^2 u(0,t)\right|^2 = 0.$$

Since $b > 0$, this gives for all $t > 0$,

$$\left|u(t)\right|^2 \leqslant \|u_0\|^2. \tag{4.33}$$

To obtain more estimates, we consider the following scalar product:

$$\begin{aligned}(Lu, \phi u)(t) = {} & \frac{1}{2}\frac{d}{dt}\left(\phi, u^2\right)(t) + (\phi u Du, u)(t) + \frac{3a}{2}\left(D\phi, (Du)^2\right)(t) - \frac{1}{2}a\left(D^3\phi, u^2\right)(t) - 5b\left(D^3\phi, (Du)^2\right)(t) \\ & + 5b\left(D\phi, \left(D^2 u\right)^2\right)(t) + b\left(D^5\phi, u^2\right)(t) + b\left|D^2 u(0,t)\right|^2 = 0,\end{aligned} \tag{4.34}$$

where

$$\phi = \phi(x) = 1 + 4x - x^3, \quad x \in [0,1]. \tag{4.35}$$

It is easy to see that

$$\phi(x) > 1, \qquad D\phi(x) \geqslant 1, \qquad D^3\phi(x) = -6, \qquad \max_{[0,1]} D\phi(x) = 4. \tag{4.36}$$

Using (4.33), we estimate

$$I_1 = (\phi u Du, u)(t) = -\frac{1}{3}\big(D\phi, u^3\big)(t) \geqslant -\frac{1}{3}\max_{[0,1]} D\phi\|u_0\|\big|Du(t)\big|^2.$$

Substituting I_1 into (4.34) and making use of (4.35), (4.36), we come to the inequality

$$\frac{d}{dt}\big(\phi, u^2\big)(t) + \left(\left[3a + 30b - \frac{8}{3}\|u_0\|\right], (Du)^2\right)(t) + 5b\big|D^2u(t)\big|^2 + 6a\big|u(t)\big|^2 \leqslant 0. \tag{4.37}$$

Because $b > 0$ and $35b + 9a - \frac{8}{3}\|u_0\| > 0$, then (4.37) can be rewritten as

$$\frac{d}{dt}\big(\phi, u^2\big)(t) + \left[35b + 9a - \frac{8}{3}\|u_0\|\right]\big|u(t)\big|^2 \leqslant 0.$$

Denoting $35b + 9a - \frac{8}{3}\|u_0\| = b_0$, we get

$$\frac{d}{dt}\big(\phi, u^2\big)(t) + \frac{b_0}{4}\big(\phi, u^2\big)(t) \leqslant 0.$$

Integration and (4.36) give

$$\big|u(t)\big|^2 \leqslant 4\|u_0\|^2 \exp(-\kappa t)$$

where

$$\kappa = \frac{b_0}{4}.$$

This proves Theorem 2. □

5. KdV equation as a limit of the Kawahara equation

In considerations above we had no restrictions on a sign of the coefficient a, but the coefficient b had to be strictly positive. Now we want to prove analogous results in the case $b = 0$, $a > 0$, i.e., for the KdV equation. Due to results of Section 3, this is an equation of the type (3.1) with $l = 1$. In this case, by Lemma 6, we must put one condition at $x = 0$ and two conditions at $x = 1$:

$$u_t + uDu + aD^3u = 0; \quad x \in (0,1),\ t > 0, \tag{5.1}$$

$$u(0,t) = u(1,t) = Du(1,t) = 0, \quad t > 0, \tag{5.2}$$

$$u(x,0) = u_0(x), \quad x \in (0,1). \tag{5.3}$$

We prove the following result.

Lemma 8. *Let $a > 0$ and $u_0 \in H^3(0,1)$. Then for all finite $T > 0$ there exists a unique solution $u(x,t)$:*

$$u \in C\big(0,T; H^3(0,1)\big) \cap L^2\big(0,T; H^5(0,1)\big), \qquad u_t \in L^\infty\big(0,T; L^2(0,1)\big)$$

to (5.1)–(5.3) *which is a limit of solutions to* (4.1)–(4.4) *as b tends to* 0.

Proof. First, passing to the limits as $m \to \infty$, $\varepsilon \to 0$ in estimates (4.20), (4.24), and combining the results, we obtain

$$\big|u_t(t)\big|^2 + a\int_0^t \big\{\big|Du(s)\big|^2 + \big|Du_s(s)\big|^2\big\}\,ds + b\int_0^t \big\{\big|D^2u(s)\big|^2 + \big|D^2u_s(s)\big|^2\big\}\,ds \leqslant C\big(\|u_0\|^2_{H^3(0,1)} + b^2\big\|D^5u_0\big\|^2\big),$$

where the constant C does not depend on $b > 0$. When b tends to 0, we get a sequence of solutions to (4.1)–(4.4) $\{u_b(x,t)\}$ which uniformly in $b > 0$ satisfies the estimates:

$$\big|u_{bt}(t)\big|^2 + a\int_0^t \big\{\big|Du_b(s)\big|^2 + \big|Du_{bs}(s)\big|^2\big\}\,ds \leqslant C\big(\|u_0\|^2_{H^3(0,1)} + b^2\big\|D^5u_0\big\|^2\big), \tag{5.4}$$

$$\lim_{b\to 0} b\|u_b\|_{C(0,T;H^2(0,1))} = 0. \tag{5.5}$$

It means that there exists a function $u(x,t)$ such that

$$u_b \rightharpoonup u \quad \text{weak star in } C\big(0,T;H^1(0,1)\big),$$
$$u_{bt} \rightharpoonup u_t \quad \text{weak star in } L^\infty\big(0,T;L^2(0,1)\big).$$

Let $v(x)$ be an arbitrary function from $H^3(0,1)$ such that

$$v(0) = Dv(0) = D^2v(0) = v(1) = Dv(1) = 0.$$

Multiplying (4.1) by v and integrating by parts, taking into account (4.2), (4.3), we get

$$(u_{bt}, v)(t) + (u_b Du_b, v)(t) + a\big(Du_b, D^2v\big)(t) + b\big(D^2u_b, D^3v\big)(t) = 0.$$

Making use of (5.4), (5.5), we can pass to the limit as $b \to 0$ and obtain

$$(u_t, v)(t) + (uDu, v)(t) + a\big(Du, D^2v\big)(t) = 0, \quad \text{a.e. in } (0,T),$$

which is valid for any $v \in H^2(0,1)$ such that $v(0) = Dv(0) = v(1) = 0$. The last identity means that $u(x,t)$ is a weak solution to the following problem:

$$aD^3u = F(x,t), \quad x \in (0,1),\ t > 0, \tag{5.6}$$
$$u(0,t) = u(1,t) = Du(1,t) = 0, \quad t > 0, \tag{5.7}$$

where

$$F(x,t) = -u_t - uDu \in L^\infty\big(0,T;L^2(0,1)\big),$$

whence, taking into account (5.4):

$$u \in C\big(0,T;H^3(0,1)\big) \cap L^2\big(0,T;H^4(0,1)\big),$$
$$u_t \in L^\infty\big(0,T;L^2(0,1)\big) \cap L^2\big(0,T;H_0^1(0,1)\big).$$

This implies

$$u_t + uDu + aD^3u = 0, \quad x \in (0,1),\ t > 0,$$
$$u(0,t) = u(1,t) = Du(1,t) = 0, \quad t > 0,$$
$$u(x) = u_0(x).$$

Uniqueness and exponential decay of this solution is a known fact [22]. This proves Lemma 8. □

Remark 2. To use solvability of (4.1)–(4.4), we need $u_0 \in H^5(0,1)$. Hence, to prove Lemma 8, we approximate $u_0 \in H^3(0,1)$ by a sequence of functions $u_{0m} \in H^5(0,1)$. Passing to the limit as $b \to 0$ in (5.4) at a fixed m and then passing to the limit as $m \to \infty$, we complete the proof of Lemma 8.

6. Kawahara equation with $b < 0$

For technical reasons, we consider in the domain $Q^- = (-1,0) \times (0,T)$ the following problem:

$$u_t + uD_xu + aD_x^3u + bD_x^5u = 0 \quad \text{in } Q^-,\ b > 0, \tag{6.1}$$
$$u(-1,t) = D_xu(-1,t) = D_x^2u(-1,t) = 0, \quad t > 0, \tag{6.2}$$
$$u(0,t) = D_xu(0,t) = 0, \quad t > 0, \tag{6.3}$$
$$u(x,0) = u_0(x). \tag{6.4}$$

Differently from the considerations above, now we set three conditions at the left boundary and two conditions at the right boundary. Correctness of this problem also could be studied as had been made for (4.1)–(4.4), but is easier to change the variables $y = -x$ which gives

$$u_t - uD_yu - aD_y^3u - bD_y^5u = 0, \quad \text{in } Q = (0,1) \times (0,T),$$
$$u(0,t) = D_yu(0,t) = 0, \quad t > 0,$$
$$u(1,t) = D_yu(1,t) = D_y^2u(1,t) = 0, \quad t > 0,$$
$$u(y,0) = u_0(-y).$$

This problem differs from (4.1)–(4.4) only by the signs of uDu and aD^3u that did not play any important part in proofs of solvability and stability. It means that all the results standing for (4.1)–(4.4) are valid also for (6.1)–(6.4) inclusive passage to the limit as $b \to 0$ for $a < 0$.

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