Existence-uniqueness and iterative methods for third-order boundary value problems

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Abstract: In this paper we shall provide necessary and sufficient conditions for the existence and uniqueness of solutions of third-order nonlinear differential equations satisfying three-point boundary conditions. For the linear case, we propose a constructive method which is a variation of the method of chasing. For the nonlinear problems sufficient conditions are provided to ensure the convergence of a general class of iterative methods. Several examples are also included.

Keywords: Third-order differential equations, three-point boundary conditions, existence and uniqueness, method of chasing, iterative methods.

1. Introduction

In this paper we shall consider the following third-order nonlinear differential equation
\[ x''' = f(t, x, x', x''), \] (1.1)
together with the three-point boundary conditions
\[ x'(a) = A, \quad x(b) = B, \quad x'(c) = C \] (1.2)
where \( a \leq b \leq c \) (\( a < c \)) and \( f : C[[a, c] \times \mathbb{R}^3 \rightarrow \mathbb{R}] \). This problem as it is or with slightly different boundary conditions has also been included in the recent work, e.g., see [3–5,7,12–14,17,21]. More so, imposing some ideal conditions sandwich beam analysis leads to a third-order linear differential equation
\[ x''' - k^2 x' + a = 0, \] (1.3)
together with three-point boundary conditions
\[ x'(0) = x(\frac{1}{2}) = x'(1) = 0 \] (1.4)
where \( k \) and \( a \) are some physical constants, e.g., see [15].

The problem (1.3), (1.4) has a unique solution and can be expressed in terms of elementary functions
\[ x(t) = \frac{a}{k^3} \left[ (\sinh \frac{1}{2} k - \sinh k t) + k (t - \frac{1}{2}) + \tanh \frac{1}{2} k \left( \cosh k t - \cosh \frac{1}{2} k \right) \right]. \] (1.5)
However, in general even if it is known that (1.1), (1.2) has a unique solution it is not possible to find it explicitly. Faced with this difficulty we resort to numerical methods and in [18] some methods for linear problems have been discussed. In this paper first we shall provide necessary and sufficient conditions for the existence and uniqueness of the solutions of (1.1), (1.2). The proofs of all these results are based on several inequalities obtained in Section 2. These results are of great help before applying any numerical technique. In Section 4, we shall consider $f$ to be linear and give very general sufficient conditions to ensure the existence of a unique solution of (4.1), (1.2). For this problem we also propose a constructive method which is a variation of the method of chasing. In Section 5, we provide a priori estimate on the length of the interval $(c - a)$ so that the iterative scheme (5.1), (5.2) converges to the unique solution of (1.1), (1.2). A priori conditions for the quadratic convergence are also given. Finally, in Section 6 some examples are illustrated.

2. Preliminary results

**Lemma 2.1.** Let $x(t) \in C^{(3)}[a, c]$, satisfying

$$x'(a) - x(b) - x'(c) - 0 \tag{2.1}$$

where $a \leq b \leq c$ $(a < c)$. Then,

$$|x(t)| \leq \frac{1}{12} (c - a)^3 \max_{a \leq t \leq c} |x'''(t)|, \tag{2.2}$$

$$|x'(t)| \leq \frac{1}{8} (c - a)^2 \max_{a \leq t \leq c} |x'''(t)| \tag{2.3}$$

and

$$|x''(t)| \leq \frac{1}{2} (c - a) \max_{a \leq t \leq c} |x'''(t)|. \tag{2.4}$$

**Proof.** Any function $x(t) \in C^{(3)}[a, c]$ satisfying (2.1) can be written as

$$x(t) = \int_a^c g(t, s)x'''(s) \, ds \tag{2.5}$$

where

$$g(t, s) = g_1(t, s) = \frac{\frac{1}{2} (b - t)(s - a)(2c - b - t)}{(c - a)}, \quad a \leq s \leq b, \quad s \leq t \leq c;$$

$$g_2(t, s) = g_1(t, s) - \frac{1}{2} (t - s)^2, \quad a \leq s \leq b, \quad a \leq t \leq s;$$

$$g_3(t, s) = g_4(t, s) + \frac{1}{2} (t - s)^2, \quad b \leq s \leq c, \quad s \leq t \leq c;$$

$$g_4(t, s) = \frac{\frac{1}{2} (b - t)(c - s)(b + t - 2a)}{(c - a)}, \quad b \leq s \leq c, \quad a \leq t \leq s.$$
\[ |x(t)| \leq \left[ \int_a^t g_1(t, s) \, ds + \int_t^b g_2(t, s) \, ds + \int_b^c g_3(t, s) \, ds \right] \max_{a \leq t \leq c} |x'''(t)| \]

if \( a \leq t \leq b \)

and

\[ |x(t)| \leq \left[ \int_a^b g_1(t, s) \, ds + \int_t^c g_2(t, s) \, ds + \int_b^c g_3(t, s) \, ds \right] \max_{a \leq t \leq c} |x'''(t)| \]

if \( b \leq t \leq c \).

Now, a direct computation provides that

\[ |x(t)| \leq \left| \phi_1(t, b) \right| \max_{a \leq t \leq c} |x'''(t)| \]  \hspace{1cm} (2.6)

where

\[ \phi_1(t, b) = \frac{1}{12} \left( \frac{b-t}{c-a} \right) \left[ 3(2c-h-t)(h-a)^2 + 3(h+t-2a)(c-b)^2 \right. \]

\[ \left. -2(b-t)^2(c-a) \right]. \]  \hspace{1cm} (2.7)

The function \( \phi_1(t, b) \) satisfies the inequality

\[ |\phi_1(t, b)| \leq \frac{1}{12} \max \left\{ (c-b)^2(c+2b-3a), (b-a)^2(3c-2b-a) \right\} \]

\[ \leq \frac{1}{12} (c-a)^3. \]  \hspace{1cm} (2.8)

Using (2.9) in (2.6) the first inequality (2.2) follows.

By similar calculations

\[ |x'(t)| \leq \left| \int_a^c \left[ \frac{\partial g(t, s)}{\partial t} \right] \, ds \times \max_{a \leq t \leq c} |x'''(t)| \right| \]

\[ = \phi_2(t) \max_{a \leq t \leq c} |x'''(t)| \]  \hspace{1cm} (2.10)

where

\[ \phi_2(t) = \frac{1}{8} (t-a)(c-t) \leq \frac{1}{8} (c-a)^2 \]

which implies (2.3). Finally, we have

\[ |x''(t)| \leq \left| \int_a^c \left[ \frac{\partial^2 g(t, s)}{\partial t^2} \right] \, ds \times \max_{a \leq t \leq c} |x'''(t)| \right| \]

\[ = \phi_3(t) \max_{a \leq t \leq c} |x'''(t)| \]  \hspace{1cm} (2.11)

where

\[ \phi_3(t) = \frac{1}{2} \frac{(t-a)^2 + (c-t)^2}{(c-a)} \leq \frac{1}{2} (c-a) \]

which gives (2.4). \( \Box \)

**Remark 2.2.** The function \( \phi_1(t, b) \) defined in (2.7) satisfies (2.1) and for this function in (2.2)–(2.4) equality holds. Thus, inequalities (2.2)–(2.4) are the best possible.
Lemma 2.3. The following estimates hold

\[
\int_a^c |g(t, s)||\phi_1(s, b)| \, ds \leq \frac{1}{24}(c-a)^3|\phi_1(t, b)|,
\]

\[
\int_a^c |g(t, s)||\phi_2(s)| \, ds \leq \left(\frac{3}{4} - \frac{1}{12} \sqrt{3}\right)(c-a)^2|\phi_1(t, b)|,
\]

\[
\int_a^c |g(t, s)||\phi_3(s)| \, ds \leq \frac{1}{3}(c-a)|\phi_1(t, b)|,
\]

\[
\int_a^c \frac{\partial g(t, s)}{\partial t} |\phi_1(s, b)| \, ds \leq \frac{7}{120}(c-a)^3\phi_2(t),
\]

\[
\int_a^c \frac{\partial g(t, s)}{\partial t} \phi_2(s) \, ds \leq \left(\frac{3}{4} - \frac{1}{12} \sqrt{3}\right)(c-a)^2\phi_2(t),
\]

\[
\int_a^c \frac{\partial g(t, s)}{\partial t} \phi_3(s) \, ds \leq \frac{1}{3}(c-a)\phi_2(t),
\]

\[
\int_a^c \frac{\partial^2 g(t, s)}{\partial t^2} |\phi_1(s, b)| \, ds \leq \frac{7}{120}(c-a)^3\phi_3(t),
\]

\[
\int_a^c \frac{\partial^2 g(t, s)}{\partial t^2} \phi_2(s) \, ds \leq \left(\frac{3}{4} - \frac{1}{12} \sqrt{3}\right)(c-a)^2\phi_3(t),
\]

\[
\int_a^c \frac{\partial^2 g(t, s)}{\partial t^2} \phi_3(s) \, ds \leq \frac{1}{3}(c-a)\phi_3(t).
\]

Proof. The proof involves computation which is tedious, though elementary. \(\square\)

3. Existence and uniqueness

Theorem 3.1. Suppose that

(i) \(K_0, K_1, K_2 > 0\) be given real numbers and let \(Q\) be the maximum of \(|f(t, u_0, u_1, u_2)|\) on the compact set: \([a, c] \times D_0\), where \(D_0 = \{(u_0, u_1, u_2): |u_0| \leq 2K_0, |u_1| \leq 2K_1, |u_2| \leq 2K_2\}\).

(ii) \((c-a) \leq \min\{(12K_0/Q)^{1/3}, (8K_1/Q)^{1/2}, (2K_2/Q)\}\).

(iii) \(|B| + \frac{1}{2}(c-a)(|A| + |C|) \leq K_0\),

\[
\max\{|A|, |C|\} \leq K_1 \quad \text{and} \quad \frac{1}{(c-a)}|C-A| \leq K_2.
\]

Then, the boundary value problem (1.1), (1.2) has a solution in \(D_0\).

Proof. We note that any solution \(x(t)\) of the problem (1.1), (1.2) is also a solution of

\[
x(t) = P_2(t) + \int_a^c g(t, s)f(s, x(s), x'(s), x''(s)) \, ds
\]
where

\[ P_2(t) = \frac{(t + b - 2c)(b - t)}{2(c - a)} A + B + \frac{(t - 2a + b)(t - b)}{2(c - a)} C \]

and conversely any solution of (3.1) is a solution of (1.1), (1.2).

Next, we define the set

\[ S[a, c] = \{ x(t) \in C^2[a, c]: \| x \| \leq 2K_0, \| x' \| \leq 2K_1, \| x'' \| \leq 2K_2 \} \]

where \( \| x^{(i)} \| = \max_{a \leq t \leq c} |x^{(i)}(t)|; \ i = 0, 1, 2 \). It is easy to verify that \( S[a, c] \) is a closed convex subset of the Banach space \( C^2[a, b] \). The mapping defined by

\[ (Tx)(t) = P_2(t) + \int_a^c g(t, s)f(s, x(s), x'(s), x''(s)) \, ds \]  

is completely continuous. Obviously any fixed point of (3.2) is a solution of (1.1), (1.2).

Let \( x(t) \in S[a, c] \), then \( (Tx)(t) - P_2(t) \) satisfies the conditions of Lemma 2.1 and

\[ (Tx)'''(t) - P_2'''(t) = (Tx)'''(t) = f(t, x(t), x'(t), x''(t)) \]

thus

\[ \max_{a \leq t \leq c} |(Tx)'''(t)| \leq Q. \]

Hence, from Lemma 2.1 we have

\[ \|(Tx)(t) - P_2(t)\| \leq \frac{1}{12}(c - a)^3Q \]

which gives

\[ \|(Tx)\| \leq |B| + \frac{1}{2}(c - a)(|A| + |C|) + \frac{1}{12}(c - a)^3Q \]

\[ \leq K_0 + K_0 = 2K_0. \]  

Further, we have

\[ \left| (Tx)'(t) - \frac{(c - t)}{(c - a)} A - \frac{(t - a)}{(c - a)} C \right| \leq \frac{1}{6}(c - a)^2Q \]

which also implies that

\[ \|(Tx)''\| \leq \max\{ |A|, |C| \} + \frac{1}{3}(c - a)^2Q \]

\[ \leq K_1 + K_1 = 2K_1. \]

Similarly, we get

\[ \|(Tx)'''\| \leq \frac{1}{(c - a)} |C - A| + \frac{1}{2}(c - a)Q \]

\[ \leq K_2 + K_2 = 2K_2. \]

Thus, \( T \) maps \( S[a, c] \) into itself. Further, the inequalities (3.3)–(3.5) imply that the sets \( \{(Tx)(t): x(t) \in S[a, c]\}, \{(Tx)'(t): x(t) \in S[a, c]\} \) and \( \{(Tx)'''(t): x(t) \in S[a, c]\} \) are uniformly bounded and equicontinuous on \([a, c]\). Hence, that \( TS[a, c] \) is compact follows from the Ascoli–Arzelà theorem. Thus, the Schauder fixed point theorem applies and a fixed point of (3.2) in \( D_0 \) exists. \( \square \)
Remark 3.2. If \( b = \frac{1}{3}(a + c) \) then, from (2.8) the inequality (2.9) can be replaced by \( |\phi_1(t, \frac{1}{3}(a + c))| \leq \frac{1}{24}(c - a)^3 \). Thus, in Theorem 3.1 the condition (ii) can be changed to \( (c - a) \leq \min\{(24K_0/Q)^{1/3}, (8K_1/Q)^{1/2}, (2K_2/Q)\} \).

Corollary 3.3. Let the conditions (i), (ii) of Theorem 3.1 be satisfied and let \( g(t) \in C^2[a, c] \) be a given function. Then, the differential equation (1.1) together with
\[
\begin{align*}
  x'(a) &= g'(a), \\
  x(b) &= g(b), \\
  x'(c) &= g'(c)
\end{align*}
\]  
has a solution if \( M_0 + M_1(c - a) \leq K_0, \ M_1 \leq K_1 \) and \( M_2 \leq K_2 \) where \( |g(t)| \leq M_0, \ |g'(t)| \leq M_1 \) and \( |g''(t)| \leq M_2 \) for all \( t \in [a, c] \).

Proof. We need to verify that the condition (iii) of Theorem 3.1 is satisfied. For this, let \( A = g'(a), \ B = g(b) \) and \( C = g'(c) \) then, we have
\[
\left| \frac{1}{(c - a)} |C - A| \right| = \frac{|g'(c) - g'(a)|}{c - a} = |g''(p)|, \quad a < p < c, \quad \text{but} \quad |g''(t)| \leq M_2.
\]
Next, \( \max\{|A|, |C|\} = \max\{|g'(a)|, |g'(c)|\} \leq M_1 \). Finally, we find \( |B| + \frac{1}{2}(c - a)(|A| + |C|) \leq M_0 + (c - a)M_1 \). \( \square \)

Corollary 3.4. Assume that the function \( f(t, u_0, u_1, u_2) \) on \( [a, c] \times \mathbb{R}^3 \) satisfies the following condition
\[
|f(t, u_0, u_1, u_2)| \leq c_0 + c_1 |u_0|^{\alpha_1} + c_2 |u_1|^{\alpha_2} + c_3 |u_2|^{\alpha_3}
\]  
(3.7)
where \( c_i \geq 0, \ 0 \leq i \leq 3 \) and \( 0 \leq \alpha_1, \alpha_2, \alpha_3 < 1 \). Then, for any function \( g(t) \in C^2[a, c] \) the boundary value problem (3.1), (3.6) has a solution.

Proof. Let \( x(t) \in S[a, c] \) then, condition (3.7) provides
\[
|f(t, x(t), x'(t), x''(t))| \leq c_0 + c_1 K_0^{\alpha_1} + c_2 K_1^{\alpha_2} + c_3 K_2^{\alpha_3} = M \quad \text{(say)}.
\]
Now Corollary 3.3 is applicable by choosing \( K_0, K_1, K_2 \) sufficiently large so that
\[
\frac{1}{12}(c - a)^3 M \leq K_0, \quad \frac{1}{8}(c - a)^2 M \leq K_1, \quad \frac{1}{3}(c - a)M \leq K_2
\]
and
\[
M_0 + M_1(c - a) \leq K_0, \quad M_1 \leq K_1, \quad M_2 \leq K_2. \quad \square
\]

Corollary 3.5. Let the conditions of Theorem 3.1 be satisfied. Then, for any given \( \epsilon > 0 \) there is a solution \( x(t) \) of (1.1), (1.2) such that \( |x^{(i)}(t) - P_2^{(i)}(t)| < \epsilon; \ i = 0, 1, 2 \) provided \( (c - a) \) is sufficiently small.

Proof. Let \( x(t) \) be a solution of (1.1), (1.2) and \( x(t) \in S[a, c] \), then
\[
\begin{align*}
  |x(t) - P_2(t)| &\leq \frac{1}{12}(c - a)^3 Q, \\
  |x'(t) - P_2'(t)| &\leq \frac{1}{8}(c - a)^2 Q, \\
  |x''(t) - P_2''(t)| &\leq \frac{1}{3}(c - a) Q.
\end{align*}
\]
Thus, if
\[(c - a) \leq \min\{(12\varepsilon/Q)^{1/3}, (8\varepsilon/Q)^{1/2}, 2\varepsilon/Q\}\]
the corollary follows. □

**Theorem 3.6.** Assume that the function \(f(t, u_0, u_1, u_2)\) on \([a, c] \times D_1\) satisfies the following condition
\[
|f(t, u_0, u_1, u_2)| \leq L + L_0|u_0| + L_1|u_1| + L_2|u_2| \tag{3.8}
\]
where
\[
D_1 = \left\{(u_0, u_1, u_2) : |u_0| \leq \frac{L + l}{1 - \theta} |\phi_1(t, b)| + \max_{a \leq t \leq c} |P_2(t)|, |u_1| \leq \frac{L + l}{1 - \theta} |\phi_2(t)| + \max_{a \leq t \leq c} |P'_2(t)|, |u_2| \leq \frac{L + l}{1 - \theta} |\phi_3(t)| + \max_{a \leq t \leq c} |P''_2(t)|\right\}
\]
and
\[
\theta = \frac{7}{120}(c - a)^3L_0 + \left(\frac{1}{4} - \frac{1}{12\sqrt{3}}\right)(c - a)^2L_1 + \frac{1}{3}(c - a)L_2 < 1,
\]
\[
l = L_0 \max_{a \leq t \leq c} |P_2(t)| + L_1 \max_{a \leq t \leq c} |P'_2(t)| + L_2 \max_{a \leq t \leq c} |P''_2(t)|.
\]
Then, the boundary value problems (1.1), (1.2) has a solution in \(D_1\).

**Proof.** The boundary value problem (1.1), (1.2) is equivalent to the following problem
\[
z'''(t) = f(t, z(t) + P_2(t), z'(t) + P'_2(t), z''(t) + P''_2(t)), \tag{3.9}
\]
\[
z'(a) = z(b) = z'(c) = 0. \tag{3.10}
\]
We define \(S_1[a, c]\) as the set of functions twice continuously differentiable on \([a, c]\) satisfying the boundary conditions (3.10). If we introduce in \(S_1[a, c]\) the finite norm
\[
\|z\| = \max\left\{\frac{1}{a \leq t \leq c} |z(t)|, \frac{1}{a \leq t \leq c} |z'(t)|, \frac{1}{a \leq t \leq c} |z''(t)|\right\}
\]
then it becomes a Banach space. We shall show that the mapping \(T: S_1[a, c] \to S_1[a, c]\) defined by
\[
(Tz)(t) = \int_a^c g(t, s)f(s, z(s) + P_2(s), z'(s) + P'_2(s), z''(s) + P''_2(s)) \, ds \tag{3.11}
\]
maps the ball \(B = \{z(t) \in S_1[a, c] : \|z\| \leq (L + l)/(1 - \theta)\}\) into itself. For this, let \(x(t) \in B\). Then we have
\[
|z(t)| \leq \frac{L + l}{1 - \theta} |\phi_1(t, b)|, \quad |z'(t)| \leq \frac{L + l}{1 - \theta} |\phi_2(t)|, \quad |z''(t)| \leq \frac{L + l}{1 - \theta} |\phi_3(t)|
\]
and hence \((z(t) + P_2(t), z'(t) + P'_2(t), z''(t) + P''_2(t)) \in D_1\). Thus, using (3.8) into (3.11), we get
\[
||(Tz)(t)|| \leq \int_a^c |g(t, s)||L + L_0(|z(s)| + |P_2(s)|)
\]
\[+ L_1(|z'(s)| + |P'_2(s)|) + L_2(|z''(s)| + |P''_2(s)|)\, ds
\]
\[\leq (L + l)|\phi_1(t, b)| + \frac{L + l}{1 - \theta} \int_a^c |g(t, s)||L_0|\phi_1(t, s)| + L_1\phi_2(s) + L_2\phi_3(s)\, ds.
\]
Using Lemma 2.3 in the above inequality, we obtain

\[(Tz)(t) \leq (L + l) \phi_1(t, b) + \frac{L + l}{1 - \theta} \theta \phi_1(t, b)\]

\[= \frac{L + l}{1 - \theta} \phi_1(t, b).\]

Similarly, we find

\[(Tz)'(t) \leq \frac{L + l}{1 - \theta} \phi_2(t) \quad \text{and} \quad (Tz)''(t) \leq \frac{L + l}{1 - \theta} \phi_3(t).\]

Thus, \([Tz] \leq (L + l)/(1 - \theta)\) and from the Schauder fixed point theorem it follows that \(T\) has a fixed point in \([a, c]\). This fixed point is a solution of (3.9), (3.10).

**Theorem 3.7.** Suppose that the boundary value problem (1.1), (2.1) has a nontrivial solution \(x(t)\) and the condition (3.8) with \(L = 0\) is satisfied for all \((t, u_0, u_1, u_2) \in [a, c] \times D_2\), where

\[D_2 = \{(u_0, u_1, u_2): |u_0| \leq m|\phi_1(t, b)|, |u_1| \leq m\phi_2(t), |u_2| \leq m\phi_3(t)\}\]

and \(m = \max_{a \leq t \leq c}|x'''(t)|\). Then, it is necessary that \(\theta \geq 1\).

**Proof.** Since \(x(t)\) is a nontrivial solution of (1.1), (2.1) it is necessary that \(m \neq 0\). Further, as a consequence of inequalities (2.6), (2.10) and (2.11) we note that \(x(t) \in D_2\). Next, the solution \(x(t)\) satisfy

\[x(t) = \int_a^c g(t, s)f(s, x(s), x'(s), x''(s)) \, ds\]

and hence

\[|x(t)| \leq \int_a^c |g(t, s)| \left[ L_0 |x(s)| + L_1 |x'(s)| + L_2 |x''(s)| \right] \, ds\]

\[\leq m \int_a^c |g(t, s)| \left[ L_0 \phi_1(s, b) + L_1 \phi_2(s) + L_2 \phi_3(s) \right] \, ds. \tag{3.12}\]

Using Lemma 2.3 in the above inequality, we find

\[|x(t)| \leq m\theta |\phi_1(t, b)| \tag{3.13}\]

and, similarly

\[|x'(t)| \leq m\theta \phi_2(t), \tag{3.14}\]

\[|x''(t)| \leq m\theta \phi_3(t). \tag{3.15}\]

Using (3.13)–(3.15) into (3.12) successively, we find

\[|x(t)| \leq m\theta^k |\phi_1(t, b)|, \quad |x'(t)| \leq m\theta^k \phi_2(t) \]

and

\[|x''(t)| \leq m\theta^k \phi_3(t), \quad k = 1, 2, \ldots. \]

Thus, if \(\theta < 1\) then for every given \(\epsilon > 0\), we can find sufficiently large \(k\) such that \(|x(t)| < \epsilon\). However, this contradicts our assumption that \(x(t)\) is nontrivial and hence it is necessary that \(\theta \geq 1\).
Remark 3.8. The conditions of Theorem 3.7 ensure that at least one of \( L_0, L_1, L_2 \) will not be zero, otherwise on \([a, c]\) the solution \( x(t) \) will coincide with a polynomial of degree two and will not be a nontrivial solution of (1.1), (2.1). Further, \( x(t) \equiv 0 \) is obviously a solution of (1.1), (2.1) and, if \( \theta < 1 \), then it is also unique.

4. Solution of linear problems

For \( p(t), q(t), r(t) \) and \( h(t) \in C[a, c], \) Theorem 3.6 ensures that the linear differential equation

\[
 x''' = p(t)x'' + q(t)x' + r(t)x + h(t)
\]

satisfying (1.2) has a unique solution provided \( \theta < 1 \) where

\[
 L_0 = \max_{a \leq t \leq c} |r(t)|, \quad L_1 = \max_{a \leq t \leq c} |q(t)| \quad \text{and} \quad L_2 = \max_{a \leq t \leq c} |p(t)|.
\]

It is of interest to note that (4.1), (1.2) has a unique solution provided (i) \( q(t) > 0 \) for all \( t \in (a, c) \) (ii) \( r(t) < 0 \) if \( t \in (a, b) \) and \( r(t) > 0 \) if \( t \in (b, c) \). For this, from the linearity of the problem (4.1), (1.2) it is sufficient to show that the linear differential equation

\[
 z''' = p(t)z'' + q(t)z' + r(t)z
\]

satisfying (2.1) has only the trivial solution. If \( z(t) \) is a nontrivial solution of (4.2), (2.1) then, \( z'(b) = z''(b) = 0 \) is not possible follows from the uniqueness of the initial value problems. We shall show that \( z'(b) \neq 0 \) and \( z''(b) \neq 0. \) Assume without loss of generality that \( z'(b) = 0, \) \( z''(b) > 0 \) then, there exists a \( t_1 \in (b, c) \) such that \( z''(t) > 0 \) in \( [b, t_1) \) and \( z''(t_1) = 0 \). From this it follows that \( z(t), z'(t) > 0 \) in \( (b, t_1). \) Thus, at the point \( t_1 \) the differential equation (4.2) provides \( z''(t_1) > 0 \) whereas, \( (z''(t_1) - \epsilon) - z''(t_1))/ - \epsilon \leq 0 \) for all small \( \epsilon > 0 \) and hence \( z''(t_1) \leq 0. \) This contradiction shows that \( z'(b) = 0, \) \( z''(b) > 0 \) is not possible. Next, we assume that \( z'(b) > 0, \) \( z''(b) = 0 \) then, from (4.2) we get \( z''(b) > 0 \) and hence there exists \( t_2 \in (b, c) \) such that \( z''(t) = 0 \) and \( z''(t) > 0 \) in \( (b, t_2). \) This again leads to a contradiction and \( z'(b) = 0 \) \( (\leq 0), \) \( z''(b) = 0 \) is also not possible. Thus, if we assume that \( z'(b) > 0 \) then, the only possibilities are \( z''(b) > 0 \) or \( z''(b) < 0 \) however, in both of these cases also we get a similar contradiction and hence our assumption that \( z(t) \neq 0 \) is not correct.

In the rest of this section, we shall assume that \( p(t), q(t) \) and \( r(t) \) satisfy some of the above sufficient conditions so that (4.1), (1.2) has a unique solution. This solution can be constructed (directly or converting (4.1), (1.2) into its equivalent first-order system form) by using one of the several known methods e.g. the method of complementary functions [1,2,8,11], the method of adjoints [1,6], the method of particular solutions [2], the method of chasing [9,10], the method of invariant imbedding [16] the initial-value adjusting method [19,20,22] etc. Here, we shall formulate a variation of the method of chasing which is different from the one given in [18]. For this, let \( x(t) \) be the solution of (4.1), (1.2) and introduce a second order linear differential equation

\[
 x'(t) = \alpha_1(t)x(t) + \alpha_2(t)x''(t) + \alpha_3(t)
\]

where \( \alpha_1(t), \alpha_2(t) \) and \( \alpha_3(t) \) are yet to be determined. Differentiating equation (4.3) and eliminating \( x''(t) \) from (4.1), we get

\[
 x''(t) = \alpha_1(t)x(t) + \alpha_2(t)x'(t) + \alpha_3(t)x''(t) + \alpha_4(t)(p(t)x''(t) + q(t)x'(t) + r(t)x(t) + h(t)) + \alpha_5(t)
\]
and hence from (4.3), we find
\[ x''(t) = \left( \alpha'_1(t) + \alpha_2(t)r(t) + \alpha'_2(t) + \alpha_1(t)\alpha_2(t)q(t) \right)x(t) \]
\[ + \left( \alpha'_3(t) + \alpha_2(t)p(t) + \alpha_1(t)\alpha_2(t) + \alpha'_2(t)q(t) \right)x'(t) \]
\[ + \left( \alpha'_4(t) + \alpha_2(t)h(t) + \alpha_1(t)\alpha_3(t) + \alpha_2(t)\alpha_3(t)q(t) \right). \]

(4.4)

Comparing the coefficients in equation (4.4), we obtain the nonlinear first-order system
\[ \alpha'_1(t) = -\alpha_2(t)r(t) - \alpha'_2(t) - \alpha_1(t)\alpha_2(t)q(t), \]
\[ \alpha'_2(t) = 1 - \alpha_2(t)p(t) - \alpha_1(t)\alpha_2(t) - \alpha'_2(t)q(t), \]
\[ \alpha'_3(t) = -\alpha_2(t)h(t) - \alpha_1(t)\alpha_3(t) - \alpha_2(t)\alpha_3(t)q(t). \]

(4.5)

The system (4.5) can be integrated forward with the initial conditions
\[ \alpha_1(a) = 0, \quad \alpha_2(a) = 0, \quad \alpha_3(a) = A \]

only up to the point \( b \), to obtain \( \alpha_1(b, 4.6), \alpha_2(b, 4.6) \) and \( \alpha_3(b, 4.6) \). The initial conditions (4.6) are so chosen that the solution \( y(t) \) of (4.1) represented by (4.3) satisfy \( y'(a) = A \). Similarly, we integrate (4.5) backward with the terminal conditions
\[ \alpha_1(c) = 0, \quad \alpha_2(c) = 0, \quad \alpha_3(c) = C \]

only up to the point \( b \), to get \( \alpha_1(b, 4.7), \alpha_2(b, 4.7) \) and \( \alpha_3(b, 4.7) \).

From these known values the relation (4.3) provides
\[ x'(b) = \alpha_1(b, 4.6)B + \alpha_2(b, 4.6)x''(b) + \alpha_3(b, 4.6), \]
\[ x'(b) = \alpha_1(b, 4.7)B + \alpha_2(b, 4.7)x''(b) + \alpha_3(b, 4.7). \]

(4.8)

The linear system (4.8) can easily be solved to find the values of \( x'(b) \) and \( x''(b) \). The solution \( x(t) \) of (4.1), (1.2) is now obtained by integrating backward (forward) from the point \( b \) to \( a(c) \) from the known \( x(b) - B \) and the calculated values of \( x'(b) \) and \( x''(b) \). If \( b = a(c) \) then, we need to integrate (4.5) only once with the initial (terminal) condition (4.6) (resp. (4.7)) up to the point \( c(a) \) and the first (second) equation of (4.8) provides \( x''(a) \) (resp. \( x''(c) \)).

5. Solution of nonlinear problems

Here, we shall provide sufficient conditions for the convergence of the iterative scheme
\[ x''_{m+1}(t) = f(t, x_m(t), x'_m(t), x''_m(t)) \]
\[ + \beta(t) \sum_{i=0}^{2} \left( x''_{m+1}(t) - x''_{m+1}(t) \right) \frac{\partial f(t, x_m(t), x'_m(t), x''_m(t))}{\partial x''_m(t)}, \]

(5.1)

\[ x_{m+1}'(a) = A, \quad x_{m+1}(b) = B, \quad x_{m+1}'(c) = C; \quad m = 0, 1, \ldots \]

(5.2)

to the solution \( x(t) \) of (1.1), (1.2). In (5.1), \( \beta(t) \in \mathbb{C}[a, b] \) and \( x_0(t) = P_2(t) \).
Theorem 5.1. Suppose that the function \( f(t, u_0, u_1, u_2) \) is continuously differentiable with respect to \( u_0, u_1, u_2 \) on \([a, c] \times D_3\), where

\[
D_3 = \{ (u_0, u_1, u_2) \colon |u_0 - P_2(t)| \leq \overline{Q}(1 - \theta_\beta)^{-1} |\phi_1(t, b)|,
\]

\[
|u_1 - P_2'(t)| \leq \overline{Q}(1 - \theta_\beta)^{-1} \phi_2(t),
\]

\[
|u_2 - P_2''(t)| \leq \overline{Q}(1 - \theta_\beta)^{-1} \phi_3(t)
\]

and

\[
\overline{Q} = \max_{a \leq t \leq c} |f(t, P_2(t), P_2'(t), P_2''(t))|,
\]

\[
\sup_{[u, v] \times D_3} \left| \frac{\partial f(t, u_0, u_1, u_2)}{\partial u_i} \right| \leq L_i, \quad i = 0, 1, 2,
\]

\[
\theta_\beta = (1 + 2\beta)\theta < 1, \quad \beta = \max_{a \leq t \leq c} |\beta(t)|.
\]

Then,

(i) the sequence \( \{x_m(t)\} \) generated by (5.1), (5.2) remains in \( D_3 \);

(ii) the sequence \( \{x_m(t)\} \) converges to the unique solution \( x(t) \) of the boundary value problem (1.1), (1.2);

(iii) a bound on the error is given by

\[
|x(t) - x_m(t)| \leq \overline{Q} \left( \frac{(1 + \beta)\theta}{1 - \beta \theta} \right)^m (1 - \theta_\beta)^{-1} |\phi_1(t, b)|,
\]

\[
|x'(t) - x'_m(t)| \leq \overline{Q} \left( \frac{(1 + \beta)\theta}{1 - \beta \theta} \right)^m (1 - \theta_\beta)^{-1} \phi_2(t),
\]

\[
|x''(t) - x''_m(t)| \leq \overline{Q} \left( \frac{(1 + \beta)\theta}{1 - \beta \theta} \right)^m (1 - \theta_\beta)^{-1} \phi_3(t).
\]

Proof. Obviously, \( x_0(t) - P_2(t) \in D_3 \). Thus, it suffices to show that \( x_m(t) \in D_3 \) implies that \( x_{m+1}(t) \in D_3 \). For this, we begin with the integral representation of (5.1), (5.2)

\[
x_{m+1}(t) = P_2(t) + \int_a^c g(t, s) \left[ f(s, x_m(s), x'_m(s), x''_m(s))
\right.
\]

\[
+ \beta(s) \sum_{i=0}^2 (x^{(i)}_{m+1}(s) - x^{(i)}_m(s)) \frac{\partial f(s, x_m(s), x'_m(s), x''_m(s))}{\partial x^{(i)}_m(s)} ds
\]

and hence

\[
|x_{m+1}(t) - P_2(t)|
\]

\[
\leq \int_a^c |g(t, s)| \left[ |f(s, x_m(s), x'_m(s), x''_m(s)) - f(s, P_2(s), P_2'(s), P_2''(s))| + |f(s, P_2(s), P_2'(s), P_2''(s))| + \beta \sum_{i=0}^2 |x^{(i)}_{m+1}(s) - x^{(i)}_m(s)| L_i \right] ds
\]
\[ \| x_{m+1} - P_2 \| = \max \left\{ \sup_{a \leq t \leq c} \frac{|x_{m+1}(t) - P_2(t)|}{|\phi_1(t, b)|}, \sup_{a \leq t \leq c} \frac{|x_{m+1}'(t) - P_2'(t)|}{\phi_2(t)}, \sup_{a \leq t \leq c} \frac{|x_{m+1}''(t) - P_2''(t)|}{\phi_3(t)} \right\} \]

is finite. Thus, on using Lemma 2.3 in (5.4), we get

\[ |x_{m+1}(t) - P_2(t)| \leq \overline{Q} \left[ 1 + (1 + \beta)(1 - \theta_\beta)^{-1} \right] |\phi_1(t, b)| + \beta \theta |\phi_1(t, b)| \| x_{m+1} - P_2 \| \]

which is same as

\[ \frac{|x_{m+1}(t) - P_2(t)|}{|\phi_1(t, b)|} \leq \overline{Q}(1 - \beta \theta)(1 - \theta_\beta)^{-1} + \beta \theta \| x_{m+1} - P_2 \|. \] (5.5)

Similar calculations also give

\[ \frac{|x_{m+1}'(t) - P_2'(t)|}{\phi_2(t)} \leq \overline{Q}(1 - \beta \theta)(1 - \theta_\beta)^{-1} + \beta \theta \| x_{m+1} - P_2 \|, \] (5.6)

\[ \frac{|x_{m+1}''(t) - P_2''(t)|}{\phi_3(t)} \leq \overline{Q}(1 - \beta \theta)(1 - \theta_\beta)^{-1} + \beta \theta \| x_{m+1} - P_2 \|. \] (5.7)

Combining (5.5)–(5.7), to obtain

\[ \| x_{m+1} - P_2 \| \leq \overline{Q}(1 - \beta \theta)(1 - \theta_\beta)^{-1} + \beta \theta \| x_{m+1} - P_2 \| \]

which provides

\[ \| x_{m+1} - P_2 \| \leq \overline{Q}(1 - \theta_\beta)^{-1} \]

and hence \( x_{m+1}(t) \subset D_3 \).
Next, from (5.1), (5.2) we have
\[x_{m+1}(t) - x_m(t) = \int_a^c g(t, s) \left[ f(s, x_m(s), x'_m(s), x''_m(s)) - f(s, x_{m-1}(s), x'_{m-1}(s), x''_{m-1}(s)) \right. \]
\[\left. + \beta(s) \sum_{i=0}^{2} \left\{ \left( x^{(i)}_{m+1}(s) - x^{(i)}_m(s) \right) \frac{\partial f(s, x_m(s), x'_m(s), x''_m(s))}{\partial x^{(i)}_m(s)} \right. \right. \]
\[\left. - \left( x^{(i)}_{m}(s) - x^{(i)}_{m-1}(s) \right) \frac{\partial f(s, x_{m-1}(s), x'_{m-1}(s), x''_{m-1}(s))}{\partial x^{(i)}_{m-1}(s)} \right \} ds. \quad (5.8)\]

Thus, from the fact that \( \{ x_m(t) \} \subset D_3 \), we get
\[|x_{m+1}(t) - x_m(t)| \leq \int_a^c g(t, s) \left[ \sum_{i=0}^{2} L_i |x^{(i)}_m(s) - x^{(i)}_{m-1}(s)| \right. \]
\[\left. + \beta \sum_{i=0}^{2} L_i \left( |x^{(i)}_{m+1}(s) - x^{(i)}_m(s)| + |x^{(i)}_m(s) - x^{(i)}_{m-1}(s)| \right) \right] ds \]
\[= \int_a^c g(t, s) \left[ (1 + \beta) \left\{ L_0 \frac{|x'_m(s) - x'_{m-1}(s)|}{\phi_1(s, b)} \left| \phi_1(s, b) \right| \right. \right. \]
\[\left. + L_1 \frac{|x'_m(s) - x'_{m-1}(s)|}{\phi_2(s)} \phi_2(s) + L_2 \frac{|x''_m(s) - x''_{m-1}(s)|}{\phi_3(s)} \phi_3(s) \right) \]
\[\left. + \left( L_0 \frac{|x_{m+1}(s) - x_m(s)|}{\phi_1(s, b)} \left| \phi_1(s, b) \right| \right. \right. \]
\[\left. + L_1 \frac{|x_{m+1}(s) - x_m(s)|}{\phi_2(s)} \phi_2(s) + L_2 \frac{|x''_{m+1}(s) - x''_m(s)|}{\phi_3(s)} \phi_3(s) \right) \]
\[\leq \int_a^c g(t, s) \left[ (1 + \beta) \| x_m - x_{m-1} \| + \beta \| x_{m+1} - x_m \| \right] \]
\[\times (L_0 \phi_1(s, b) + L_1 \phi_2(s) + L_2 \phi_3(s)) ds. \quad (5.9)\]

Using Lemma 2.3 in to (5.9), we find
\[|x_{m+1}(t) - x_m(t)| \leq \left[ (1 + \beta) \| x_m - x_{m-1} \| + \beta \| x_{m+1} - x_m \| \right] \theta |\phi_1(t, b)| \]
which is same as
\[|x_{m+1}(t) - x_m(t)| / \phi_1(t, b) \mid \leq \left[ (1 + \beta) \| x_m - x_{m-1} \| + \beta \| x_{m+1} - x_m \| \right] \theta. \quad (5.10)\]

Following the similar arguments, we also obtain
\[|x'_{m+1}(t) - x'_m(t)| / \phi_2(t) \leq \left[ (1 + \beta) \| x_m - x_{m-1} \| + \beta \| x_{m+1} - x_m \| \right] \theta \quad (5.11)\]
and
\[|x''_{m+1}(t) - x''_m(t)| / \phi_3(t) \leq \left[ (1 + \beta) \| x_m - x_{m-1} \| + \beta \| x_{m+1} - x_m \| \right] \theta. \quad (5.12)\]
Combining (5.10)–(5.12), we have
\[ \| x_{m+1} - x_m \| \leq \left[ (1 + \beta) \| x_m - x_{m-1} \| + \beta \| x_{m+1} - x_m \| \right] \theta \]
which also gives
\[ \| x_{m+1} - x_m \| \leq \frac{(1 + \beta) \theta}{1 - \beta \theta} \| x_m - x_{m-1} \|. \]

Finally, an easy induction gives
\[ \| x_{m+1} - x_m \| \leq \left( \frac{(1 + \beta) \theta}{1 - \beta \theta} \right)^m \| x_1 - x_0 \|. \]  

(5.13)

Since, \( \theta \beta - (1 + 2 \beta \theta) < 1 \) inequality (5.13) implies that the sequence \( \{ x_m(t) \} \) is Cauchy and hence converges to \( x(t) \in D_3 \). That this \( x(t) \) is indeed the unique solution of (1.1), (1.2) can easily be verified.

To find the error bound (5.3), we use (5.13) in the triangular inequality
\[ \| x_{m+p} - x_m \| \leq \| x_{m+p} - x_{m+p-1} \| + \| x_{m+p-1} - x_{m+p-2} \| + \cdots + \| x_{m+1} - x_m \| \]
\[ \leq \left( \frac{(1 + \beta) \theta}{1 - \beta \theta} \right)^m \left( 1 - \frac{(1 + \beta) \theta}{1 - \beta \theta} \right)^{-1} \| x_1 - x_0 \| \]

and taking \( p \to \infty \), to obtain
\[ \| x - x_m \| \leq \left( \frac{(1 + \beta) \theta}{1 - \beta \theta} \right)^m \left( 1 - \frac{(1 + \beta) \theta}{1 - \beta \theta} \right)^{-1} \| x_1 - x_0 \|. \]  

(5.14)

Next, from (5.1), (5.2) we have
\[
| x_1'(t) - x_0'(t) | \leq \int_a^c | g(t, s) | \left[ | f(s, x_0(s), x_0'(s), x_0''(s)) | + \beta \sum_{i=0}^2 L_i | x_1^{(i)}(s) - x_0^{(i)}(s) | \right] \, ds
\]
\[ \leq \int_a^c | g(t, s) | \left[ (\widetilde{Q} + \beta (L_0 | \phi_1(s, b) | + L_1 \phi_2(s) + L_2 \phi_3(s)) \| x_1 - x_0 \| \right] \, ds
\]
\[ \leq (\widetilde{Q} + \beta \theta \| x_1 - x_0 \| ) \| \phi_1(t, b) \|. \]  

Similarly,
\[ | x_1''(t) - x_0''(t) | \leq (\widetilde{Q} + \beta \theta \| x_1 - x_0 \| ) \phi_2(t) \]
and
\[ | x_1'''(t) - x_0'''(t) | \leq (\widetilde{Q} + \beta \theta \| x_1 - x_0 \| ) \phi_3(t). \]

The above inequalities imply that
\[ \| x_1 - x_0 \| \leq \frac{\widetilde{Q}(1 + \beta \theta)^{-1}}{1 - \beta \theta}. \]  

(5.15)
Using (5.15) into (5.14), we get
\[ \| x - x_m \| < \hat{Q} \left( \frac{(1 + \beta)\theta}{1 - \beta\theta} \right)^m (1 - \theta\beta)^{-1} \]
which implies that (5.3) follows from the definition of \( \| x - x_m \| \).

**Theorem 5.2.** Let the conditions of Theorem 5.1 be satisfied and \( \beta(t) = 1 \). Further, let \( f(t, u_0, u_1, u_2) \) be twice continuously differentiable with respect to \( (u_0, u_1, u_2) \) on \([a, c] \times D_3\), and for all \( (t, u_0, u_1, u_2) \in [a, c] \times D_3 \),
\[ \left| \frac{\partial^2}{\partial u_i \partial u_j} f(t, u_0, u_1, u_2) \right| \leq L_i L_j K, \quad 0 \leq i, j \leq 2. \]
Then,
\[ |x_{m+1}(t) - x_m(t)| \leq \frac{1}{\delta} \Delta^2 \phi_1(t, b), \]
\[ |x'_{m+1}(t) - x'_m(t)| \leq \frac{1}{\delta} \Delta^2 \phi_2(t), \quad |x''_{m+1}(t) - x''_m(t)| \leq \frac{1}{\delta} \Delta^2 \phi_3(t) \]
where
\[ \delta = \frac{K\theta}{2(1 - \theta)} \left[ \frac{1}{12} L_0 (c - a)^3 + \frac{1}{6} L_1 (c - a)^2 + \frac{1}{2} L_2 (c - a) \right] \]
and
\[ \Delta = (\hat{Q}/(1 - \theta)) \delta. \]
**Thus. the sequence \( \{x_m(t)\} \) generated by (5.1), (5.2) converges quadratically if \( \Delta < 1 \).**

**Proof.** In Theorem 5.1 we have already proved that \( \{x_m(t)\} \subseteq D_3 \). Since, \( f \) is twice continuously differentiable, we have
\[ f(t, x_m(t), x'_m(t), x''_m(t)) = f(t, x_{m-1}(t), x'_{m-1}(t), x''_{m-1}(t)) \]
\[ + \sum_{i=0}^{2} \left( x^{(i)}_{m}(t) - x^{(i)}_{m-1}(t) \right) \frac{\partial f(t, x_{m-1}(t), x'_{m-1}(t), x''_{m-1}(t))}{\partial x^{(i)}_{m-1}(t)} \]
\[ + \frac{1}{2} \left[ \sum_{i=0}^{2} \left( x^{(i)}_{m}(t) - x^{(i)}_{m-1}(t) \right) \frac{\partial}{\partial p_i(t)} \right]^2 f(t, p_0(t), p_1(t), p_2(t)) \]
where \( p_i(t) \) lies between \( x^{(i)}_{m-1}(t) \) and \( x^{(i)}_{m}(t) \), \( 0 \leq i \leq 2 \).
Using (5.17) into (5.8), we get
\[ x_{m+1}(t) - x_m(t) \]
\[ = \int_a^c g(t, s) \left[ \sum_{i=0}^{2} \left( x^{(i)}_{m+1}(s) - x^{(i)}_m(s) \right) \frac{\partial f(s, x_m(s), x'_m(s), x''_m(s))}{\partial x^{(i)}_m(s)} \right. \]
\[ + \frac{1}{2} \left. \left[ \sum_{i=0}^{2} \left( x^{(i)}_{m}(s) - x^{(i)}_{m-1}(s) \right) \frac{\partial}{\partial p_i(s)} \right]^2 f(s, p_0(s), p_1(s), p_2(s)) \right] ds. \]
Thus, as earlier, we find
\[ |x_{m+1}(t) - x_m(t)| \leq \int_a^b |g(t, s)| \left[ \left( L_0 \| \phi_1(s, b) \| + L_1 \phi_2(s) + L_2 \phi_3(s) \right) \| x_{m+1} - x_m \| + \frac{1}{2} K \left( L_0 \| \phi_1(s, b) \| + L_1 \phi_2(s) + L_2 \phi_3(s) \right)^2 \| x_{m+1} - x_m \|^2 \right] ds \]
\[ \leq \theta \| x_{m+1} - x_m \| \| \phi_1(t, b) \| + \frac{1}{2} K \theta \| x_{m+1} - x_m \|^2 \]
\[ \times \left[ \frac{1}{12} L_0 (c-a)^3 + \frac{1}{3} L_1 (c-a)^2 + \frac{1}{2} L_2 (c-a) \right] \| \phi_1(t, b) \| \]

and hence
\[ |x_{m+1}(t) - x_m(t)| \| \phi_1(t, b) \| \leq \theta \| x_{m+1} - x_m \| + (1 - \theta) \delta \| x_{m+1} - x_m \|^2. \] (5.18)

A similar computation also provides
\[ |x_{m+1}'(t) - x_m'(t)| \| \phi_2(t) \| \leq \theta \| x_{m+1} - x_m \| + (1 - \theta) \delta \| x_{m+1} - x_m \|^2 \] (5.19)

and
\[ |x_{m+1}''(t) - x_m''(t)| \| \phi_3(t) \| \leq \theta \| x_{m+1} - x_m \| + (1 - \theta) \delta \| x_{m+1} - x_m \|^2. \] (5.20)

Combining (5.18)--(5.20), we have
\[ \| x_{m+1} - x_m \| \leq \theta \| x_{m+1} - x_m \| + (1 - \theta) \delta \| x_{m+1} - x_m \|^2, \]
which is same as
\[ \| x_{m+1} - x_m \| \leq \delta \| x_{m+1} - x_m \|^2. \]

Now, an easy induction gives
\[ \| x_{m+1} - x_m \| \leq \frac{1}{\delta} \left[ \delta \| x_1 - x_0 \| \right]^{2^m}. \] (5.21)

Using (5.15) into (5.21) we get
\[ \| x_{m+1} - x_m \| \leq \frac{1}{\delta} \Delta^{2^m}, \]
which is same as (5.16). \[ \square \]

6. Some examples

**Example 6.1.** For the boundary value problem
\[ x''' = x \sinh x + \cosh t, \] (6.1)
\[ x'(0) = 1, \quad x(b) = x'(1) = 0, \quad 0 < b < 1, \] (6.2)
we have \( P_2(t) = \frac{1}{2} (b-t)(b+t-2), \ Q = 2K_0 \sinh 2K_0 + \cosh 1. \) Thus the conditions of Theorem 3.1 are satisfied provided
\[ \frac{1}{2} \leq K_0 \quad \text{and} \quad 1 \leq \left( \frac{12K_0}{2K_0 \sinh 2K_0 + \cosh 1} \right)^{1/3}. \] (6.3)

Both the inequalities (6.3) are satisfied if \( 0.5 < K_0 < 1.189562286. \) Hence there exists at least one
solution of (6.1), (6.2) in the region $S = \{(t, x): 0 \leq t \leq 1, \ |x| \leq K_0, \text{ where } 0.5 \leq K_0 \leq 1.189562286\}$. 

**Example 6.2.** For the differential equation 
\[ x''' = x^a \sin x + e^{-t^2} \cos t, \quad 0 < \alpha < 1, \] 
(6.4) 
together with (1.2), Corollary 3.4 ensures the existence of at least one solution in the region $S = \{(t, x): 0 < t < 1, \ |x| < K_0\}$ as long as $A$, $B$, $C$ and $(c - a)$ are finite.

**Example 6.3.** For the function $f(t, x) = x \sin x + e^{-t^2} \cos t$ the condition (3.8) is satisfied for all $(t, x) \in \mathbb{R}^2$ with $L = 1$ and $L_0 = 1$. Thus, the boundary value problem (6.4), (1.2) with $\alpha = 1$ has a solution $|x(t)| < \infty$ for all $A$, $B$ and $C$ provided $\frac{2}{120}(c - a)^3 < 1$ or $(c - a) < 2.578463976\ldots$. 

**Example 6.4.** Following the considerations of Section 4, we find that the boundary value problem (1.3), (1.4) has a unique solution. Further, for this problem the system (4.5) reduces to 
\[
\begin{align*}
\alpha_1'(t) &= -\alpha_1^2(t) - k^2 \alpha_1(t) \alpha_2(t), \\
\alpha_2'(t) &= 1 - \alpha_1(t) \alpha_2(t) - k^2 \alpha_1^2(t), \\
\alpha_3'(t) &= \alpha_2(t) - \alpha_1(t) \alpha_3(t) - k^2 \alpha_2(t) \alpha_3(t).
\end{align*}
\] 
(6.5) 
We integrate (6.5) forward with the initial conditions 
\[
\alpha_1(0) = \alpha_2(0) = \alpha_3(0) = 0
\] 
(6.6) 
up to $t = \frac{1}{2}$. Next, we integrate (6.5) backward with the terminal conditions 
\[
\alpha_1(1) = \alpha_2(1) = \alpha_3(1) = 0
\] 
(6.7) 
up to $t = \frac{1}{2}$. 

The system (4.8) for the boundary conditions (1.4) reduces to 
\[
x'(\frac{1}{2}) = \alpha_2(\frac{1}{2}, 6.6)x''(\frac{1}{2}) + \alpha_3(\frac{1}{2}, 6.6), \quad x'(\frac{1}{2}) = \alpha_2(\frac{1}{2}, 6.7)x''(\frac{1}{2}) + \alpha_3(\frac{1}{2}, 6.7).
\] 
(6.8) 
Thus, to find $x'(\frac{1}{2})$ and $x''(\frac{1}{2})$ we need to know only $\alpha_2(\frac{1}{2}, 6.6)$, $\alpha_3(\frac{1}{2}, 6.6)$, $\alpha_2(\frac{1}{2}, 6.7)$ and $\alpha_3(\frac{1}{2}, 6.7)$. 

Numerical computation using Runge–Kutta method of order four for several different choices of $k$ and $a$ provides the values of $x'(\frac{1}{2})$ and $x''(\frac{1}{2})$ which agree with the values obtained from the exact solution (1.5). This procedure has advantage over the method given in [18, p. 43–48]. In particular, we need to integrate only one system (6.5) instead of two different differential systems. Moreover, the forward integration is necessary only in the interval $[0, \frac{1}{2}]$ whereas backward in $[\frac{1}{2}, 1]$ compare to forward integrations over $[0, 1]$ and $[\frac{1}{2}, 1]$. 

**Example 6.5.** Consider the differential equation 
\[ x''' = x \sin x + \sin t \] 
(6.9) 
together with (0.2). Since $P_3(t) \equiv 0$, we find that $Q = \max_{0 \leq t \leq 1} |\sin t| < 1$. Further, $\theta = \frac{\beta}{4k} L_0$ and hence if we take $\beta(t) = 1$ then $\theta_\beta = 3\theta = \frac{3}{4k} L_0$. Next, as long as $3\theta < 1$ we have 
\[
D_3 = \left\{ u_0: \ |u_0| \leq (1 - 3\theta)^{-1} |\phi_1(t, \frac{1}{2})| \right\} \subseteq \left\{ u_0: \ |u_0| \leq \frac{5}{3(40 - 7L_0)} \right\} = \overline{D}_3 \quad \text{(say)},
\]
and hence
\[ L_0 = \sup_{u_0 \in \overline{D}_3} (1 + |u_0|) \leq 1 + 5/3(40 - 7L_0). \] (6.10)

The inequality (6.10) subject to \( \frac{\theta}{2} L_0 < 1 \) is satisfied as long as \( 0 \leq L_0 \leq 1.05105803 \ldots \). Thus, from Theorem 5.1 the iterative scheme
\[ x_{m+1}(t) = (x_m(t) \sin x_m(t) + \sin t) \]
\[ + (x_{m+1}(t) - x_m(t))(x_m(t) \cos x_m(t) + \sin x_m(t)), \] (6.11)
\[ x_{m+1}(0) = x_{m+1}(\frac{1}{2}) = x_{m+1}(1) = 0, \quad m = 0, 1, \ldots \]
with \( x_0(t) \equiv 0 \) converges to the unique solution of (6.9), (0.2) in
\[ D_3 = \{ u_0 : |u_0| \leq (1.2253928 \ldots) |\phi_1(t, \frac{1}{2})| \}. \]

Also, from (5.3) the error bound is
\[ |x(t) - x_m(t)| \leq (0.130632755 \ldots)^m (1.2253928 \ldots) |\phi_1(t, \frac{1}{2})|. \]

Next, an easy computation provides that \( K = 1.85662656 \ldots \) and hence
\[ \Delta = \frac{K \theta}{2(1 - \theta)^2} \cdot \frac{1}{24} L_0 = 0.00282885 \ldots < 1. \]

Thus, the conditions of Theorem 5.2 are satisfied and (6.11) converges quadratically to the unique solution of (6.9), (0.2).

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References


(Additional reference)