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Aggregation of Markov chains

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For a collection of Markov chains the aggregated process, that is a process for which the transition rates are a mixture of the transition rates of the Markov chains in the collection, is introduced. A sufficient condition is given, called cross-balance, a generalization of global balance to a collection of processes, under which the equilibrium distribution of the aggregated process is shown to be the same mixture of the equilibrium distributions of the Markov chains in the collection. A number of examples are discussed including a construction method for constructing the equilibrium distribution.

aggregated process * cross-balance * equilibrium distribution * collection of Markov chains

1. Introduction

Over the last decades considerable attention has been paid to the determination of equilibrium distributions of stochastic processes arising from queueing networks. However, most of this work considers product form equilibrium distributions only. The Jackson network (Jackson, 1957) was found to possess a product form solution. Since then, the class of networks that possess a product form equilibrium distribution has been extended considerably. As of today, this class is known to contain BCMP networks (Baskett et al., 1975), networks with blocking (cf. Hordijk and van Dijk, 1983; Serfozo, 1989), networks with batch movements (cf. Henderson et al., 1990; Henderson and Taylor, 1990) and networks with batch movements and blocking (cf. Boucherie and van Dijk, 1990, Boucherie and van Dijk, 1991). Also, a lot of work has been done on understanding why a stochastic process possesses a product form equilibrium distribution. Kelly (1976) introduces the notion of quasi-reversibility, Walrand and Varaiya (1980) connect quasi-reversible queues, Whittle (1984) introduces weak coupling and Pollett (1986) connects reversible Markov processes to give an explanation of the existence of product form equilibrium distributions. At this moment, for a wide class of stochastic processes product form equilibrium distributions are proven to exist, however, the class of stochastic processes with a product form equilibrium distribution is a very restricted class. This paper aims to extend this class to a class with a more general form of equilibrium distribution. In

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particular, we extend the class of stochastic processes with a single underlying transition structure leading to one product form equilibrium distribution to a class of stochastic processes with an equilibrium distribution that is a sum of product forms such as arising from processes which are subject to various underlying transition structures. We consider an amalgamation of stochastic processes each of which has an equilibrium distribution and provide a so-called cross-balance condition such that the equilibrium distribution of this amalgamation is itself an amalgamation of the equilibrium distributions of the underlying stochastic processes.

A well-known result in the theory of stochastic processes is the following. Consider a Markov chain that can move according to K different sets of transition rates and chooses the kth with probability $r^{(k)}$. Then, without any constraints on the transition rates of the process, the equilibrium distribution π is given by

$$\pi = \sum_{k=1}^{K} r^{(k)} \pi^{(k)}, \tag{1.1}$$

where $\pi^{(k)}$ is the equilibrium distribution for the process with the kth set of transition rates. However, this result is valid only if the process selects a set of transition rates at the start and always remains using this set. If the process can, independent of the previous or successive transitions, select upon each transition from a collection of transition rates via which the transition will be made, i.e., if the transition rates q are a mixture of the sets of transition rates

$$q = \sum_{k=1}^{K} r^{(k)} q^{(k)}, \tag{1.2}$$

where $q^{(k)}$ is the kth set of transition rates, then the equilibrium distribution will, in general, not be of the form (1.1) with $\pi^{(k)}$ the equilibrium distribution for the kth set of transition rates. This paper gives a sufficient condition on the transition rates $q^{(k)}$ for the aggregated process, that is the process with transition rates (1.2), to have an equilibrium distribution (1.1). This sufficient condition is cross-balance, a generalization of global balance to a collection of processes. It relates the transition rates $q^{(k)}$ for process k in the collection to the transition rates $q^{(k')}$ for process k' in the collection.

Section 2 presents the model and main result of this paper. Section 3 gives some examples to the theory, in particular, Example 3.6 presents a construction method for constructing the equilibrium distribution for a stochastic process. This construction method is based on cross-balance and divides the state space of the process in possibly overlapping state spaces for the processes in the collection. Finally, Section 4 gives some concluding remarks.

2. Model

Consider a collection of K stable, regular, continuous-time Markov chains, labelled $k = 1, \ldots, K$, at finite or countable state space S. A state of a Markov chain in the collection is denoted by $\bar{n}, \bar{n} \in S$. The transition rate from state $\bar{n}$ to state $\bar{n}'$ for Markov chain k is denoted by $q^{(k)}(\bar{n}, \bar{n}')$, $k = 1, \ldots, K$. The irreducible set $V^{(k)}$ for the Markov chain with transition rates $q^{(k)}$ is the set $V^{(k)} \subseteq S$ at which the Markov chain is irreducible and at which there exists a unique equilibrium distribution $\pi^{(k)}$, $k = 1, \ldots, K$, that is $\pi^{(k)} = \{\pi^{(k)}(\bar{n}) \mid \pi^{(k)}(\bar{n}) > 0, \bar{n} \in V^{(k)}, \pi^{(k)}(\bar{n}) = 0, \bar{n} \notin V^{(k)}\}$ is the unique solution to the global balance equations at S (cf. Kelly, 1979),

$$\sum_{\bar{n}' \neq \bar{n}} \{\pi^{(k)}(\bar{n}) q^{(k)}(\bar{n}, \bar{n}') - \pi^{(k)}(\bar{n}') q^{(k)}(\bar{n}', \bar{n})\} = 0, \quad \bar{n} \in V^{(k)}. \tag{2.1}$$

Note that (2.1) implies that the irreducible set $V^{(k)}$ is a closed set, that is if a solution $\pi^{(k)}$ exists to (2.1) then it must be that $q^{(k)}(\bar{n}, \bar{n}') = 0$ if $\bar{n} \in V^{(k)}$, $\bar{n}' \notin V^{(k)}$, $k = 1, \ldots, K$.

For a collection of Markov chains define the following process.

Definition 2.1 (*Aggregated process*). Consider a collection of Markov chains with transition rates $q^{(k)}$, $k = 1, \ldots, K$. The aggregated process with aggregation coefficients $r^{(k)} \in \mathbb{R}$, the real numbers, $k = 1, \ldots, K$, such that for all $\bar{n}, \bar{n}' \in S$, $\bar{n} \neq \bar{n}'$,

$$\sum_{k=1}^{K} r^{(k)} q^{(k)}(\bar{n}, \bar{n}') \geq 0, \tag{2.2}$$

is the Markov chain at state space S with transition rates q defined as

$$q(\bar{n}, \bar{n}') = \sum_{k=1}^{K} r^{(k)} q^{(k)}(\bar{n}, \bar{n}'), \quad \bar{n}, \bar{n}' \in S. \tag{2.3}$$

Remark 2.2 (*Aggregation coefficients*). Note that in the definition above it is not assumed that $r^{(k)} \geq 0$. Therefore, condition (2.2) is necessary for the transition rates q to be properly defined. If $r^{(k)} \geq 0$ for all k then (2.2) is trivially satisfied.

Remark 2.3 (*Irreducible set*). The irreducible set $V \subseteq S$ of the aggregated process cannot be immediately obtained from the irreducible sets $V^{(k)}$ of the processes in the collection. For example, consider a collection of two Markov chains such that $V^{(1)} \cap V^{(2)} \neq \emptyset$ and $V^{(1)} \neq V^{(2)}$. Let $\bar{n}_0 \in V^{(2)}$ and define a sequence of states $\bar{n}_0, \bar{n}_1, \ldots, \bar{n}_{j-1}, \bar{n}_j$ such that $\bar{n}_j \in V^{(1)}$ and $\bar{n}_i \notin V^{(1)} \cup V^{(2)}$, $i = 1, \ldots, j-1$. If $q^{(1)}(\bar{n}_i, \bar{n}_{i+1}) > 0$, $i = 0, \ldots, j-1$, then the irreducible set of the aggregated process contains the states $\bar{n}_1, \ldots, \bar{n}_{j-1}$ which are not elements of the irreducible sets of the processes in the collection. This implies that, at least in some cases, $V \supset \bigcup_{k=1}^{K} V^{(k)}$. Also, the case where $V \subset \bigcup_{k=1}^{K} V^{(k)}$ is possible. For example, consider the following collection of three Markov chains. Assume that $q^{(1)} = q^{(2)}$, $V^{(1)} = V^{(2)} \supset V^{(3)}$. Then the aggregated process with aggregation coefficients $r^{(1)} = 1$, $r^{(2)} = -1$, $r^{(3)} = 1$ satisfies (2.2) and is given by $q(\bar{n}, \bar{n}') = q^{(3)}(\bar{n}, \bar{n}')$ with irreducible set $V = V^{(3)} \subset V^{(1)} = \bigcup_{k=1}^{3} V^{(k)}$.

If we define $q^{(k)}(\bar{n}, \bar{n}') = 0$ if $\bar{n}$ or $\bar{n}' \notin V^{(k)}$ then the irreducible set of the aggregated process is determined by the irreducible sets of the processes in the collection. However, this seems to be an unnecessary assumption. In the sequel we reconsider the problem of determining V when the notion of cross-balance is introduced. It will be shown that the irreducible set V is a subset of the union of the $V^{(k)}$ if the collection satisfies cross-balance (see Lemma 2.8).

The remaining part of this section relates the irreducible set V and the equilibrium distribution π of the aggregated process to the irreducible sets $V^{(k)}$ and the equilibrium distributions $\pi^{(k)}$ of the Markov chains in the collection. In order to avoid problems with the normalizing constant when deriving this relation we first consider invariant measures rather than equilibrium distributions. As a consequence of the assumption on the existence of $\pi^{(k)}$ there exists an invariant measure at $V^{(k)}$, i.e., a set of non-negative numbers $m^{(k)}=\{m^{(k)}(\bar{n}), \bar{n}\in V^{(k)}\}$ that satisfies the global balance equations (2.1), for the processes in the collection, $k=1,\dots,K$. The following lemma gives a sufficient condition for the aggregated process to have an invariant measure m that is a sum of the invariant measures for the Markov chains in the collection. This sufficient condition will be interpreted in Remark 2.6. The irreducible set V of the aggregated process is not determined in this lemma.

Lemma 2.4. *Consider a collection of K Markov chains with transition rates $q^{(k)}$, irreducible sets $V^{(k)}\subseteq S$ and invariant measures $m^{(k)}$, $k=1,\dots,K$. Then the aggregated process with aggregation coefficients $r^{(k)}$, $k=1,\dots,K$, such that*

$$\sum_{k=1}^{K} r^{(k)}m^{(k)}(\bar{n})\geqslant 0, \quad \bar{n}\in S,$$

has an invariant measure m given by

$$m(\bar{n})=\sum_{k=1}^{K} r^{(k)}m^{(k)}(\bar{n}), \quad \bar{n}\in S, \tag{2.4}$$

if for all k, k', $k, k'=1,\dots,K$, the following relation holds for all $\bar{n}\in S$,

$$\begin{aligned}\sum_{\bar{n}'\neq\bar{n}} &\{m^{(k)}(\bar{n})q^{(k')}(\bar{n},\bar{n}')+m^{(k')}(\bar{n})q^{(k)}(\bar{n},\bar{n}')\}\\ &=\sum_{\bar{n}'\neq\bar{n}}\{m^{(k)}(\bar{n}')q^{(k')}(\bar{n}',\bar{n})+m^{(k')}(\bar{n}')q^{(k)}(\bar{n}',\bar{n})\}.\end{aligned} \tag{2.5}$$

Proof. It is sufficient to prove that m defined in (2.4) satisfies the global balance equations for the aggregated process for all $\bar{n}\in S$,

$$\sum_{\bar{n}'\neq\bar{n}}\{m(\bar{n})q(\bar{n},\bar{n}')-m(\bar{n}')q(\bar{n}',\bar{n})\}=0. \tag{2.6}$$

Substitution of (2.3) and (2.4) into the global balance equations gives

$$\begin{aligned}\sum_{\bar{n}'\neq\bar{n}}&\{m(\bar{n})q(\bar{n},\bar{n}')-m(\bar{n}')q(\bar{n}',\bar{n})\}\\ &=\sum_{\bar{n}'\neq\bar{n}}\sum_{k=1}^{K}\sum_{k'=1}^{K} r^{(k)}r^{(k')}\{m^{(k)}(\bar{n})q^{(k')}(\bar{n},\bar{n}')-m^{(k)}(\bar{n}')q^{(k')}(\bar{n}',\bar{n})\}\\ &=\frac{1}{2}\sum_{k=1}^{K}\sum_{k'=1}^{K} r^{(k)}r^{(k')}\Bigg(\sum_{\bar{n}'\neq\bar{n}}\{m^{(k)}(\bar{n})q^{(k')}(\bar{n},\bar{n}')+m^{(k')}(\bar{n})q^{(k)}(\bar{n},\bar{n}')\}\\ &\qquad\qquad -\sum_{\bar{n}'\neq\bar{n}}\{m^{(k)}(\bar{n}')q^{(k')}(\bar{n}',\bar{n})+m^{(k')}(\bar{n}')q^{(k)}(\bar{n}',\bar{n})\}\Bigg)\\ &=0,\end{aligned}$$

where the second equality is obtained via changing the order of summation and the last equality by using (2.5). □

Remark 2.5 ($K=2$). If $K=2$, Lemma 2.4 gives a necessary and sufficient condition for the existence of an invariant measure of the form (2.4) for the aggregated process. This can easily be seen by substitution of (2.3) and (2.4) in the global equations (2.6):

$$
\begin{aligned}
&(r^{(1)}+r^{(2)}) \sum_{\bar n'\neq \bar n} \{m(\bar n)q(\bar n,\bar n') - m(\bar n')q(\bar n',\bar n)\} \\
&= \sum_{\bar n\neq \bar n} (\{r^{(1)}m^{(1)}(\bar n)+r^{(2)}m^{(2)}(\bar n)\}\{r^{(1)}q^{(1)}(\bar n,\bar n')+r^{(2)}q^{(2)}(\bar n,\bar n')\} \\
&\qquad - \{r^{(1)}m^{(1)}(\bar n')+r^{(2)}m^{(2)}(\bar n')\}\{r^{(1)}q^{(1)}(\bar n',\bar n)+r^{(2)}q^{(2)}(\bar n',\bar n)\}) \\
&= \sum_{\bar n'\neq \bar n} (r^{(1)}r^{(1)}\{m^{(1)}(\bar n)q^{(1)}(\bar n,\bar n') - m^{(1)}(\bar n')q^{(1)}(\bar n',\bar n)\} \\
&\qquad + r^{(1)}r^{(2)}\{m^{(1)}(\bar n)q^{(2)}(\bar n,\bar n') + m^{(2)}(\bar n)q^{(1)}(\bar n,\bar n') \\
&\qquad\qquad - m^{(1)}(\bar n')q^{(2)}(\bar n',\bar n) - m^{(2)}(\bar n')q^{(1)}(\bar n',\bar n)\} \\
&\qquad + r^{(2)}r^{(2)}\{m^{(2)}(\bar n)q^{(2)}(\bar n,\bar n') - m^{(2)}(\bar n')q^{(2)}(\bar n',\bar n)\}) \\
&= r^{(1)}r^{(2)} \sum_{\bar n'\neq \bar n} \{m^{(1)}(\bar n)q^{(2)}(\bar n,\bar n') + m^{(2)}(\bar n)q^{(1)}(\bar n,\bar n') \\
&\qquad\qquad - m^{(1)}(\bar n')q^{(2)}(\bar n',\bar n) - m^{(2)}(\bar n')q^{(1)}(\bar n',\bar n)\},
\end{aligned}
$$

where the last equality is obtained by using global balance for both Markov chains separately. This implies that global balance for the aggregated process is equivalent to (2.5).

Remark 2.6 (*Interpretation of* (2.5)). (2.5) is a technical relation. However, as can be seen from Remark 2.5, there is an obvious way to interpret this relation. When we add the global equations for processes k and k' to (2.5) we obtain after rearranging terms

$$
\begin{aligned}
&\sum_{\bar n'\neq \bar n} \{(m^{(k)}(\bar n)+m^{(k')}(\bar n))(q^{(k)}(\bar n,\bar n')+q^{(k')}(\bar n,\bar n'))\} \\
&= \sum_{\bar n'\neq \bar n} \{(m^{(k)}(\bar n')+m^{(k')}(\bar n'))(q^{(k)}(\bar n',\bar n)+q^{(k')}(\bar n',\bar n))\}.
\end{aligned}
$$

Under the assumptions of Lemma 2.4 this relation is equivalent to (2.5). Therefore, (2.5) expresses that for all k, k', $m^{(k)}+m^{(k')}$ is an invariant measure for the process with transition rates $q^{(k)}+q^{(k')}$.

As can be seen from Remark 2.5, (2.5) gives a general condition for the existence of an invariant measure m for the aggregated process. However, (2.5) seems to be rather a complicated condition to verify. Therefore, in the following definition we give a more practical form of balance, so-called cross-balance, which implies (2.5).

In the definition below we do not make any assumptions on the irreducible set of the processes in the collection. We start afresh with a collection of processes with transition rates $q^{(k)}$, $k=1, \ldots, K$, at state space S. If a collection satisfies cross-balance, the irreducible sets $V^{(k)}$ of the processes are determined and also, as can be seen from Lemma 2.8, the irreducible set V of the aggregated process is determined by the irreducible sets $V^{(k)}$ of the processes in the collection.

Definition 2.7 (*Cross-balance*). Consider a collection of Markov chains with transition rates $q^{(k)}$, $k=1, \ldots, K$. If there exists a collection of measures $m^{(k)}=\{m^{(k)}(\bar{n}), \bar{n} \in S\}$, $k=1, \ldots, K$, such that for all k, k', $k, k'=1, \ldots, K$, and for all $\bar{n} \in S$,

$$\sum_{\bar{n}' \neq \bar{n}} \{m^{(k)}(\bar{n}) q^{(k')}(\bar{n}, \bar{n}') - m^{(k')}(\bar{n}') q^{(k)}(\bar{n}', \bar{n})\} = 0, \tag{2.7}$$

then the collection of Markov chains satisfies cross-balance with measures $m^{(k)}$, $k=1, \ldots, K$.

Note that in the definition above it is not assumed that the measures $m^{(k)}$ are invariant measures for the processes in the collection. However, since (2.7) must hold for all k, k', for $k=k'$ this implies that $m^{(k)}$ is an invariant measure for process k. This implies that the irreducible sets $V^{(k)}$ of the processes in the collection are determined by cross-balance also. Therefore, cross-balance is a generalization of global balance to collections of processes.

Based on the assumptions made on the uniqueness of the equilibrium distributions, the following lemma reduces the irreducible set of the aggregated process to the union of the irreducible sets of the processes in the collection.

Lemma 2.8. *Consider a collection of Markov chains with transition rates $q^{(k)}$, irreducible sets $V^{(k)}$ and unique equilibrium distributions $\pi^{(k)}$ at $V^{(k)}$. Then, if there exists a set of constants $c^{(k)}>0$, $k=1, \ldots, K$, such that the collection satisfies cross-balance with measures $c^{(k)}\pi^{(k)}$, $k=1, \ldots, K$, then the aggregated process cannot have transitions out of the set $V=\bigcup_{k=1}^{K} V^{(k)}$, i.e., for all k it must be the case that if $\bar{n} \in V$ and $\bar{n}' \notin V$ then $q^{(k)}(\bar{n}, \bar{n}')=0$.*

Proof. Let $\bar{n}_0 \in V$, say $\bar{n}_0 \in V^{(k_0)}$, and $\bar{n}_1 \notin V$, then $\pi^{(k_0)}(\bar{n}_0)>0$ and $\pi^{(k)}(\bar{n}_1)=0$ for all k, $k=1, \ldots, K$. Assume that $q(\bar{n}_0, \bar{n}_1)>0$, then for some k, say k_1, we must have that $q^{(k_1)}(\bar{n}_0, \bar{n}_1)>0$. Now consider cross-balance for k_0 and k_1 at $\bar{n}_1$:

$$\begin{aligned}
&\sum_{\bar{n}' \neq \bar{n}_1} \{c^{(k_1)}\pi^{(k_1)}(\bar{n}_1) q^{(k_0)}(\bar{n}_1, \bar{n}') - c^{(k_0)}\pi^{(k_0)}(\bar{n}') q^{(k_1)}(\bar{n}', \bar{n}_1)\} \\
&\quad = -\sum_{\bar{n}' \neq \bar{n}_1} c^{(k_0)}\pi^{(k_0)}(\bar{n}') q^{(k_1)}(\bar{n}', \bar{n}_1) \\
&\quad \leqslant -c^{(k_0)}\pi^{(k_0)}(\bar{n}_0) q^{(k_1)}(\bar{n}_0, \bar{n}_1) < 0,
\end{aligned}$$

which is in contradiction with the assumption that the collection satisfies cross-balance. □

In this paper we consider collections of Markov chains satisfying cross-balance only. Therefore, without loss of generality, we may now assume that *the initial distribution of the aggregated process is such that with probability 1 the process starts at* $V=\bigcup_{k=1}^{K} V^{(k)}$.

The following theorem states the main result of this paper. In this theorem the equilibrium distribution of the aggregated process is related to the equilibrium distributions of the processes in the collection.

Theorem 2.9 (*Main result*). *Consider a collection of Markov chains with transition rates* $q^{(k)}$, *irreducible sets* $V^{(k)}$, *and unique equilibrium distributions* $\pi^{(k)}$ *at* $V^{(k)}$, $k=1,\ldots,K$. *If there exists a set of constants* $c^{(k)}>0$, $k=1,\ldots,K$, *such that the collection satisfies cross-balance with measures* $c^{(k)}\pi^{(k)}$, $k=1,\ldots,K$, *then the aggregated process with aggregation coefficients* $r^{(k)}$ *such that*

$$\sum_{k=1}^{K} r^{(k)}c^{(k)}=C, \quad C>0, \tag{2.8}$$

has an equilibrium distribution π *at irreducible set* $V=\bigcup_{k=1}^{K} V^{(k)}$ *given by*

$$\pi(\bar{n})=\frac{1}{C}\sum_{k=1}^{K} r^{(k)}c^{(k)}\pi^{(k)}(\bar{n}), \quad \bar{n}\in V. \tag{2.9}$$

Proof. By Lemma 2.8, the irreducible set of the aggregated process is given by V as defined in the theorem. It is sufficient to prove that π defined in (2.9) is a probability distribution at V and satisfies the global balance equations (2.6).

Assume that for some $\bar{n}_0$ $\pi(\bar{n}_0)<0$. Then (2.2) and cross-balance imply

$$\begin{aligned}
0&=\sum_{k=1}^{K} r^{(k)}\sum_{\bar{n}'\neq\bar{n}_0}\{c^{(k)}\pi^{(k)}(\bar{n}_0)q^{(k')}(\bar{n}_0,\bar{n}')-c^{(k')}\pi^{(k')}(\bar{n}')q^{(k)}(\bar{n}',\bar{n}_0)\}\\
&=\sum_{\bar{n}'\neq\bar{n}_0}\left\{\sum_{k=1}^{K} r^{(k)}c^{(k)}\pi^{(k)}(\bar{n}_0)q^{(k')}(\bar{n}_0,\bar{n}')-c^{(k')}\pi^{(k')}(\bar{n}')\sum_{k=1}^{K} r^{(k)}q^{(k)}(\bar{n}',\bar{n}_0)\right\}\\
&=\sum_{\bar{n}'\neq\bar{n}_0}\{C\pi(\bar{n}_0)q^{(k')}(\bar{n}_0,\bar{n}')-c^{(k')}\pi^{(k')}(\bar{n}')q(\bar{n}',\bar{n}_0)\}<0.
\end{aligned}$$

Thus, $\pi(\bar{n})\geqslant 0$ for all $\bar{n}\in V$. Summation of π yields

$$\sum_{\bar{n}\in V}\pi(\bar{n})=\sum_{\bar{n}\in V}\frac{1}{C}\sum_{k=1}^{K} r^{(k)}c^{(k)}\pi^{(k)}(\bar{n})=\frac{1}{C}\sum_{k=1}^{K} r^{(k)}c^{(k)}\sum_{\bar{n}\in V}\pi^{(k)}(\bar{n})=1,$$

which implies $0\leqslant\pi(\bar{n})\leqslant 1$ for all $\bar{n}\in V$ and $\pi(V)=1$. Now consider a sequence of mutually exclusive events $E_i\subseteq S$ then, since for all k $\pi^{(k)}$ is a probability distribution

$$\begin{aligned}
\pi\left(\bigcup_{i=1}^{\infty}E_i\right)&=\frac{1}{C}\sum_{k=1}^{K} r^{(k)}c^{(k)}\pi^{(k)}\left(\bigcup_{i=1}^{\infty}E_i\right)=\frac{1}{C}\sum_{k=1}^{K} r^{(k)}c^{(k)}\sum_{i=1}^{\infty}\pi^{(k)}(E_i)\\
&=\sum_{i=1}^{\infty}\frac{1}{C}\sum_{k=1}^{K} r^{(k)}c^{(k)}\pi^{(k)}(E_i)=\sum_{i=1}^{\infty}\pi(E_i).
\end{aligned}$$

Note that (2.7) implies (2.5). Now apply Lemma 2.4 with $m^{(k)}=c^{(k)}\pi^{(k)}$. $\square$

Remark 2.10 (*Interpretation*). Consider a Markov chain that can start with K different sets of transition rates. If with probability $r^{(k)}$ the Markov chain starts with the set of transition rates $q^{(k)}$ with equilibrium distribution $\pi^{(k)}$ then the equilibrium distribution of this Markov chain is given by

$$\pi = \sum_{k=1}^{K} r^{(k)} \pi^{(k)},$$

which is exactly the form we obtain by inserting $c^{(k)} \equiv 1$, $r^{(k)} > 0$ such that $\sum_k r^{(k)} = 1$ into Theorem 2.9. This form is obvious when we select once and for all a process with corresponding transition rates in advance. In contrast, the aggregated process presented here allows to select from a collection of transition rates $q^{(k)}$ at any transition. With probability $r^{(k)}$ it selects transition rate $q^{(k)}$ for a particular transition, independent of the previous or successive transitions. In this case the above form is no longer obvious. For the process to have this form for the equilibrium distribution there will be some restrictions on the transition rates $q^{(k)}$. These conditions are given by cross-balance.

Remark 2.11 (*Aggregation coefficients $r^{(k)}$ and coefficients $c^{(k)}$*). The coefficients $c^{(k)}$ introduced in the main result are not essential for the theory, for example with $c^{(k)} \equiv 1$ Theorem 2.9 remains valid. In the applications, however, these coefficients play a very important role. In many cases a collection of Markov chains satisfies cross-balance for a special choice of the $c^{(k)}$ only (cf. Examples 3.2, 3.3). In some applications the coefficients $c^{(k)}$ will replace the normalizing constant and will be chosen such that at the union of the irreducible sets of the processes in the collection the invariant measures for the processes are the same (cf. Examples 3.5, 3.6).

In the main result above the aggregation coefficients are not necessarily positive, for example see Remark 2.3 and Example 3.6. Note that the aggregation coefficients may be chosen such that $C = 1$. This can, without loss of generality, be obtained by replacing $r^{(k)} := r^{(k)}/C$. Note, however, that (2.9) does not express a mixture of the distributions $\pi^{(k)}$. This would be the case if $r^{(k)} > 0$ for all k, which in the general setting is not necessary.

Remark 2.12 (*Uniqueness of π*). Although the initial condition of the aggregated process is such that with probability 1 the process starts at $\bigcup_{k=1}^{K} V^{(k)}$, the aggregated process is not necessarily irreducible, and thus, the equilibrium distribution of the aggregated process is not necessarily unique. In general, conditions on the processes in the collection and the aggregation coefficients which guarantee that the equilibrium distribution of the aggregated process is unique are hard to give. These conditions will depend on the specific form of the transition rates (cf. Examples 3.1 and 3.2). However, in some cases general conditions are possible. For example, in each of the following two cases it can easily be verified that the equilibrium distribution of the aggregated process is unique.

(1) If $r^{(k)} > 0$ for all k and the irreducible sets are such that

$$V^{(i)} \cap V^{(i+1)} \neq \emptyset, \quad i = 1, \ldots, K-1.$$

(2) If the irreducible sets are such that

$$V^{(i)} \cap V^{(i+1)} \neq \emptyset, \quad i = 1, \ldots, K-1,$$

$$V^{(i)} \backslash \{V^{(i-1)} \cup V^{(i+1)}\} \neq \emptyset, \quad i = 2, \ldots, K-1.$$

3. Examples

This section gives some examples of collections of processes that satisfy cross-balance. The aim of this section is to illustrate some applications, such as the construction method for the equilibrium distribution in Example 3.6, and to give some examples of the implications of cross-balance on the transition rates of the processes in the collection. These examples show that the notion of cross-balance unifies various known special situations and leads to possible new examples. First, in Example 3.1, we consider a standard simple example that can directly be incorporated in the theory. This example combined with Example 3.2 shows that the uniqueness of the equilibrium distribution of the aggregated process depends on the specific form of the transition rates of the processes in the collection. In particular, it depends on the transition rates between the irreducible sets of the processes in the collection. Examples 3.3 and 3.4 consider some well-known processes from the literature. In Example 3.3 we consider the truncated process, and in Example 3.4 we show that a process and its time-reversed process satisfy cross-balance. Example 3.5 gives a novel example. In this example two processes are combined into one aggregated process. The implications on the transition rates of the two processes are worked out in detail as to illustrate the implications of cross-balance. In Example 3.6 the approach is different. Here we start with a process with given transition rates and irreducible set. We construct a collection of processes such that the aggregated process has the same transition rates and irreducible set as the original process and we use this collection of processes to derive the equilibrium distribution of the original process.

3.1. Disjoint irreducible sets; reducible aggregated process

Consider a collection of K Markov chains at state space S with transition rates $q^{(k)}$, irreducible sets $V^{(k)}$ and unique equilibrium distribution $\pi^{(k)}$ at $V^{(k)}$, $k = 1, \ldots, K$. Assume that $V^{(k)} \cap V^{(k')} = \emptyset$ for all $k \neq k'$ and define $V = \bigcup_{k=1}^{K} V^{(k)}$. If the transition rates $q^{(k)}$, $k = 1, \ldots, K$, are such that

$$q^{(k)}(\bar{n}, \bar{n}') = 0 \quad \text{if } \bar{n}' \notin V^{(k)} \text{ or } \bar{n}' \in V^{(k)} \text{ and } \bar{n} \in V \backslash V^{(k)}, \tag{3.1}$$

the collection of Markov chains trivially satisfies cross-balance with measures $\pi^{(k)}$, $k = 1, \ldots, K$. The aggregated process cannot make any transitions between the irreducible sets $V^{(k)}$. Therefore, the aggregated process is reducible and has an equilibrium distribution

$$\pi(\bar{n}) = \sum_{k=1}^{K} r^{(k)}\pi^{(k)}(\bar{n}), \quad \bar{n} \in V, \tag{3.2}$$

for arbitrary coefficients $r^{(k)} > 0$, $\sum_k r^{(k)} = 1$.

The following example is of interest. Consider a closed queueing network consisting of J stations, labelled $j = 1, \ldots, J$. The state space of the Markov chain representing this queueing network is $S = \mathbb{N}_0^J$. Let process k be the process with irreducible set $V^{(k)} = \{\bar{n}: \bar{n} = (n_1, \ldots, n_J), \sum_{j=1}^{J} n_j = k\}$, $k = 1, \ldots, K$, and $q^{(k)}$ be the transition rates of process k and $\pi^{(k)}$ the corresponding equilibrium distribution. This collection satisfies cross-balance with measures $\pi^{(k)}$. The equilibrium distribution of the aggregated process is given by (3.2), where $r^{(k)}$ represents the probability that the queueing network starts with k jobs. An example similar to this example is given in Walrand (1988, p. 6).

3.2. Disjoint irreducible sets; irreducible aggregated process

The essential assumption in the example above is not that the irreducible sets of the processes in the collection are disjoint, but that the aggregated process cannot make any transitions between the irreducible sets of the processes in the collection. For example, consider a collection of 2 Markov chains with transition rates $q^{(1)}$, $q^{(2)}$, irreducible set $V^{(1)}$, $V^{(2)}$ and unique equilibrium distribution $\pi^{(1)}$ at $V^{(1)}$, $\pi^{(2)}$ at $V^{(2)}$. Assume that $V^{(1)} \cap V^{(2)} = \emptyset$. Let the transition rates be as in (3.1) but add for fixed $\bar{n}_1 \in V^{(1)}$, $\bar{n}_2 \in V^{(2)}$,

$$q^{(1)}(\bar{n}_2, \bar{n}_1) = \alpha_1 > 0, \qquad q^{(2)}(\bar{n}_1, \bar{n}_2) = \alpha_2 > 0.$$

Then the collection satisfies cross-balance with measures

$$m^{(1)} = \frac{\alpha_1}{\pi^{(1)}(\bar{n}_1)}\pi^{(1)}, \qquad m^{(2)} = \frac{\alpha_2}{\pi^{(2)}(\bar{n}_2)}\pi^{(2)}.$$

The aggregated process with aggregation coefficients $r^{(1)}, r^{(2)} > 0$ has a unique equilibrium distribution π at $V = V^{(1)} \cup V^{(2)}$ given by

$$\pi(\bar{n}) = \frac{1}{r^{(1)}\alpha_1/\pi^{(1)}(\bar{n}_1) + r^{(2)}\alpha_2/\pi^{(2)}(\bar{n}_2)}\left(r^{(1)}\frac{\alpha_1}{\pi^{(1)}(\bar{n}_1)}\pi^{(1)}(\bar{n}) + r^{(2)}\frac{\alpha_2}{\pi^{(2)}(\bar{n}_2)}\pi^{(2)}(\bar{n})\right).$$

3.3. Truncation

Consider a Markov chain with transition rates $q^{(1)}$, irreducible set $V^{(1)}$ and unique equilibrium distribution $\pi^{(1)}$ at $V^{(1)}$. Assume that there exists a set $V^{(2)} \subset V^{(1)}$ such that for each state in $V^{(2)}$ separately the rate out of $V^{(2)}$ is balanced by the rate into $V^{(2)}$, i.e., for all $\bar{n} \in V^{(2)}$,

$$\sum_{\bar{n}' \in V^{(1)} \setminus V^{(2)}} \{\pi^{(1)}(\bar{n})q^{(1)}(\bar{n}, \bar{n}') - \pi^{(1)}(\bar{n}')q^{(1)}(\bar{n}', \bar{n})\} = 0.$$

Then the truncated process with transition rates $q^{(2)}$ defined as

$$q^{(2)}(\bar{n}, \bar{n}') = \begin{cases} q^{(1)}(\bar{n}, \bar{n}'), & \text{if } \bar{n}' \in V^{(2)}, \\ 0, & \text{otherwise,} \end{cases}$$

has an equuilbrium distribution $\pi^{(2)}$ at $V^{(2)}$ defined by

$$\pi^{(2)}(\bar{n}) = \frac{\pi^{(1)}(\bar{n})}{\sum_{\bar{n} \in V^{(2)}} \pi^{(1)}(\bar{n})}.$$

The collection $q^{(1)}, q^{(2)}$ satisfies cross-balance with measures

$$m^{(1)} = \pi^{(1)}, \qquad m^{(2)} = \pi^{(2)} \sum_{\bar{n} \in V^{(2)}} \pi^{(1)}(\bar{n}).$$

3.4. Time reversal

Consider a Markov chain with transition rates $q^{(1)}$, irreducible set V and unique equilibrium distribution $\pi^{(1)}$. In reversed time, the Markov chain has the same equilibrium distribution $\pi^{(2)} = \pi^{(1)}$ at irreducible set V. The transition rates $q^{(2)}$ of the time-reversed process are defined as the set of numbers that satisfies (cf. Kelly, 1979)

$$\pi^{(1)}(\bar{n})q^{(1)}(\bar{n}, \bar{n}') = \pi^{(1)}(\bar{n}')q^{(2)}(\bar{n}', \bar{n}). \tag{3.3}$$

By summation of (3.3) one directly verifies that the collection of a process and its time-reversed process satisfies cross-balance with measures $m^{(1)} = \pi^{(1)}$, $m^{(2)} = \pi^{(2)}$ and the aggregated process has equilibrium distribution $\pi = \pi^{(1)}$.

An intuitive interpretation of the aggregated process with aggregation coefficients $r^{(1)}, r^{(2)}$ is the following. For process $q^{(1)}$ the time passes by at rate 1, therefore for the process with transition rates $r^{(1)}q^{(1)}$ time passes at rate $r^{(1)}$. For the time-reversed process with rates $q^{(2)}$ time passes at rate -1, therefore for the process with rates $r^{(2)}q^{(2)}$ time passes at rate $-r^{(2)}$. For the aggregated process time passes at rate $r^{(1)} - r^{(2)}$, but since the process is stationary the speed at which time passes does not play a role in determining the equilibrium distribution. Therefore, the process for which time passes at rate $r^{(1)} - r^{(2)}$ has the same equilibrium distribution as the process for which time passes at rate 1.

3.5. Nearly disjoint irreducible sets

In this example we consider a collection of 2 processes. We will modify the transition rates of these processes such that the collection satisfies cross-balance under the restriction that the equilibrium distributions of the processes in the collection remain unchanged. To this end, note that for a process with transition rates $q^{(k)}$, irreducible set $V^{(k)}$ and equilibrium distribution $\pi^{(k)}$ at $V^{(k)}$ the transition rates $q^{(k)}(\bar{n}, \bar{n}')$ for $\bar{n} \notin V^{(k)}$ can be arbitrarily changed without affecting the equilibrium distribution $\pi^{(k)}$.

Consider a collection of 2 processes derived from a queueing network, that is, the transitions allowed for the processes are those allowed in a queueing network only, i.e., a job is allowed to enter the system at station i corresponding to a transition from state $\bar{n}$ to state $\bar{n}+e_i$, a job is allowed to leave station i and route to station j corresponding to a transition from state $\bar{n}$ to state $\bar{n}-e_i+e_j$ and a job is allowed to leave the system from station i corresponding to a transition from state $\bar{n}$ to state $\bar{n}-e_i$. The rates at which jobs enter or leave the stations is given by $\phi^{(k)}$ for process k, $k=1,2$. The transition rates $q^{(k)}$ for the processes are then given by

$$\begin{aligned} q^{(1)}(\bar{n},\bar{n}') &= \phi^{(1)}(\bar{n},\bar{n}')1(0 \le n_i' \le J_i^{(1)}), \quad \text{if } \bar{n}'=\bar{n}+e_i,\ \bar{n}-e_i,\ \bar{n}-e_i+e_j, \\ q^{(2)}(\bar{n},\bar{n}') &= \phi^{(2)}(\bar{n},\bar{n}')1(J_i^{(1)} \le n_i' \le J_i^{(2)}), \quad \text{if } \bar{n}'=\bar{n}+e_i,\ \bar{n}-e_i,\ \bar{n}-e_i+e_j. \end{aligned} \tag{3.4}$$

Then the irreducible sets of the processes in the collection are given by

$$V^{(1)} = \{\bar{n}: 0 \le n_i \le J_i^{(1)},\ i=1,\ldots,N\},$$

$$V^{(2)} = \{\bar{n}: J_i^{(1)} \le n_i \le J_i^{(2)},\ i=1,\ldots,N\}.$$

Assume that equilibrium distributions $\pi^{(1)}$ and $\pi^{(2)}$ exist, i.e., $\pi^{(1)}$, $\pi^{(2)}$ satisfy

$$\sum_{\bar{n}'\neq\bar{n},\bar{n}'\in V^{(k)}} \{\pi^{(k)}(\bar{n})\phi^{(k)}(\bar{n},\bar{n}') - \pi^{(k)}(\bar{n}')\phi^{(k)}(\bar{n}',\bar{n})\} = 0, \quad k=1,2. \tag{3.5}$$

Then, for some arbitrary coefficients $c^{(1)}$, $c^{(2)}$, the collection satisfies cross-balance with measures $m^{(k)}=c^{(k)}\pi^{(k)}$ if and only if the transition rates $q^{(k)}(\bar{n},\bar{n}')$ for $\bar{n}\notin V^{(k)}$ are defined as

$$q^{(1)}(\bar{n}_0+e_i,\bar{n}_0) = \frac{c^{(1)}\pi^{(1)}(\bar{n}_0)}{c^{(2)}\pi^{(2)}(\bar{n}_0+e_i)}\, q^{(2)}(\bar{n}_0,\bar{n}_0+e_i), \quad i=1,\ldots,N, \tag{3.6a}$$

$$q^{(1)}(\bar{n},\bar{n}')=0, \quad \text{if } \bar{n}'\in V^{(2)}\setminus V^{(1)}, \tag{3.6b}$$

$$q^{(2)}(\bar{n}_0-e_i,\bar{n}_0) = \frac{c^{(2)}\pi^{(2)}(\bar{n}_0)}{c^{(1)}\pi^{(1)}(\bar{n}_0-e_i)}\, q^{(1)}(\bar{n}_0,\bar{n}_0-e_i), \quad i=1,\ldots,N, \tag{3.6c}$$

$$q^{(2)}(\bar{n},\bar{n}')=0, \quad \text{if } \bar{n}'\in V^{(1)}\setminus V^{(2)}, \tag{3.6d}$$

where e_i denotes the ith unit vector, i.e., the vector with ith entry 1 and all other entries 0 and $\bar{n}_0=(J_1^{(1)},\ldots,J_N^{(1)})$. Note that (3.6a) and (3.6c) are well-defined since the equilibrium distributions $\pi^{(k)}$ are known. (3.6a), (3.6b) determine the transition rates of process 1 at $V^{(2)}$ and (3.6c), (3.6d) determine the transition rates of process 2 at $V^{(1)}$.

We will now show that the collection of processes with transition rates defined in (3.4) satisfies cross-balance with measures $m^{(k)}=c^{(k)}\pi^{(k)}$, $k=1,2$, if and only if the transition rates are modified as given in (3.6a)-(3.6d). To this end, first note that we have to check (2.7) for $k\neq k'$ only, since for $k=k'$ (2.7) is already given by (3.5). If $k=1$, $k'=2$ and $\bar{n}\in V^{(2)}\setminus V^{(1)}$ we have that $m^{(1)}(\bar{n})=0$ and (2.7) reduces to

$$\sum_{\bar{n}'\neq\bar{n}} m^{(2)}(\bar{n}')q^{(1)}(\bar{n}',\bar{n})=0.$$

Since $m^{(2)}(\bar{n}')>0$ for all $\bar{n}'\in V^{(2)}$ this relation can hold if and only if (3.6b) holds. For $k=2$, $k'=1$ we find that cross-balance can hold for $\bar{n}\in V^{(1)}\setminus V^{(2)}$ if and only if (3.6d) holds. If $k=1$, $k'=2$ and $\bar{n}\in V^{(1)}\setminus V^{(2)}$ we obtain by using (3.6d),

$$\begin{aligned}
&\sum_{\bar{n}'\neq\bar{n}}\{m^{(1)}(\bar{n})q^{(2)}(\bar{n},\bar{n}')-m^{(2)}(\bar{n}')q^{(1)}(\bar{n}',\bar{n})\}\\
&\quad=\sum_{\bar{n}'\in V^{(2)}}\{m^{(1)}(\bar{n})q^{(2)}(\bar{n},\bar{n}')-m^{(2)}(\bar{n}')q^{(1)}(\bar{n}',\bar{n})\}\\
&\quad=\begin{cases} m^{(1)}(\bar{n})q^{(2)}(\bar{n},\bar{n}_0)-m^{(2)}(\bar{n}_0)q^{(1)}(\bar{n}_0,\bar{n}), & \text{if } \bar{n}=\bar{n}_0-e_i,\\ 0, & \text{otherwise,}\end{cases}
\end{aligned}$$

where the last equality is obtained by observing that the process can make transitions allowed in (3.4) only. This implies that cross-balance can hold if and only if (3.6c) holds. The argument for (3.6a) can be given in a similar way. It remains to check that with the transition rates defined in (3.6a)–(3.6d) the collection satisfies cross-balance for $\bar{n}=\bar{n}_0$. For $k=1$, $k'=2$ and $\bar{n}=\bar{n}_0$ we obtain

$$\begin{aligned}
&\sum_{\bar{n}'\neq\bar{n}_0}\{m^{(1)}(\bar{n}_0)q^{(2)}(\bar{n}_0,\bar{n}')-m^{(2)}(\bar{n}')q^{(1)}(\bar{n}',\bar{n}_0)\}\\
&\quad=\sum_{\bar{n}'\in V^{(2)}}\{m^{(1)}(\bar{n}_0)q^{(2)}(\bar{n}_0,\bar{n}')-m^{(2)}(\bar{n}')q^{(1)}(\bar{n}',\bar{n}_0)\}\\
&\quad=\sum_{i=1}^{N}\{m^{(1)}(\bar{n}_0)q^{(2)}(\bar{n}_0,\bar{n}_0+e_i)-m^{(2)}(\bar{n}_0+e_i)q^{(1)}(\bar{n}_0+e_i,\bar{n}_0)\}=0,
\end{aligned}$$

where the last equality is obtained by using (3.6a). For $k=2$, $k'=1$ and $\bar{n}=\bar{n}_0$ we find that (2.7) holds from (3.6c).

The aggregated process with aggregation coefficients $r^{(1)}=r$, $r^{(2)}=1-r$, $0\leq r\leq 1$ has a unique equilibrium distribution at $V=V^{(1)}\cup V^{(2)}$ given by

$$\pi(\bar{n})=r\pi^{(1)}(\bar{n})+(1-r)\pi^{(2)}(\bar{n}),\quad \bar{n}\in V^{(1)}\cup V^{(2)}.$$

Remark 3.1 (*Discussion*). In this example, the irreducible sets $V^{(1)}$, $V^{(2)}$ intersect in exactly one point. This is crucial for the simple analysis presented above. For example, there are no restrictions on the transition $q^{(k)}$ at $V^{(k)}$, which, in general, will be the case. However, this example can be generalized to irreducible sets that intersect in several points. The analyses will become more complex and also there will be restrictions on the transition rates of the processes in the collection. However, this example does reflect some of the key features of a collection that satisfies cross-balance:

- For $\bar{n}'\notin V^{(k)}$ cross-balance implies that $q^{(k)}(\bar{n},\bar{n}')=0$ for all $\bar{n}$.
- Relation (2.7) in the definition of cross-balance may be replaced by: For all $\bar{n}\in V^{(k)}$,

$$\sum_{\bar{n}'\neq\bar{n},\bar{n}'\in V^{(k')}}\{m^{(k)}(\bar{n})q^{(k')}(\bar{n},\bar{n}')-m^{(k')}(\bar{n}')q^{(k)}(\bar{n}',\bar{n})\}=0.$$

To illustrate the implications of (3.6a) and (3.6c) on the transition rates of the aggregated process, consider the following explicit example, where each process is a queueing network consisting of single-server queues with Poisson arrivals and state-independent routing. Let $J_i^{(2)} = \infty$, $i = 1, \ldots, N$, and

$$\phi(\bar{n}, \bar{n}') = \begin{cases} \lambda p_{0i}, & \text{if } \bar{n}' = \bar{n} + e_i, \\ \mu_i p_{ij}, & \text{if } \bar{n}' = \bar{n} - e_i + e_j, \\ \mu_i p_{i0}, & \text{if } \bar{n}' = \bar{n} - e_i. \end{cases}$$

Furthermore, define the transition rates for process 1 and 2 as

$$\phi^{(2)}(\bar{n}, \bar{n}') = \phi(\bar{n}, \bar{n}'),$$

$$\phi^{(1)}(\bar{n}, \bar{n}') = \phi(\bar{n}, \bar{n}'), \quad \text{if } n_i < J_i^{(1)}, \ i = 1, \ldots, N,$$

$$\phi^{(1)}(\bar{n}, \bar{n} - e_i + e_j) = \phi(\bar{n}, \bar{n} - e_i + e_j), \quad \text{if } n_i = J_i^{(1)}, \ n_j < J_j^{(1)},$$

$$\phi^{(1)}(\bar{n}, \bar{n} + e_j) = \lambda p_{0j}^{(1)}, \quad \text{if } n_i = J_i^{(1)}, \ n_j < J_j^{(1)},$$

$$\phi^{(1)}(\bar{n} + e_j, \bar{n}) = \mu_j p_{j0}^{(1)}, \quad \text{if } n_i = J_i^{(1)}, \ n_j < J_j^{(1)},$$

where $p^{(1)}$ will be chosen such that process 1 is reversible at the boundary. With $\{\gamma_i\}_{i=1}^N$ the solution of the traffic equations

$$\gamma_i = \lambda p_{0i} + \sum_{j=1}^{N} \gamma_j p_{ji}, \quad i = 1, \ldots, N,$$

and $p_{j0}^{(1)}, p_{0j}^{(1)}$ such that $\lambda p_{0j}^{(1)} = \mu_j p_{j0}^{(1)}$, $j = 1, \ldots, N$, both process 1 and 2 have a unique product form equilibrium distribution

$$\pi^{(k)} = \frac{1}{c^{(k)}} m,$$

where m is given by

$$m(\bar{n}) = \prod_{i=1}^{N} \left(\frac{\gamma_i}{\mu_i}\right)^{n_i},$$

and $1/c^{(k)}$ is the normalizing constant for process k. (3.6a), (3.6c) give the following relations for the transition rates.

$$q^{(1)}(\bar{n}_0 + e_i, \bar{n}_0) = \mu_i \frac{\lambda}{\gamma_i} p_{0i}, \tag{3.7a}$$

$$q^{(2)}(\bar{n}_0 - e_i, \bar{n}_0) = \lambda \frac{\gamma_i}{\lambda} p_{i0}^{(1)}. \tag{3.7b}$$

If the transition rates satisfy these equations then the equilibrium distribution of the aggregated process with aggregation coefficients $r^{(1)}$, $r^{(2)}$ is given by

$$\pi(\bar{n})=\begin{cases}\dfrac{r^{(1)}}{r^{(1)}c^{(1)}+r^{(2)}c^{(2)}}\displaystyle\prod_{i=1}^{N}\left(\frac{\gamma_i}{\mu_i}\right)^{n_i}, & \text{if } \bar{n}\in V^{(1)}\setminus V^{(2)},\\[2ex] \dfrac{r^{(1)}+r^{(2)}}{r^{(1)}c^{(1)}+r^{(2)}c^{(2)}}\displaystyle\prod_{i=1}^{N}\left(\frac{\gamma_i}{\mu_i}\right)^{n_i}, & \text{if } \bar{n}\in V^{(1)}\cap V^{(2)},\\[2ex] \dfrac{r^{(2)}}{r^{(1)}c^{(1)}+r^{(2)}c^{(2)}}\displaystyle\prod_{i=1}^{N}\left(\frac{\gamma_i}{\mu_i}\right)^{n_i}, & \text{if } \bar{n}\in V^{(2)}\setminus V^{(1)}.\end{cases} \tag{3.8}$$

Interpretation 3.2. Note that, although the equilibrium distributions of the processes in the collection are of product form, the equilibrium distribution of the aggregated process is not of product form since the 'normalizing constants' are not the same for all states.

The transition rates given in (3.7a), (3.7b) seem to have a strange form. However, they can be rewritten as

$$q^{(1)}(\bar{n}_0+e_i,\bar{n}_0)=\mu_i p^*_{i0},$$
$$q^{(2)}(\bar{n}_0-e_i,\bar{n}_0)=\lambda p^{(1)}_{0i},$$

where p^* are the transition probabilities of the time-reversed process for process 2. Thus, (3.7a) represents a departure from the network and (3.7b) represents an arrival to the network. The transition rates of the aggregated process are given by

$$q(\bar{n},\bar{n}')=\begin{cases}r^{(1)}\phi(\bar{n},\bar{n}'), & \text{if } \bar{n}'\in V^{(1)}\setminus V^{(2)},\\ \mu_i(r^{(1)}p^*_{i0}+r^{(2)}p_{i0}), & \text{if } \bar{n}=\bar{n}_0+e_i,\ \bar{n}'=\bar{n}_0,\\ \lambda(r^{(1)}p^{(1)}_{0i}+r^{(2)}p^{(1)}_{0i}), & \text{if } \bar{n}=\bar{n}_0-e_i,\ \bar{n}'=\bar{n}_0,\\ r^{(2)}\phi(\bar{n},\bar{n}'), & \text{if } \bar{n}'\in V^{(2)}\setminus V^{(1)}.\end{cases}$$

The probability of leaving the system from state $\bar{n}_0+e_i$ is changed and also the probability of entering the system to state $\bar{n}_0$ is changed. The form of the transition rates of the aggregated process, however, is exactly the same as the form of the transition rates of the processes in the collection.

3.6. Construction method

In the previous example, the irreducible sets $V^{(1)}$ and $V^{(2)}$ intersect in exactly one point. In that case, we were able to construct transition rates $q^{(1)}$ at $V^{(2)}$ and $q^{(2)}$ at $V^{(1)}$ such that the collection satisfies cross-balance. In the case of identical invariant measures for both processes in the collection one would expect that the aggregated process allows the same invariant measure. However, as can be seen from (3.8) for the special case of product form invariant measures, this is not true. In this example, we extend the previous example to state spaces that intersect in several points. Also, in the case of identical invariant measures for the processes in the collection, we will show that it is possible to construct an aggregated process such that the aggregated process allows the same invariant measure. Moreover, this example shows that for a given process, we can construct a collection of processes such that the aggregated process has the same transition rates as the original process. This implies that the equilibrium distribution for the original process is given by the equilibrium distribution of the aggregated process. Thus, this section gives a construction method for constructing the equilibrium distribution for a process via the equilibrium distributions for the processes in the collection. For simplicity, we restrict our attention to a Markov chain representing a two station queueing network. This example can be generalized to queueing networks with N queues ($N \geq 1$).

Consider a Markov chain representing a two station queueing network with transition rates

$$q(\bar{n}, \bar{n}') = \begin{cases} \phi(\bar{n}, \bar{n}'), & \text{if } \bar{n}, \bar{n}' \in V, \\ & \quad \text{and } \bar{n}' = \bar{n} + e_i,\ \bar{n}' = \bar{n} - e_i,\ \bar{n}' = \bar{n} - e_i + e_j, \\ 0, & \text{otherwise}, \end{cases} \tag{3.9}$$

where V is given by

$$V = \{\bar{n}: 0 \leq n_i \leq J_i\} \cup \{J_i - 1 \leq n_i\}.$$

We will now construct a collection of queueing networks that satisfies cross-balance such that the aggregated process has transition rates q as defined above and give the equilibrium distribution for the aggregated process and thus for the process with transition rates q explicitly.

First, consider the following collection of 2 queueing networks with transition rates $q^{(k)}$ of network k defined by

$$q^{(1)}(\bar{n}, \bar{n}') = q(\bar{n}, \bar{n}')\mathbf{1}(0 \leq n_i, n_i' \leq J_i),$$

$$q^{(2)}(\bar{n}, \bar{n}') = q(\bar{n}, \bar{n}')\mathbf{1}(J_i - 1 \leq n_i, n_i').$$

Then the irreducible sets of the networks in the collection are given by

$$V^{(1)} = \{\bar{n}: 0 \leq n_i \leq J_i\}, \qquad V^{(2)} = \{\bar{n}: J_i - 1 \leq n_i\}.$$

Assume that there exist invariant measures for these processes, i.e., some sets of non-negative numbers $m^{(1)}$, $m^{(2)}$ that satisfy

$$\sum_{\bar{n}' \neq \bar{n}, \bar{n}' \in V^{(k)}} \{m^{(k)}(\bar{n}) q^{(k)}(\bar{n}, \bar{n}') - m^{(k)}(\bar{n}') q^{(k)}(\bar{n}', \bar{n})\} = 0, \quad k = 1, 2.$$

As can be seen from Figure 1, the state spaces $V^{(1)}$ and $V^{(2)}$ intersect in exactly four points. In order to guarantee that the collection satisfies cross-balance we have to assume that $q^{(1)}, q^{(2)}$ satisfy the following relations at the intersection of the irreducible sets. In these relations the states are labelled as depicted in Figure 1, where, for example $\bar{n}_3 = (J_1 - 1, J_2)$, $\bar{n}_4 = (J_1, J_2)$, $\bar{n}_7 = (J_1 - 1, J_2 - 1)$, $\bar{n}_8 = (J_1, J_2 - 1)$. The first relation (3.10a) represents cross-balance for state $\bar{n}_4$, the second relation

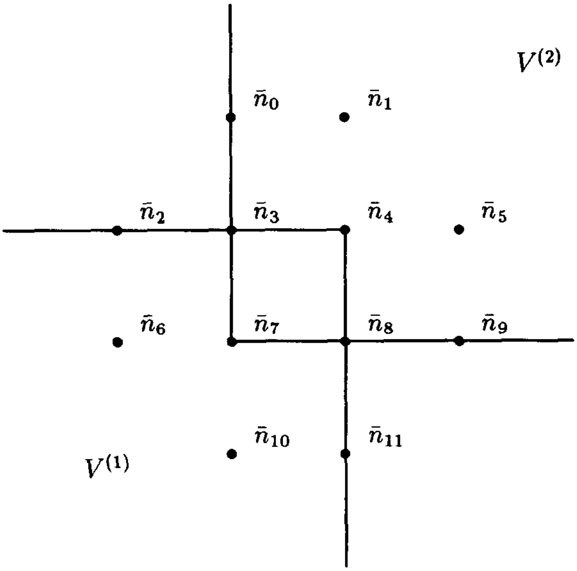

Fig. 1. State spaces and labelling of states.

(3.10b) represents cross-balance for state $\bar{n}_7$ and the third relation (3.10c) expresses that the total flow in the box consisting of $\bar{n}_3, \bar{n}_4, \bar{n}_7, \bar{n}_8$ is balanced.

$$m^{(2)}(\bar{n}_4)\{q^{(1)}(\bar{n}_4, \bar{n}_3) + q^{(1)}(\bar{n}_4, \bar{n}_8)\}$$
$$= m^{(1)}(\bar{n}_3)q^{(2)}(\bar{n}_3, \bar{n}_4) + m^{(1)}(\bar{n}_8)q^{(2)}(\bar{n}_8, \bar{n}_4), \tag{3.10a}$$
$$m^{(1)}(\bar{n}_7)\{q^{(2)}(\bar{n}_7, \bar{n}_3) + q^{(2)}(\bar{n}_7, \bar{n}_8)\}$$
$$= m^{(2)}(\bar{n}_3)q^{(1)}(\bar{n}_3, \bar{n}_7) + m^{(2)}(\bar{n}_8)q^{(1)}(\bar{n}_8, \bar{n}_7), \tag{3.10b}$$
$$\begin{aligned} & m^{(1)}(\bar{n}_3)\{q^{(2)}(\bar{n}_3, \bar{n}_4) + q^{(2)}(\bar{n}_3, \bar{n}_7) + q^{(2)}(\bar{n}_3, \bar{n}_8)\} \\ &\quad + m^{(1)}(\bar{n}_4)\{q^{(2)}(\bar{n}_4, \bar{n}_3) + q^{(2)}(\bar{n}_4, \bar{n}_8)\} \\ &\quad + m^{(1)}(\bar{n}_7)\{q^{(2)}(\bar{n}_7, \bar{n}_3) + q^{(2)}(\bar{n}_7, \bar{n}_8)\} \\ &\quad + m^{(1)}(\bar{n}_8)\{q^{(2)}(\bar{n}_8, \bar{n}_3) + q^{(2)}(\bar{n}_8, \bar{n}_4) + q^{(2)}(\bar{n}_8, \bar{n}_7)\} \\ &= m^{(2)}(\bar{n}_3)\{q^{(1)}(\bar{n}_3, \bar{n}_4) + q^{(1)}(\bar{n}_3, \bar{n}_7) + q^{(1)}(\bar{n}_3, \bar{n}_8)\} \\ &\quad + m^{(2)}(\bar{n}_4)\{q^{(1)}(\bar{n}_4, \bar{n}_3) + q^{(1)}(\bar{n}_4, \bar{n}_8)\} \\ &\quad + m^{(2)}(\bar{n}_7)\{q^{(1)}(\bar{n}_7, \bar{n}_3) + q^{(1)}(\bar{n}_7, \bar{n}_8)\} \\ &\quad + m^{(2)}(\bar{n}_8)\{q^{(1)}(\bar{n}_8, \bar{n}_3) + q^{(1)}(\bar{n}_8, \bar{n}_4) + q^{(1)}(\bar{n}_8, \bar{n}_7)\}. \end{aligned} \tag{3.10c}$$

Furthermore, as in Example 3.5, we have to define the transition rates $q^{(1)}$ at $V^{(2)}$ and $q^{(2)}$ at $V^{(1)}$ such that the collection satisfies cross-balance. Note that this does not affect the invariant measures $m^{(k)}$. To this end, by analogy with (3.6a) and (3.6c), define the following transition rates.

$$q^{(1)}(\bar{n}_0,\bar{n}_3)=\frac{m^{(1)}(\bar{n}_3)\{q^{(2)}(\bar{n}_3,\bar{n}_0)+q^{(2)}(\bar{n}_3,\bar{n}_4)+q^{(2)}(\bar{n}_3,\bar{n}_7)+q^{(2)}(\bar{n}_3,\bar{n}_8)\}}{m^{(2)}(\bar{n}_0)}$$

$$-\frac{\{m^{(2)}(\bar{n}_4)q^{(1)}(\bar{n}_4,\bar{n}_3)+m^{(2)}(\bar{n}_7)q^{(1)}(\bar{n}_7,\bar{n}_3)+m^{(2)}(\bar{n}_8)q^{(1)}(\bar{n}_8,\bar{n}_3)\}}{m^{(2)}(\bar{n}_0)},$$

$$q^{(1)}(\bar{n}_0,\bar{n}_4)=\frac{m^{(1)}(\bar{n}_3)q^{(2)}(\bar{n}_3,\bar{n}_0)+m^{(1)}(\bar{n}_4)q^{(2)}(\bar{n}_4,\bar{n}_0)}{m^{(2)}(\bar{n}_0)}-q^{(1)}(\bar{n}_0,\bar{n}_3),$$

$$q^{(1)}(\bar{n}_1,\bar{n}_4)=\frac{m^{(1)}(\bar{n}_4)q^{(2)}(\bar{n}_4,\bar{n}_1)}{m^{(2)}(\bar{n}_1)},$$

$$q^{(1)}(\bar{n}_5,\bar{n}_4)=\frac{m^{(1)}(\bar{n}_4)q^{(2)}(\bar{n}_4,\bar{n}_5)}{m^{(2)}(\bar{n}_5)},$$

$$q^{(1)}(\bar{n}_9,\bar{n}_8)=\frac{m^{(1)}(\bar{n}_8)\{q^{(2)}(\bar{n}_8,\bar{n}_3)+q^{(2)}(\bar{n}_8,\bar{n}_4)+q^{(2)}(\bar{n}_8,\bar{n}_7)+q^{(2)}(\bar{n}_8,\bar{n}_9)\}}{m^{(2)}(\bar{n}_9)}$$

$$-\frac{\{m^{(2)}(\bar{n}_3)q^{(1)}(\bar{n}_3,\bar{n}_8)+m^{(2)}(\bar{n}_4)q^{(1)}(\bar{n}_4,\bar{n}_8)+m^{(2)}(\bar{n}_7)q^{(1)}(\bar{n}_7,\bar{n}_8)\}}{m^{(2)}(\bar{n}_9)},$$

$$q^{(1)}(\bar{n}_9,\bar{n}_4)=\frac{m^{(1)}(\bar{n}_4)q^{(2)}(\bar{n}_4,\bar{n}_9)+m^{(1)}(\bar{n}_8)q^{(2)}(\bar{n}_8,\bar{n}_9)}{m^{(2)}(\bar{n}_9)}-q^{(1)}(\bar{n}_9,\bar{n}_8),$$

$$q^{(2)}(\bar{n}_2,\bar{n}_3)=\frac{m^{(2)}(\bar{n}_3)\{q^{(1)}(\bar{n}_3,\bar{n}_2)+q^{(1)}(\bar{n}_3,\bar{n}_4)+q^{(1)}(\bar{n}_3,\bar{n}_7)+q^{(1)}(\bar{n}_3,\bar{n}_8)\}}{m^{(1)}(\bar{n}_2)}$$

$$-\frac{(m^{(1)}(\bar{n}_4)q^{(2)}(\bar{n}_4,\bar{n}_3)+m^{(1)}(\bar{n}_7)q^{(2)}(\bar{n}_7,\bar{n}_3)+m^{(1)}(\bar{n}_8)q^{(2)}(\bar{n}_8,\bar{n}_3)\}}{m^{(1)}(\bar{n}_2)},$$

$$q^{(2)}(\bar{n}_2,\bar{n}_7)=\frac{m^{(2)}(\bar{n}_3)q^{(1)}(\bar{n}_3,\bar{n}_2)+m^{(2)}(\bar{n}_7)q^{(1)}(\bar{n}_7,\bar{n}_2)}{m^{(1)}(\bar{n}_2)}-q^{(2)}(\bar{n}_2,\bar{n}_3),$$

$$q^{(2)}(\bar{n}_6,\bar{n}_7)=\frac{m^{(2)}(\bar{n}_7)q^{(1)}(\bar{n}_7,\bar{n}_6)}{m^{(1)}(\bar{n}_6)},$$

$$q^{(2)}(\bar{n}_{10},\bar{n}_7)=\frac{m^{(2)}(\bar{n}_7)q^{(1)}(\bar{n}_7,\bar{n}_{10})}{m^{(1)}(\bar{n}_{10})},$$

$$q^{(2)}(\bar{n}_{11},\bar{n}_8)=\frac{m^{(2)}(\bar{n}_8)\{q^{(1)}(\bar{n}_8,\bar{n}_3)+q^{(1)}(\bar{n}_8,\bar{n}_4)+q^{(1)}(\bar{n}_8,\bar{n}_7)+q^{(1)}(\bar{n}_8,\bar{n}_{11})\}}{m^{(1)}(\bar{n}_{11})}$$

$$-\frac{\{m^{(1)}(\bar{n}_3)q^{(2)}(\bar{n}_3,\bar{n}_8)+m^{(1)}(\bar{n}_4)q^{(2)}(\bar{n}_4,\bar{n}_8)+m^{(1)}(\bar{n}_7)q^{(2)}(\bar{n}_7,\bar{n}_8)\}}{m^{(1)}(\bar{n}_{11})},$$

$$q^{(2)}(\bar{n}_{11},\bar{n}_7)=\frac{m^{(2)}(\bar{n}_7)q^{(1)}(\bar{n}_7,\bar{n}_{11})+m^{(2)}(\bar{n}_8)q^{(1)}(\bar{n}_8,\bar{n}_{11})}{m^{(1)}(\bar{n}_{11})}-q^{(2)}(\bar{n}_{11},\bar{n}_8).$$

The collection of processes with transition rates $q^{(1)}$, $q^{(2)}$ as specified above satisfies cross-balance and the aggregated process with aggregation coefficients $r^{(1)}$, $r^{(2)}$ has a unique equilibrium distribution π at V given by

$$\pi = B(r^{(1)}m^{(1)} + r^{(2)}m^{(2)}),$$

where B is a normalizing constant. The transition rates of the aggregated process, however, are not equal to the transition rates of the original process (3.9). Therefore, we have to add another network to the collection to correct for this difference. To this end, define the process with transition rates $q^{(3)}$ given by

$$q^{(3)}(\bar{n}, \bar{n}') = \begin{cases} q^{(1)}(\bar{n}, \bar{n}'), & \text{if } \bar{n} \in V^{(2)} \setminus V^{(1)}, \\ q^{(2)}(\bar{n}, \bar{n}'), & \text{if } \bar{n} \in V^{(1)} \setminus V^{(2)}, \\ q(\bar{n}, \bar{n}'), & \text{if } \bar{n}, \bar{n}' \in V^{(1)} \cap V^{(2)}, \\ 0, & \text{otherwise.} \end{cases}$$

The aggregated process with aggregation coefficients $r^{(1)} = 1$, $r^{(2)} = 1$, $r^{(3)} = -1$ has transition rates q as given in (3.9). Therefore, we have now constructed a collection of queueing networks such that the transition rates of the aggregated process equal the transition rates of the original process. Now we have to construct the equilibrium distribution for the aggregated process. To this end, assume that process 3 allows an invariant measure $m^{(3)}$ at $V^{(3)} = V^{(1)} \cap V^{(2)}$. As before, we can give general condition on $q^{(1)}, q^{(2)}, q^{(3)}$ such that the collection satisfies cross-balance with measures $m^{(1)}, m^{(2)}, m^{(3)}$. Under these conditions we can then conclude the equilibrium distribution of the aggregated process and thus the equilibrium distribution of the original process. However, to illustrate the implications of these assumptions on the transition rates of the original process, we will consider a special case. Assume that for $\bar{n} \in V^{(3)}$ the invariant measures satisfy

$$m^{(1)}(\bar{n}) = m^{(2)}(\bar{n}) = m^{(3)}(\bar{n}) = m(\bar{n}).$$

Then it is obvious that under the assumptions previously made for $q^{(1)}$ and $q^{(2)}$, without any further assumptions on $q^{(3)}$, the collection satisfies cross-balance. The transition rates for the aggregated process with aggregation coefficients $r^{(1)} = 1$, $r^{(2)} = 1$, $r^{(3)} = -1$ are given in (3.9). Furthermore, the aggregated process has a unique equilibrium distribution π at V given by

$$\pi = B(m^{(1)} + m^{(2)} - m^{(3)}),$$

where B is a normalizing constant.

In order to derive the equilibrium distribution of the aggregated process we only need the following assumption on the processes in the collection and thus on the original process.

- There exists a measure m for the original process that satisfies the global balance equations at $V^{(k)}$, $k = 1, 2, 3$, i.e., for all $\bar{n} \in V^{(k)}$ and for $k = 1, 2, 3$, m is a solution to

$$\sum_{\bar{n}' \neq \bar{n}, \bar{n}' \in V^{(k)}} \{m(\bar{n})q(\bar{n}, \bar{n}') - m(\bar{n}')q(\bar{n}', \bar{n})\} = 0.$$

The other assumptions made on the transition rates of the processes in the collection, i.e., (3.10a)-(3.10c) are implied by this assumption. (3.10a) is implied by global balance at $\bar{n}_4$ for process 1, (3.10b) by global balance at $\bar{n}_7$ for process 2, and (3.10c) is trivially satisfied since $m^{(1)} = m^{(2)}$ and $q^{(1)} = q^{(2)}$ at $V^{(3)}$.

4. Concluding remarks

In this paper we have considered collections of Markov chains. For a collection of Markov chains we have introduced the aggregated process. Based on cross-balance we have shown that the equilibrium distribution for the aggregated process is a mixture of the equilibrium distributions of the processes in the collection. In the examples we have shown that collections such as a process and its truncated process or a process and its time-reversed process satisfy cross-balance. Also, for an explicit example, we have given a construction method for constructing the equilibrium distribution based on cross-balance.

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