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A trace formula for the forcing relation of braids

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Abstract

The forcing relation of braids has been introduced for a 2-dimensional analogue of the Sharkovskii order on periods for maps of the interval. In this paper, by making use of the Nielsen fixed point theory and a representation of braid groups, we deduce a trace formula for the computation of the forcing order.

MSC: 20F36; 37E30

Keywords: Braid groups; Braid forcing; Nielsen theory

1. Introduction

The influential statement "Period three implies chaos" of Li and Yorke [21] turns out to be a consequence of a much earlier theorem

Theorem 1.1 (*Sharkovskii [23]*). *On the set of natural numbers, define a linear order*

$$3 \succ 5 \succ 7 \succ \cdots \succ 2 \cdot 3 \succ 2 \cdot 5 \succ \cdots \succ 4 \cdot 3 \succ 4 \cdot 5 \succ \cdots \succ 8 \succ 4 \succ 2 \succ 1.$$

For any continuous map $f : [0, 1] \to [0, 1]$ of the interval, if f has a periodic orbit of period n, then f must have a periodic orbit of period m for every $m \prec n$.

However, in dimension 2, the same statement cannot be true, as shown by the $(2\pi/3)$-rotation of the unit disk. It was not until the work of Matsuoka [22] and Boyland [5], that the role of braids was revealed in the problem of the forcing relation of periodic orbits for homeomorphisms of the plane.

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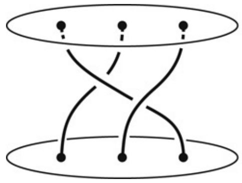

Fig. 1. A geometric braid.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation-preserving homeomorphism, and let $\{h_t : \mathbb{R}^2 \to \mathbb{R}^2\}_{0 \le t \le 1}$ be an isotopy with $h_0 = \mathrm{id}$ and $h_1 = f$. An f-invariant set $P = \{x_1, \dots, x_n\} \subset \mathbb{R}^2$ gives rise to a geometric braid (Ref. [3], see Fig. 1)

$$\{(h_t(x_i), t) \mid 0 \le t \le 1, 1 \le i \le n\}$$

in the cylinder $\mathbb{R}^2 \times [0, 1]$. Indeed, the closed curve $\{[h_t(x_1), \dots, h_t(x_n)] \mid 0 \le t \le 1\}$ in the configuration space

$$\mathcal{X}_n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}^2, x_i \neq x_j, \forall i \neq j\}/\Sigma_n,$$

where Σ_n denotes the symmetric group of n symbols, gives rise to a braid β_P in the n-strand braid group $B_n = \pi_1(\mathcal{X}_n)$. With another connecting isotopy $\{h_t\}$, the resulting braid β_P may differ by a power of the "full-twist". Matsuoka obtained lower bounds for the number of m-periodic points of $f|_{\mathbb{R}^2 \setminus P}$, in terms of the trace of the reduced Burau representation of the braid $(\beta_P)^m$.

Later, Kolev [20] (see also [12]) found that a 3-periodic orbit P guarantees the existence of m-periodic orbits for every m, unless the braid β_P is conjugate to a power of the braid $\sigma_1\sigma_2$. Roughly speaking, this means that the $(2\pi/3)$-rotation mentioned above is the only exceptional case. Therefore, Li–Yorke's statement still holds in a subtle way under 2-dimensional dynamics. The analogue of the Sharkovskii order naturally leads to the notion of forcing relation of (conjugacy classes of) braids.

In the following, the notation $[\beta]$ stands for the conjugacy class (in the group which is specified by the context) of a braid β.

Definition 1.2. A braid β *forces* a braid γ if, for any orientation-preserving homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ and any isotopy $\{h_t\} : \mathrm{id} \simeq f$, the existence of an f-invariant set P with $[\beta_P] = [\beta]$ guarantees the existence of an f-invariant set Q with $[\beta_Q] = [\gamma]$.

Remark 1.3. There is a homomorphism from the braid group B_n onto the mapping class group of the pair $(\mathbb{R}^2, P)$ (acting on $(\mathbb{R}^2, P)$ *from the right*), its kernel being generated by the "full-twist". Via this homomorphism, $[\beta_P]$ is sent to the conjugacy class of the mapping class represented by f, which is independent of the choice of the isotopy $\{h_t\}$. Following Boyland [5], this invariant is referred to as the *braid type* of (f, P) in the literature. It is clear that the forcing relation of braids defined above naturally descends to that of braid types.

The forcing relation is essentially a problem concerning plane homeomorphisms. So the Bestvina–Handel theory of train-track maps [2] comes in naturally. By analyzing the symbolic dynamics of train-track maps, Handel [14] was able to totally solve the forcing relation among 3-strand pseudo-Anosov braids, and de Carvalho and Hall [7,8] have managed to do the same for horseshoe braids. This approach is, theoretically speaking, powerful enough to be extended to mapping classes of all punctured surfaces. On the other hand, there still is the challenging task of recovering the braiding information encoded in the symbolic dynamics.

In this paper, we take another approach. Besides the Thurston classification of surface homeomorphisms, we apply the Nielsen fixed point theory. As a powerful tool for studying fixed points and periodic orbits of self maps, the Nielsen theory has been well developed and successful in many mathematical problems. It turns out that there are plenty of coincidences between the notions in the Nielsen theory and the forcing theory, providing a more direct bridge between the topological and the algebraic aspects of braids.

We start by slightly expanding the language of forcing.

Definition 1.4. A braid β' is an *extension* of β if β' is a (disjoint but possibly intertwined) union of β and another braid γ. An extension β' is *forced by* β if, for any orientation-preserving homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ and any isotopy $\{h_t\} : \mathrm{id} \simeq f$, the existence of an f-invariant set P with $[\beta_P] = [\beta]$ guarantees the existence of an additional f-invariant set $Q \subset \mathbb{R}^2 \setminus P$ with $[\beta_{P\cup Q}] = [\beta']$.

The advantage of considering $[\beta_{P\cup Q}]$ is that it contains the extra information about how the forced braid $\gamma = \beta_Q$ winds around the original braid β.

Our main result is stated as follows.

Theorem 1.5. *Suppose a braid $\beta' \in B_{n+m}$ is an extension of $\beta \in B_n$. Then β' is forced by β if and only if β' is neither collapsible nor peripheral relative to β, and the conjugacy class $[\beta']$ has a nonzero coefficient in* $\mathrm{tr}_{B_{n+m}}\zeta_{n,m}(\beta)$.

In the theorem, $\zeta_{n,m}$ is a matrix representation of B_n over a free $\mathbb{Z}B_{n+m}$-module, and the trace $\mathrm{tr}_{B_{n+m}}$ is meant to take value in the free Abelian group generated by the *conjugacy classes in B_{n+m}* (see Section 4). In addition, β' is said to be collapsible or peripheral relative to β if, roughly speaking, some strands of β' may be merged or moved to infinity while keeping β untouched (see Definition 3.3 and the figures therein).

Thus, to obtain the $(n+m)$-strand forced extensions of a braid $\beta \in B_n$, it suffices to compute the trace $\mathrm{tr}_{B_{n+m}}\zeta_{n,m}(\beta)$ and then drop off certain irrelevant terms.

The paper is organized as follows. In Section 2, we propose a Nielsen theory tailored to finite invariant sets of self embeddings. In Section 3, we apply this Nielsen theory to the forcing problem of braids, and reduce it to the computation of a generalized Lefschetz number. In Section 4, the representation $\zeta_{n,m}$ is defined and a trace formula for the generalized Lefschetz number is derived. Section 5 is devoted to the proof of Theorem 1.5. In the final section, we discuss the algorithmic aspects of the trace formula and give some concrete examples.

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2. Nielsen theory

2.1. Nielsen fixed point theory

Throughout the paper, all maps between topological spaces are assumed to be continuous. The material in this subsection is standard, see [15,16]. We assume that X is a compact, connected polyhedron and $f : X \to X$ is a self map.

Consider the *mapping torus* $T_f = X \times \mathbb{R}_+/(x, t+1) \sim (f(x), t)$ of f. Denote by Γ the fundamental group of T_f, and by Γ_c the set of conjugacy classes of Γ. Then Γ_c is independent of the base point of T_f and can be regarded as the set of free homotopy classes of closed curves in T_f.

Note that $x \in \operatorname{Fix} f$ if and only if on the mapping torus T_f its time-1 orbit curve $\{[x, t] \mid 0 \le t \le 1\}$ is closed. Define $x, y \in \operatorname{Fix} f$ to be in the same *fixed point class* if and only if their time-1 orbit curves are freely homotopic in T_f. Therefore, every fixed point class $\mathbf{F}$ gives rise to a conjugacy class $\mathrm{cd}(\mathbf{F})$ in Γ, called the *coordinate* of $\mathbf{F}$. A fixed point class $\mathbf{F}$ is called *essential* if its index $\mathrm{ind}(f, \mathbf{F})$ is nonzero.

The *generalized Lefschetz number* is defined as

$$L_\Gamma(f) = \sum_{\mathbf{F}} \mathrm{ind}(f, \mathbf{F}) \cdot \mathrm{cd}(\mathbf{F}) \in \mathbb{Z}\Gamma_c$$

which takes value in the free abelian group $\mathbb{Z}\Gamma_c$ generated by Γ_c.

The number of nonzero terms in $L_\Gamma(f)$ is called the *Nielsen number* of f. It is the number of essential fixed point classes, a lower bound for the number of fixed points of f.

The generalized Lefschetz number is a homotopy invariant, i.e. if $f \simeq g : X \to X$ then, identifying the fundamental groups of T_f and T_g in the standard way, we have $L_\Gamma(f) = L_\Gamma(g)$.

2.2. *Stratified maps*

The Nielsen theory for stratified maps is a version of relative Nielsen theory. Readers are referred to [18] for a detailed treatment of this subject.

Definition 2.1. Let W be a compact, connected polyhedron and let $\emptyset = W^0 \subset W^1 \subset \cdots \subset W^{m-1} \subset W^m = W$ be a filtration of compact subpolyhedra. For $1 \le k \le m$, the subspace $W_k = W^k \setminus W^{k-1}$ is called the k-th stratum. A map $f : W \to W$ is called a *stratified map* if $f(W_k) \subset W_k$ for all strata W_k. Two stratified maps $f, f' : W \to W$ are called *stratified homotopic* if there is a homotopy of stratified maps $\{h_t : W \to W\}_{0 \le t \le 1}$ such that $h_0 = f$, $h_1 = f'$.

We will be concerned with fixed point classes of $f_m = f|_{W_m} : W_m \to W_m$ in the top stratum. A free homotopy class of closed curves in T_{f_m}, represented by a closed curve γ, is said to be *related* to a lower stratum W_k if there is a homotopy of closed curves $\{\gamma_s : [0, 1] \to T_f\}_{0 \le s \le 1}$ such that $\gamma_0 = \gamma$, each γ_s is in T_{f_m} for $0 \le s < 1$, and γ_1 is in $T_{f|_{W_k}}$.

Definition 2.2. A fixed point class of f_m is called *degenerate* if its coordinate is related to some lower stratum W_k. Otherwise, it is called *non-degenerate*.

Every non-degenerate fixed point class of f_m is a compact subset of W_m, and hence its fixed point index is well defined.

Denote by Γ the fundamental group of T_{f_m} and by Γ_c the set of conjugacy classes in Γ. The *generalized Lefschetz number* of the stratified map f is defined as

$$L_\Gamma(f) = \sum_{\mathbf{F}_m} \mathrm{ind}(f_m, \mathbf{F}_m) \cdot \mathrm{cd}(\mathbf{F}_m) \in \mathbb{Z}\Gamma_c$$

where the sum is taken over all non-degenerate fixed point classes $\mathbf{F}_m$ of f_m.

The Nielsen fixed point theory has a natural version for stratified maps. The main result is that $L_\Gamma(f)$ is not changed by a stratified homotopy of the map f.

The following theorem will play an important role in the analysis of the forcing relation of braids. Suppose S is a compact, connected, orientable surface, and consider homeomorphisms of S as stratified maps with respect to the filtration $\emptyset \subset \partial S \subset S$.

Theorem 2.3 (*Jiang and Guo [17], Boyland [6]*). *Every orientation-preserving homeomorphism $f : S \to S$ is isotopic (through homeomorphisms) to a homeomorphism ϕ such that, for any $n \geq 1$, any fixed point class of ϕ^n is essential and contained in a single ϕ-orbit and, moreover, no fixed point class of $\phi^n|_{\operatorname{int} S}$ is related to ∂S.*

In the theorem, the $\phi|_{\operatorname{int} S}$-orbits persist under isotopy, i.e. none of them can be merged or be eliminated by isotoping the homeomorphism $\phi|_{\operatorname{int} S}$. In particular, $\phi|_{\operatorname{int} S}$ has the minimal number of periodic orbits of period n in its isotopy class for every $n \geq 1$. In the rest of this paper, we will refer to the homeomorphism $\phi|_{\operatorname{int} S}$ as a *minimal representative* in the isotopy class of $f|_{\operatorname{int} S}$.

2.3. A Nielsen theory for finite invariant sets

In this subsection, assume X is a compact, connected, smooth manifold of dimension d and $f : X \to X$ is a self *embedding*.

Let m be a fixed natural number. Consider the symmetric product space

$$\mathrm{SP}^m X = \{(x_1, \ldots, x_m) \mid x_i \in X\}/\Sigma_m.$$

Its points are written as $[x_1, \ldots, x_m]$, with repetitions allowed. For an integer k, $0 \leq k \leq m$, define the subspace

$$\mathrm{SP}^{m,k} X = \{[x_1, \ldots, x_m] \in \mathrm{SP}^m X \mid |\{x_1, \ldots, x_m\}| \leq k\}.$$

Then we have a filtration

$$\emptyset = \mathrm{SP}^{m,0} X \subset \mathrm{SP}^{m,1} X \subset \cdots \subset \mathrm{SP}^{m,m-1} X \subset \mathrm{SP}^{m,m} X = \mathrm{SP}^m X.$$

The top stratum is sometimes called the deleted m-th symmetric product space and denoted

$$\mathrm{DSP}^m X = \mathrm{SP}^m X \setminus \mathrm{SP}^{m,m-1} X.$$

For $1 \leq k \leq m$, the k-th stratum is $W_k = \mathrm{SP}^{m,k} X \setminus \mathrm{SP}^{m,k-1} X$.

Remark 2.4. Each stratum W_k is a manifold of dimension $k \cdot d$. When $d = 2$, which is our main concern later, the m-th symmetric product $\mathrm{SP}^m X$ itself is a manifold of dimension $2m$.

The map $f : X \to X$ induces a map $\mathrm{SP}^m f : \mathrm{SP}^m X \to \mathrm{SP}^m X$ given by $\mathrm{SP}^m f([x_1, \ldots, x_m]) = [f(x_1), \ldots, f(x_m)]$. Since f is an embedding, $\mathrm{SP}^m f$ is now a stratified map with respect to the above filtration. Hence the theory in the previous subsection is applicable.

Observe that a fixed point $[x_1, \ldots, x_m]$ of $\mathrm{DSP}^m f = \mathrm{SP}^m f|_{\mathrm{DSP}^m X}$ corresponds to an f-invariant set consisting of precisely m distinct points. Thus, the number of non-degenerate, essential fixed point classes of $\mathrm{DSP}^m f$ is a lower bound for the number of such f-invariant sets, for all embeddings isotopic to f.

Below is a useful criterion for the degeneracy of a fixed point class of $\mathrm{DSP}^m f$.

Proposition 2.5. *Suppose X is a compact, connected, smooth manifold of dimension d and suppose $f : X \to X$ is a self embedding. Let $Q = \{x_1, \ldots, x_m\} \subset X$ be an f-invariant set. Let $\mathcal{D}$ denote the disjoint union of k copies of the d-dimensional disk, $1 \le k < m$. The coordinate of the fixed point $[x_1, \ldots, x_m]$ of $\mathrm{DSP}^m f$ is related to the k-th stratum W_k if and only if there exists an isotopy of embeddings $\{i_t : \mathcal{D} \to X\}_{0 \le t \le 1}$ such that $i_0 = f \circ i_1$, $Q \subset i_t(\mathcal{D})$, and each component of $i_t(\mathcal{D})$ contains at least one point of Q, for all $0 \le t \le 1$.*

Proof. Sufficiency is clear: we can use the tubes $\{(x, t) \mid x \in i_t(\mathcal{D}), t \in [0, 1]\}$ to construct a homotopy which relates the time-1 orbit curve of $[x_1, \ldots, x_m]$ to a closed curve in the mapping torus of $\mathrm{SP}^m f|_{W_k}$.

Necessity. Let $\{\gamma_s : [0, 1] \to T_{\mathrm{SP}^m f}\}_{s \in [0,1]}$ be a homotopy of closed curves, relating the time-1 orbit curve γ of $[x_1, \ldots, x_m]$ to a closed curve lying in the mapping torus of $\mathrm{SP}^m f|_{W_k}$. By sliding along $T_{\mathrm{SP}^m f}$, we may assume that $\gamma_s(t) = [[z_1(s, t), \ldots, z_m(s, t)], t]$, where $z_j(s, t)$ are continuous functions of s, t. Put $Q_{s,t} = \{z_j(s, t) \mid 1 \le j \le m\} \subset X$. Then $Q_{0,t} = Q$ and $Q_{s,0} = f(Q_{s,1})$.

Since there are precisely k distinct elements in $Q_{1,t}$, if $z_{j_1}(1, t) = z_{j_2}(1, t)$ holds for some t, then it holds for all t. Therefore, we may choose an isotopy of embeddings $\{i'_t : \mathcal{D} \to X\}_{t \in [0,1]}$ such that $i'_0 = f \circ i'_1$ and each component of $i'_t(\mathcal{D})$ contains precisely one point of $Q_{1,t}$. Extending $\{i'_t\}$ to a two-parameter isotopy $\{i''_{s,t} : \mathcal{D} \to X\}_{s,t \in [0,1]}$ such that $i''_{1,t} = i'_t$, $i''_{s,0} = f \circ i''_{s,1}$ and $Q_{s,t} \subset i''_{s,t}(\mathcal{D})$, we get the desired isotopy $\{i_t = i''_{0,t} : \mathcal{D} \to X\}_{t \in [0,1]}$. $\square$

Definition 2.6. In Proposition 2.5, the components of $i_0(\mathcal{D})$ containing more than one point of Q are called *merging disks* of Q.

Remark 2.7. The existence of merging disks of Q means the f-invariant set Q can be merged into a smaller one by isotoping f in a neighborhood of these disks.

2.4. *Index formulae*

The next two lemmas may be found in [15].

Lemma 2.8. *Suppose x is a generic fixed point of $f : X \to X$, i.e. f is differentiable at x with Jacobian A such that $\det(I - A) \ne 0$. Then x is an isolated fixed point and $\mathrm{ind}(f, x) = \operatorname{sgn}\det(I - A)$.*

Lemma 2.9. *Suppose x and y are isolated fixed points of $f : X \to X$ and $g : Y \to Y$, respectively. Then $\mathrm{ind}(f \times g, x \times y) = \mathrm{ind}(f, x) \cdot \mathrm{ind}(g, y)$.*

Lemma 2.10. *Let $f : (\mathbb{R}^n)^k \to (\mathbb{R}^n)^k$ be a map defined by*

$$(x_1, \ldots, x_k) \mapsto (f_2(x_2), \ldots, f_k(x_k), f_1(x_1))$$

where $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}^n$ are a family of maps, each admitting the origin $0 \in \mathbb{R}^n$ as a fixed point. If the origin $0 \in (\mathbb{R}^n)^k$ is an isolated fixed point of f, then $\mathrm{ind}(f, 0) = \mathrm{ind}(f_1 \circ \cdots \circ f_k, 0)$.

Proof. It suffices to consider the case of generic fixed points. Denote by $A_1, \ldots, A_k$ the Jacobians of $f_1, \ldots, f_k$ at 0. Then the Jacobian of f at 0 is

$$A = \begin{pmatrix} 0 & A_2 & & & \\ & 0 & A_3 & & \\ & & 0 & & \\ & & & \ddots & A_k \\ A_1 & & & & 0 \end{pmatrix}.$$

Therefore,

$$\operatorname{ind}(f, 0) = \operatorname{sgn}\det(I - A) = \operatorname{sgn}\det(I - A_1 \cdots A_k) = \operatorname{ind}(f_1 \circ \cdots \circ f_k, 0). \quad \square$$

The following lemma is required for the proof of Proposition 4.3.

Lemma 2.11. *Let $\lambda > 1$ be a real number, and let $B = (B_{ij})$ be an $m \times m$ matrix with*

$$B_{ij} = \begin{cases} (-1)^{n_j}, & \eta(j) = i, \\ 0, & \eta(j) \neq i, \end{cases}$$

where $n_1, \ldots, n_m \in \mathbb{Z}$ are a set of integers and $\eta \in \Sigma_m$ is a permutation. For a generic fixed point x of a map $f : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ with Jacobian $A = \begin{pmatrix} \lambda B & \\ & \lambda^{-1} B \end{pmatrix}$, we have

$$\operatorname{ind}(f, x) = (-1)^m (-1)^{n_1 + \cdots + n_m} \operatorname{sgn} \eta.$$

Proof. We may assume η is a cycle. Then

$$\det(I - A) = (1 - (-1)^{n_1 + \cdots + n_m} \lambda^m)(1 - (-1)^{n_1 + \cdots + n_m} \lambda^{-m}).$$

Hence

$$\operatorname{ind}(f, x) = \operatorname{sgn}\det(I - A) = -(-1)^{n_1 + \cdots + n_m} = (-1)^m (-1)^{n_1 + \cdots + n_m} \operatorname{sgn} \eta. \quad \square$$

3. Forced extensions of a braid

In this section, we apply Nielsen's theory to the problem of the forcing relation of braids. It turns out that the mapping tori involved here are naturally embedded into the defining configuration spaces of braid groups, and the coordinates of the fixed points can be readily interpreted as forced extensions.

3.1. Coordinates recognized as braid extensions

Let $\sigma_1, \ldots, \sigma_{n-1}$ denote the standard generators of the Artin's n-strand braid group B_n (Ref. [3], see Fig. 2), and let $A_{i,j}$, $1 \leq i < j \leq n$, denote the standard pure braid

$$A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}.$$

Let $P \subset \mathbb{R}^2$ be a prescribed set of n punctures. Define the configuration spaces

$$\mathcal{X}_n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}^2, x_i \neq x_j, \forall i \neq j\}/\Sigma_n,$$

$$\mathcal{X}_{n,m} = \{(x_1, \ldots, x_{n+m}) \mid x_i \in \mathbb{R}^2, x_i \neq x_j, \forall i \neq j\}/\Sigma_n \times \Sigma_m,$$

$$\mathcal{Y}_{n,m} = \{(y_1, \ldots, y_m) \mid y_i \in \mathbb{R}^2 \setminus P, y_i \neq y_j, \forall i \neq j\}/\Sigma_m.$$

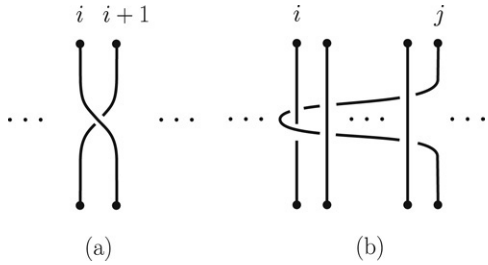


Fig. 2. The standard braids (a) σ_i and (b) $A_{i,j}$.

Clearly $\mathcal{Y}_{n,m}$ embeds into $\mathcal{X}_{n,m}$ via $[y_1, \ldots, y_m] \mapsto [P, y_1, \ldots, y_m]$, and $\mathcal{X}_{n,m}$ projects onto $\mathcal{X}_n$ via $[x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}] \mapsto [x_1, \ldots, x_n]$. Then $\mathcal{Y}_{n,m}$ is precisely the fiber of the bundle $\pi : \mathcal{X}_{n,m} \to \mathcal{X}_n$, and we have

$$\pi_1(\mathcal{X}_n) = B_n,$$
$$\pi_1(\mathcal{X}_{n,m}) = \langle \sigma_1, \ldots, \sigma_{n-1}, \sigma_n^2, \sigma_{n+1}, \ldots, \sigma_{n+m-1} \rangle \subset B_{n+m},$$
$$\pi_1(\mathcal{Y}_{n,m}) = \langle A_{1,n+1}, \ldots, A_{n,n+1}, \sigma_{n+1}, \ldots, \sigma_{n+m-1} \rangle \subset \pi_1(\mathcal{X}_{n,m}).$$

Given a nontrivial n-strand braid β, "sliding the plane down the braid β" determines (up to isotopy) a homeomorphism $f_\beta : \mathbb{R}^2 \setminus P \to \mathbb{R}^2 \setminus P$, as well as a connecting isotopy $\{h_t : \mathbb{R}^2 \to \mathbb{R}^2\}_{0 \le t \le 1} : \mathrm{id} \simeq f_\beta$ such that the curves $\{h_t(P)\}_{0 \le t \le 1}$ represent the braid β.

Now we figure out our key observations.

Proposition 3.1. (1) *The mapping torus of the induced map* $\mathrm{DSP}^m\, f_\beta : \mathcal{Y}_{n,m} \to \mathcal{Y}_{n,m}$ *can be identified with the space obtained from*

$$\{([h_t(P), y_1, \ldots, y_m], t) \mid y_i \in \mathbb{R}^2 \setminus h_t(P), 0 \le t \le 1\} \subset \mathcal{X}_{n,m} \times [0, 1]$$

by identifying the top $\mathcal{Y}_{n,m} \times 0$ *with the bottom* $\mathcal{Y}_{n,m} \times 1$.

(2) *Under the above identification, the fundamental group* $\Gamma_{\beta,m}$ *of* $T_{\mathrm{DSP}^m\, f_\beta}$ *is the subgroup in* B_{n+m} *generated by* β *and* $\pi_1(\mathcal{Y}_{n,m})$*, where* β *is regarded as an* $(n+m)$*-strand braid with m trivial strands added.*

(3) *Moreover, when a fixed point of* $\mathrm{DSP}^m\, f_\beta$ *corresponds to an* f_β*-invariant set* $Q \subset \mathbb{R}^2 \setminus P$*, the coordinate of the former is precisely* $[\beta_{P \cup Q}]$.

Proof. (1) The identification is realized by the embedding

$$\mathcal{Y}_{n,m} \times [0, 1] \to \mathcal{X}_{n,m} \times [0, 1],$$
$$([P, y_1, \ldots, y_m], t) \mapsto ([h_t(P), h_t(y_1), \ldots, h_t(y_m)], t).$$

(2) and (3) follow from (1). $\square$

3.2. Compactification

By Proposition 3.1, the coordinates of the fixed points of $\mathrm{DSP}^m\, f_\beta$ are naturally interpreted as $(n+m)$-strand extensions of β. However, $\mathrm{DSP}^m\, f_\beta$ is a self map of a noncompact space, hence falls out of the framework of the usual Nielsen fixed point theory.

So, we apply instead the theory in Section 2.3. For this, we compactify $\mathbb{R}^2 \setminus P$ to a 2-disk with n holes and denote it by Y_n, and assume further that $f_\beta = \overline{f}_\beta|_{\mathrm{int}\,Y_n}$ for some homeomorphism $\overline{f}_\beta : Y_n \to Y_n$.

Consider the symmetric product space $\mathrm{SP}^m\, Y_n$ and the induced stratified map $\mathrm{SP}^m\, \overline{f}_\beta : \mathrm{SP}^m\, Y_n \to \mathrm{SP}^m\, Y_n$ with respect to the filtration $\mathrm{SP}^{m,0}\, Y_n \subset \mathrm{SP}^{m,1}\, Y_n \subset \cdots \subset \mathrm{SP}^{m,m}\, Y_n$. Note that the interior of the manifold $\mathrm{DSP}^m\, Y_n$ is precisely $\mathcal{Y}_{n,m}$. In particular, $\pi_1(\mathrm{DSP}^m\, Y_n) = \pi_1(\mathcal{Y}_{n,m})$ and $\pi_1(T_{\mathrm{DSP}^m\, \overline{f}_\beta}) = \pi_1(T_{\mathrm{DSP}^m\, f_\beta}) = \Gamma_{\beta,m}$.

The stratified map $\mathrm{SP}^m\, \overline{f}_\beta$ is actually the desired compactification of $\mathrm{DSP}^m\, f_\beta$. But some more fixed points may arise on the boundary of the manifold $\mathrm{DSP}^m\, Y_n$. Hence the generalized Lefschetz number $L_{\Gamma_{\beta,m}}(\mathrm{SP}^m\, \overline{f}_\beta)$ may contain unwanted terms (called "peripheral" terms) which should be identified and ruled out.

In addition, the coordinates of degenerate fixed point classes of $\mathrm{DSP}^m\, \overline{f}_\beta$ also need to be identified in the computation of $L_{\Gamma_{\beta,m}}(\mathrm{SP}^m\, \overline{f}_\beta)$. These considerations lead to the notions of the next subsection.

3.3. Collapsible and peripheral extensions

Recall the Thurston classification theorem for homeomorphisms of compact surfaces. The theorem has a natural version for punctured surfaces, even for punctured planes (by regarding the plane as a once-punctured 2-sphere).

Theorem 3.2 (*Thurston [10,24]*). *Every homeomorphism $f : S \to S$ of a compact surface S is isotopic to a homeomorphism ϕ (Thurston representative) such that either*

(1) *ϕ is a periodic map, i.e. $\phi^k = \mathrm{id}$ for some positive integer k; or*

(2) *ϕ is a pseudo-Anosov map, i.e. there is a number $\lambda > 1$ and a pair of transverse measured foliations (F^s, μ^s) and (F^u, μ^u) such that $\phi(F^s, \mu^s) = (F^s, \lambda^{-1}\mu^s)$ and $\phi(F^u, \mu^u) = (F^u, \lambda\mu^u)$; or*

(3) *ϕ is a reducible map, i.e. there is a system of disjoint simple closed curves $\gamma = \{\gamma_1, \ldots, \gamma_k\}$ in $\mathrm{int}\, S$ (reducing curves) such that γ is invariant by ϕ (but γ_i's may be permuted) and γ has a ϕ-invariant tubular neighborhood U such that each component of $S \setminus U$ has negative Euler characteristic and on each ϕ-component of $S \setminus U$, ϕ satisfies* (1) *or* (2).

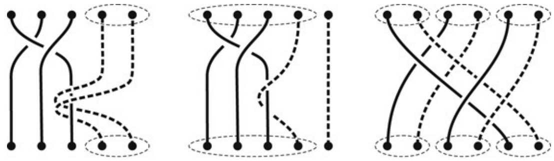
Every braid determines a unique isotopy class of homeomorphisms of a punctured plane. In this way, the braids naturally fall into three types: periodic, pseudo-Anosov and reducible.

Definition 3.3. Suppose $\beta' \in B_{n+m}$ is an extension of $\beta \in B_n$. Let ϕ be a Thurston representative determined by β'. We say β' is *collapsible* (resp. *peripheral*) relative to β if there exists a system of reducing curves of ϕ such that one of them encloses none of (resp. precisely one of or all of) the punctures corresponding to β. (See Fig. 3.)

Definition 3.4. If an extension $\beta' \in \beta \cdot \pi_1(\mathcal{Y}_{n,m})$ of a braid $\beta \in B_n$ is collapsible (resp. peripheral) relative to β, then we say the conjugacy class $[\beta']$ in $\Gamma_{\beta,m}$ is *collapsible* (resp. *peripheral*).

3.4. Forced extensions

We are ready to state the main result of this section.



(a) (b) (c)

Fig. 3. (a) Collapsible and (b), (c) peripheral braids relative to the solid braid.

Proposition 3.5. *Suppose $\beta \in B_n$ is a nontrivial braid. The $(n+m)$-strand forced extensions of β are exactly the non-peripheral terms in $L_{\Gamma_{\beta,m}}(\mathrm{SP}^m\,\overline{f}_\beta)$.*

Proof. Thanks to the homotopy invariance of the generalized Lefschetz number, we can assume $f_\beta = \overline{f}_\beta|_{\operatorname{int} Y_n} : \mathbb{R}^2 \setminus P \to \mathbb{R}^2 \setminus P$ is a minimal representative (in the sense of Theorem 2.3) in its isotopy class.

On the one hand, a fixed point $[x_1, \dots, x_m]$ of $\mathrm{DSP}^m\,\overline{f}_\beta$ falls out of $\mathcal{Y}_{n,m}$ if and only if some x_i falls into ∂Y_n, and this is equivalent by the minimality of f_β to noting that the coordinate of the fixed point is peripheral.

On the other hand, the fixed point class represented by a fixed point $[x_1, \dots, x_m]$ of $\mathrm{DSP}^m\,\overline{f}_\beta$ lying in $\mathcal{Y}_{n,m}$ is non-degenerate; otherwise, by Proposition 2.5 the f_β-invariant set $\{x_1, \dots, x_m\}$ can be merged into a smaller one (cf. Remark 2.7), contradicting the minimality of f_β.

It follows that the non-peripheral terms in $L_{\Gamma_{\beta,m}}(\mathrm{SP}^m\,\overline{f}_\beta)$ are precisely the coordinates of the fixed points of $\mathrm{DSP}^m\, f_\beta = \mathrm{DSP}^m\,\overline{f}_\beta|_{\mathcal{Y}_{n,m}}$, which by the minimality of f_β again are exactly the $(n+m)$-strand forced extensions of β. $\square$

From the theoretical point of view, constructing a minimal representative in the isotopy class of f_β is not an easy task. However, by the homotopy invariance of the generalized Lefschetz number, we can compute $L_{\Gamma_{\beta,m}}(\mathrm{SP}^m\,\overline{f}_\beta)$ from any map stratified homotopic to $\mathrm{SP}^m\,\overline{f}_\beta$. This is exactly what we will do in the next section. The following lemma is needed for this purpose.

Lemma 3.6. *Let $g : Y_n \to Y_n$ be an embedding isotopic to $\overline{f}_\beta$ and let $Q = \{x_1, \dots, x_m\}$ be a g-invariant set. Then the fixed point class of $\mathrm{DSP}^m\, g$ represented by $[x_1, \dots, x_m]$ is degenerate if and only if its coordinate is collapsible.*

Proof. There is an obvious correspondence between the merging disks of a g-invariant set and the reducing curves defining the notion of collapsibility. $\square$

4. A trace formula for $L_{\Gamma_{\beta,m}}(\mathrm{SP}^m\overline{f}_\beta)$

In this section, we define a representation $\zeta_{n,m}$ of B_n over the free (left) $\mathbb{Z}B_{n+m}$-module generated by

$$\mathcal{E}_{n,m} = \{\mu = (\mu_1, \dots, \mu_{n-1}) \mid \mu_i \in \mathbb{Z}_{\geq 0}, \mu_1 + \dots + \mu_{n-1} = m\},$$

and derive a trace formula for $L_{\Gamma_{\beta,m}}(\mathrm{SP}^m\,\overline{f}_\beta)$. The size of the basis $\mathcal{E}_{n,m}$ is $\binom{m+n-2}{m}$.

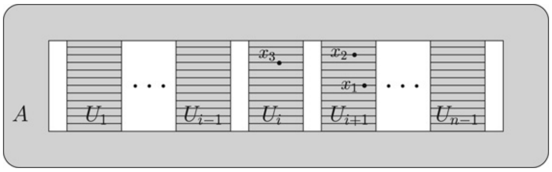


Fig. 4. Decomposition of Y_n.

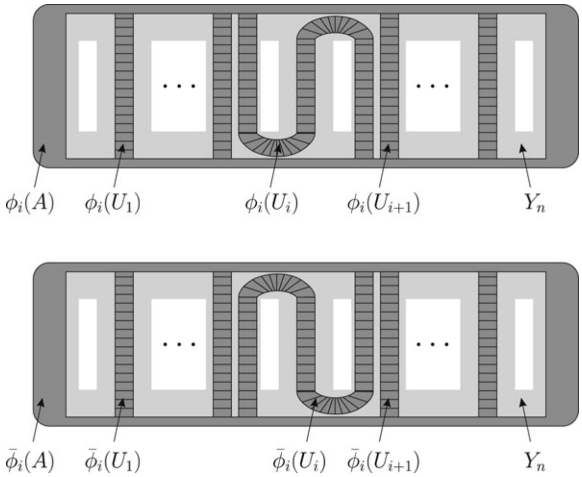


Fig. 5. The embeddings $\phi_i : Y_n \to Y_n$ and $\bar{\phi}_i : Y_n \to Y_n$.

4.1. Braid actions on Y_n

We decompose the surface Y_n into an annulus and $n-1$ rectangles, as shown in Fig. 4. Let $U = U_1 \cup \cdots \cup U_{n-1}$ be the union of the $n-1$ foliated open rectangles. Define an ordering on U such that $x_1 \prec x_2$ if either x_1 lies in a rectangle to the right of x_2 or x_1 lies in a strictly lower leaf of the same rectangle as x_2. For example, the order of the three points in Fig. 4 is $x_1 \prec x_2 \prec x_3$.

Set

$$V = \{[x_1, \ldots, x_m] \in \mathcal{Y}_{n,m} \mid x_i \in U, x_1 \prec \cdots \prec x_m\}.$$

Then $V = \bigcup_{\mu \in \mathcal{E}_{n,m}} V_\mu$, where

$$V_\mu = \{[x_1, \ldots, x_m] \in V \mid |\{x_1, \ldots, x_m\} \cap U_i| = \mu_i, 1 \le i \le n-1\}.$$

Each V_μ is connected; thus the elements of $\mathcal{E}_{n,m}$ are in one-one correspondence to the components of V.

Illustrated in Fig. 5 are two embeddings $\phi_i : Y_n \to Y_n$ and $\bar{\phi}_i : Y_n \to Y_n$, which can be understood as the action of the elementary braids σ_i andσ_i^{-1} on Y_n, respectively. Both push the annulus outward, irrationally rotate the outmost boundary, keep the foliations of $\phi_i^{-1}(U)$ or $\bar{\phi}_i^{-1}(U)$, uniformly contract along the leaves of the foliations, and uniformly expand along the transversal direction. Slightly abusing our notations, we also use ϕ_i and $\bar{\phi}_i$ to denote the induced stratified maps of $\mathrm{SP}^m\, Y_n$.

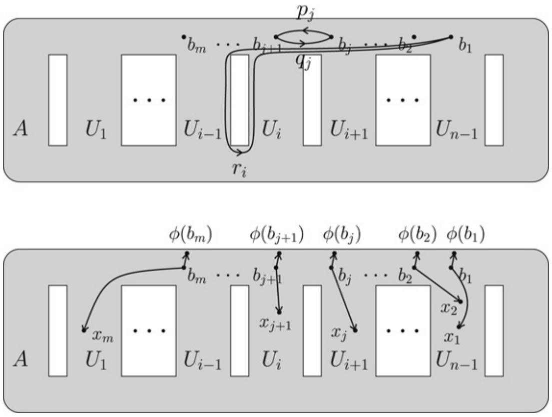


Fig. 6. Paths in Y_n.

For every $\phi \in \{\phi_1, \dots, \phi_{n-1}, \bar{\phi}_1, \dots, \bar{\phi}_{n-1}\}$, we have

$$V_\mu \cap \phi^{-1}(V_\nu) = \bigcup_{\eta \in \Sigma_m} W^{(\phi)}_{\mu\nu\eta}$$

where

$$W^{(\phi)}_{\mu\nu\eta} = \left\{ [x_1, \dots, x_m] \in V_\mu \cap \phi^{-1}(V_\nu) \,\middle|\, \begin{matrix} x_{\eta(1)} \prec \cdots \prec x_{\eta(m)}, \\ \phi(x_1) \prec \cdots \prec \phi(x_m) \end{matrix} \right\}.$$

Each $W^{(\phi)}_{\mu\nu\eta}$ is connected; thus the elements of the set $\{\eta \in \Sigma_m \mid W^{(\phi)}_{\mu\nu\eta} \neq \emptyset\}$ are in one-one correspondence to the components of $V_\mu \cap \phi^{-1}(V_\nu)$.

Further, as shown in Fig. 6, choose a base point $b = [b_1, \dots, b_m] \in \mathcal{Y}_{n,m}$. Then the generators σ_{n+j} and $A_{i,n+1}$ of $\pi_1(\mathcal{Y}_{n,m})$ are represented by the loops $[b_1, \dots, b_{j-1}, p_j, q_j, b_{j+2}, \dots, b_m]$ and $[r_i, b_2, \dots, b_m]$, respectively.

For every $x = [x_1, \dots, x_m] \in V$ with $x_1 \prec \cdots \prec x_m$, the disjoint "descending" paths connecting b_k to x_k in Y_n give rise to a path γ_x in $\mathcal{Y}_{n,m}$. Similarly, the disjoint "ascending" paths connecting b_k to $\phi(b_k)$ give rise to a path $\gamma_{\phi(b)}$ in $\mathcal{Y}_{n,m}$. For every nonempty $W^{(\phi)}_{\mu\nu\eta}$, we choose a point $x \in W^{(\phi)}_{\mu\nu\eta}$ and let $\alpha^{(\phi)}_{\mu\nu\eta}$ denote the element of $\pi_1(\mathcal{Y}_{n,m})$ represented by the loop $\gamma_{\phi(b)} \cdot \phi(\gamma_x) \cdot \gamma^{-1}_{\phi(x)}$. Note that $\alpha^{(\phi)}_{\mu\nu\eta}$ is independent of the choices of x, γ_x, $\gamma_{\phi(b)}$ and $\gamma_{\phi(x)}$.

4.2. *The representation $\zeta_{n,m}$*

Proposition 4.1. *The equations*

$$\mu \cdot \zeta_{n,m}(\sigma_i) = \sum_{\nu \in \mathcal{E}_{n,m}} c^{(i)}_{\mu\nu} \cdot \nu,$$

$$\mu \cdot \zeta_{n,m}(\sigma_i^{-1}) = \sum_{\nu \in \mathcal{E}_{n,m}} d^{(i)}_{\mu\nu} \cdot \nu,$$

where

$$c_{\mu\nu}^{(i)} = (-1)^{\nu_i} \cdot \sigma_i \cdot \sum_{\eta: W_{\mu\nu\eta}^{(\phi_i)} \neq \emptyset} \operatorname{sgn} \eta \cdot \alpha_{\mu\nu\eta}^{(\phi_i)},$$

$$d_{\mu\nu}^{(i)} = (-1)^{\nu_i} \cdot \sigma_i^{-1} \cdot \sum_{\eta: W_{\mu\nu\eta}^{(\bar{\phi}_i)} \neq \emptyset} \operatorname{sgn} \eta \cdot \alpha_{\mu\nu\eta}^{(\bar{\phi}_i)},$$

give rise to a group representation $\zeta_{n,m}$ of B_n over the free $\mathbb{Z}B_{n+m}$-module generated by $\mathcal{E}_{n,m}$.

Proof. Direct proof of this proposition is rather complicated. We refer the reader to [25] for another representation $\xi_{n,m}$ of B_n with the same basis $\mathcal{E}_{n,m}$ and with the same coefficient ring $\mathbb{Z}B_{n+m}$. Let $a_{\mu\nu}^{(i)}$, $b_{\mu\nu}^{(i)}$ denote the matrix elements of $\xi_{n,m}(\sigma_i)$, $\xi_{n,m}(\sigma_i^{-1})$, respectively. Then a direct computation shows that $c_{\mu\nu}^{(i)} = (b_{\nu\mu}^{(i)})^*$ and $d_{\mu\nu}^{(i)} = (a_{\nu\mu}^{(i)})^*$, where $* : \mathbb{Z}B_{n+m} \to \mathbb{Z}B_{n+m}$ is the involution determined by $a^* = a^{-1}$ for $a \in B_{n+m}$. Therefore, $\zeta_{n,m}$ is precisely the dual representation of $\xi_{n,m}$. $\square$

The computation of $c_{\mu\nu}^{(i)}$, $d_{\mu\nu}^{(i)}$ is straightforward, and we state the result as follows.

For each permutation $\eta \in \Sigma_m$, there exists a unique positive permutation braid (a positive braid that has a geometric representative where every pair of strands crosses at most once, see [9]) $\alpha_\eta \in \pi_1(\mathcal{Y}_{n,m}) \subset B_{n+m}$ in which the last m strands are permutated in the manner of η. Set $\eta^\pm = \operatorname{sgn} \eta \cdot (\alpha_\eta)^{\pm 1}$.

For integers $1 \le i \le j \le k \le l \le m$, let $\theta_{i,j,k,l} \in \Sigma_m$ denote the permutation that sends the sequence $i+1, i+2, \ldots, l$ to

$$k+1, k+2, \ldots, l, \qquad k, k-1, \ldots, j+1, \qquad i+1, i+2, \ldots, j.$$

Also set

$$\Theta_{j,k,l} = \left\{ \eta \in \Sigma_m \,\middle|\, \begin{array}{l} \eta(i) = i, \forall i \neq j+1, j+2, \ldots, l, \\ \eta(j+1) < \eta(j+2) < \cdots < \eta(k), \\ \eta(k+1) < \eta(k+2) < \cdots < \eta(l) \end{array} \right\}$$

and

$$\Theta_{j,k,l}^\pm = \sum_{\eta \in \Theta_{j,k,l}} \eta^\pm.$$

For integer $1 \le i \le n-1$ and elements $\mu = (\mu_1, \ldots, \mu_{n-1})$, $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathcal{E}_{n,m}$, $c_{\mu\nu}^{(i)}$ and $d_{\mu\nu}^{(i)}$ do not vanish if and only if $\mu_{i-1} \le \nu_{i-1}$, $\mu_{i+1} \le \nu_{i+1}$ and $\mu_k = \nu_k$ for all $k \neq i-1, i, i+1$. In this case,

$$c_{\mu\nu}^{(i)} = (-1)^{\nu_i} \sigma_i \left(\prod_{k=u_{i+1}+1}^{v_i} A_{i,n+k} \right) \theta_{v_{i+1}, v_{i+1}, v_i, v_i}^+ \Theta_{v_i, u_i, v_{i-1}}^+ \Theta_{v_{i+2}, u_{i+1}, v_{i+1}}^+,$$

$$d_{\mu\nu}^{(i)} = (-1)^{\nu_i} \theta_{u_{i+1}, v_{i+1}, v_i, u_i}^- \left(\prod_{k=v_{i+1}+1}^{u_i} A_{i,n+k} \right)^{-1} \Theta_{v_i, u_i, v_{i-1}}^+ \Theta_{v_{i+2}, u_{i+1}, v_{i+1}}^+ \sigma_i^{-1},$$

where $u_j = \sum_{k=j}^{n-1} \mu_k$ and $v_j = \sum_{k=j}^{n-1} \nu_k$.

4.3. The trace formula

Definition 4.2. Let Γ be a group, $\mathbb{Z}\Gamma$ its group ring, Γ_c the set of conjugacy classes, $\mathbb{Z}\Gamma_c$ the free Abelian group generated by Γ_c, and $\pi_\Gamma : \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma_c$ the obvious projection. Suppose ζ is an endomorphism of a free $\mathbb{Z}\Gamma$-module such that $\zeta(v_i) = \sum_{j=1}^{k} a_{ij} \cdot v_j$ for a basis $\{v_1, \ldots, v_k\}$. The *trace* of ζ is defined as

$$\mathrm{tr}_\Gamma \zeta = \pi_\Gamma \left(\sum_{i=1}^{k} a_{ii} \right) \in \mathbb{Z}\Gamma_c.$$

It is straightforward to verify that the definition is independent of the choice of the basis.

Note that, under the basis $\mathcal{E}_{n,m}$, all matrix elements of $\zeta_{n,m}(\beta)$ belong to $\mathbb{Z}\Gamma_{\beta,m}$. Therefore, $\zeta_{n,m}(\beta)$ can be naturally regarded as an endomorphism of the free $\mathbb{Z}\Gamma_{\beta,m}$-module generated by $\mathcal{E}_{n,m}$. In this way, the notation $\mathrm{tr}_{\Gamma_{\beta,m}} \zeta_{n,m}(\beta)$ in the following proposition makes sense.

Now we prove the main result of this section.

Proposition 4.3. *For every nontrivial braid $\beta \in B_n$, we have*

$$L_{\Gamma_{\beta,m}}(\mathrm{SP}^m\, \overline{f}_\beta) = (-1)^m \mathrm{tr}_{\Gamma_{\beta,m}} \zeta_{n,m}(\beta) - \textit{collapsible terms}.$$

Proof. Choose a word $\beta = \tau_1 \cdots \tau_k$ where $\tau_1, \ldots, \tau_k \in \{\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\}$. We put $\varphi_i = \phi_{j_i}$ if $\tau_i = \sigma_{j_i}$ or $\varphi_i = \bar{\phi}_{j_i}$ if $\tau_i = \sigma_{j_i}^{-1}$. Then the embedding $g = \varphi_k \cdots \varphi_1 : Y_n \to Y_n$ induces a map $\mathrm{SP}^m g : \mathrm{SP}^m Y_n \to \mathrm{SP}^m Y_n$ stratified homotopic to $\mathrm{SP}^m\, \overline{f}_\beta$. Hence $L_{\Gamma_{\beta,m}}(\mathrm{SP}^m\, \overline{f}_\beta) = L_{\Gamma_{\beta,m}}(\mathrm{SP}^m g)$. It is immediate from the definitions of ϕ_i and $\bar{\phi}_i$ that $\mathrm{Fix}\, \mathrm{DSP}^m g \subset V$.

Note that the components of $\bigcup_{\mu \in \mathcal{E}_{n,m}} V_\mu \cap \mathrm{SP}^m g^{-1}(V_\mu)$ are in one-one correspondence to the summands of the last expression in the following equation.

$$\begin{aligned}(-1)^m \mathrm{tr}_{\Gamma_{\beta,m}} \zeta_{n,m}(\beta) &= (-1)^m \mathrm{tr}_{\Gamma_{\beta,m}} \zeta_{n,m}(\tau_1) \cdots \zeta_{n,m}(\tau_k) \\ &= \sum_{\mu^0, \ldots, \mu^k \in \mathcal{E}_{n,m} : \mu^0 = \mu^k} \ \sum_{\eta^1, \ldots, \eta^k \in \Sigma_m : W^{(\varphi_i)}_{\mu^{i-1} \mu^i \eta^i} \neq \emptyset} (-1)^m (-1)^{\mu^1_{j_1} + \cdots + \mu^k_{j_k}} \mathrm{sgn}(\eta^1 \cdots \eta^k) \\ &\quad \cdot [\tau_1 \alpha^{(\varphi_1)}_{\mu^0 \mu^1 \eta^1} \cdots \tau_k \alpha^{(\varphi_k)}_{\mu^{k-1} \mu^k \eta^k}].\end{aligned}$$

Moreover, each of these components is homeomorphic to $\mathbb{R}^{2m}$, on which $\mathrm{SP}^m g$ acts hyperbolically, hence giving rise to precisely one fixed point of $\mathrm{SP}^m g$, either in V or $\overline{V} \cap \mathrm{SP}^{m,m-1} Y_n$. In the former case, the coordinate of the fixed point corresponding to $\mu^0, \ldots, \mu^k, \eta^1, \ldots, \eta^k$ is precisely

$$[\tau_1 \alpha^{(\varphi_1)}_{\mu^0 \mu^1 \eta^1} \cdots \tau_k \alpha^{(\varphi_k)}_{\mu^{k-1} \mu^k \eta^k}]$$

and, by Lemma 2.11, the index is

$$(-1)^m (-1)^{\mu^1_{j_1} + \cdots + \mu^k_{j_k}} \mathrm{sgn}(\eta^1 \cdots \eta^k).$$

In the latter case, the corresponding summand is always collapsible. Therefore, from Lemma 3.6 the proposition follows. $\square$

Remark 4.4. In fact, the configuration space $\mathcal{Y}_{n,m}$ has the homotopy type of a compact m-dimensional complex and the trace $(-1)^m \mathrm{tr}_{\Gamma_{\beta,m}} \zeta_{n,m}(\beta)$ is nothing but the generalized Lefschetz number of a self map of the complex induced by $\mathrm{DSP}^m f_\beta$. In this sense, the collapsible and peripheral terms in the trace both arise from the compactification issue.

5. Proof of main theorem

According to Propositions 3.5 and 4.3, the $(n+m)$-strand forced extensions of a nontrivial braid $\beta \in B_n$ are precisely those non-collapsible, non-peripheral terms in the trace $\mathrm{tr}_{\Gamma_{\beta,m}} \zeta_{n,m}(\beta)$. The following proposition states that these terms do not cancel in $\mathrm{tr}_{B_{n+m}} \zeta_{n,m}(\beta)$, and hence eventually establishes Theorem 1.5.

Proposition 5.1. *Let $\beta \in B_n$ be a nontrivial braid and suppose two extensions $\beta', \beta'' \in \beta \cdot \pi_1(\mathcal{Y}_{n,m})$ of β are conjugate in B_{n+m}. If β' is forced by β, then $[\beta']$ and $[\beta'']$ have the same coefficient in $L_{\Gamma_{\beta,m}}(\mathrm{SP}^m \overline{f}_\beta)$.*

Proof. Assume $f_\beta = \overline{f}_\beta|_{\mathrm{int}\, Y_n} : \mathbb{R}^2 \setminus P \to \mathbb{R}^2 \setminus P$ is a minimal representative (in the sense of Theorem 2.3) in its isotopy class, and assume the term $[\beta']$ in $L_{\Gamma_{\beta,m}}(\mathrm{SP}^m \overline{f}_\beta)$ is the coordinate of the fixed point of $\mathrm{DSP}^m \overline{f}_\beta$ corresponding to an $\overline{f}_\beta$-invariant set $Q \subset \mathrm{int} Y_n = \mathbb{R}^2 \setminus P$. We extend f_β to a homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose the puncture point set P splits into a disjoint union of periodic orbits $c_1 \cup \cdots \cup c_s$ of ϕ, and suppose Q splits into a disjoint union of periodic orbits $d_1 \cup \cdots \cup d_t$ of ϕ.

The conjugation between β' and β'' in B_{n+m} gives rise to a homeomorphism $\psi : \mathbb{R}^2 \to \mathbb{R}^2$, which preserves the set $P \cup Q = c_1 \cup \cdots \cup c_s \cup d_1 \cup \cdots \cup d_t$. Put $\phi' = \psi \phi \psi^{-1}$. Since β'' restricts to β on the first n strands, $\phi'|_{\mathbb{R}^2 \setminus P}$ is isotopic to f_β. Further, $\phi'|_{\mathbb{R}^2 \setminus P}$ is also a minimal representative in its isotopy class.

By conjugating β'' in $\Gamma_{\beta,m}$ if necessary, we may assume $d_1, \ldots, d_t$ are periodic orbits of ϕ'. Let m_i be the period of d_i. Then

$$\mathrm{ind}(\phi^{m_i}, \psi^j(d_i)) = \mathrm{ind}(\phi'^{m_i}, \psi^{j+1}(d_i))$$

for all j and

$$\mathrm{ind}(\phi^{m_i}, \psi^j(d_i)) = \mathrm{ind}(\phi'^{m_i}, \psi^j(d_i))$$

provided that $\psi^j(d_i) \in \{c_1, \ldots, c_s\}$.

Let n_i be the maximum positive number such that

$$\psi(d_i), \ldots, \psi^{n_i - 1}(d_i) \in \{c_1, \ldots, c_s\}.$$

Then

$$\{\psi^{n_1}(d_1), \ldots, \psi^{n_t}(d_t)\} = \{d_1, \ldots, d_t\},$$

and, by induction,

$$\mathrm{ind}(\phi^{m_i}, d_i) = \mathrm{ind}(\phi'^{m_i}, \psi^{n_i}(d_i)).$$

On the other hand, by Lemmas 2.9 and 2.10, we have

$$\operatorname{ind}(\mathrm{DSP}^m \overline{f}_\beta, [\beta']) = \prod_{i=1}^{t} \operatorname{ind}(\phi^{m_i}, d_i)/m_i,$$

$$\operatorname{ind}(\mathrm{DSP}^m \overline{f}_\beta, [\beta'']) = \prod_{i=1}^{t} \operatorname{ind}(\phi'^{m_i}, d_i)/m_i.$$

Therefore, these two indices are identical. □

6. Algorithms and examples

Thanks to Theorem 1.5, the computation of the $(n+m)$-strand forced extensions of a given braid $\beta \in B_n$ may proceed as follows.

1. By means of the representation $\zeta_{n,m}$, compute an initial formal sum for the trace $\operatorname{tr}_{B_{n+m}} \zeta_{n,m}(\beta)$.
2. Merge conjugate terms in the formal sum by solving the conjugacy problem in B_{n+m}.
3. Identify collapsible terms and peripheral terms by computing reducing curves, and drop them off.
4. Return the nonzero terms remaining after cancellation.

In the procedure described above, one has to deal with two algorithmic problems: the conjugacy problem in the braid group B_{n+m} and the computation of reducing curves. Fortunately, there have been effective algorithms for both these tasks.

For the conjugacy problem, we refer the reader to a very efficient algorithm due to Gebhardt [11]. See also [4,26] for improvements on this direction.

As to the second problem, one solution is a braid algorithm due to Bernardete, Nitecki and Gutierrez [1]. It can be improved significantly if one computes the ultra summit set [11] or its variant [26] instead of the super summit set (see the references for details). An alternative solution is given by Bestvina and Handel [2], which is also applicable for general surface homeomorphisms but apparently less efficient, because it involves a computation of train-track maps.

At the present time, we are not able to talk much about the computational complexity of the above procedure, partly because the topic of braid algorithms is a fairly new one and many questions still remain open. Nevertheless, the bulk part of running time is evidently spent on the second step. Hence it is a major issue to control the number of terms written down in the first step.

A braid is called *cyclic* if it induces a cyclic permutation on the end points of its strands. We call an extension of a braid *elementary* if it is obtained by appending a single cyclic braid.

Elementary forced extensions are the main concern of the braid forcing problem. Observe that the elementary extensions only constitute a small fraction of the terms in $L_{\Gamma_{\beta,m}}(\mathrm{SP}^m \overline{f}_\beta)$. Hence in the first step of the above procedure, we may drop off all non-elementary extensions of β to save considerably on running time.

As another example of a shortcut to facilitate the computation, when β is a pseudo-Anosov braid (the most significant case in dynamics), the identification of collapsible and peripheral terms may be reduced by the following proposition on the reducibility problem of braids, for which a polynomial solution (for a fixed number of strands) has been claimed recently by Ko and Lee [19].

Proposition 6.1. *An extension β' of a pseudo-Anosov β is collapsible or peripheral relative to β if and only if β' is a reducible braid.*

Proof. Let ϕ be a Thurston representative determined by β'. If β' is not reducible, then there are no reducing curves of ϕ; hence β' is neither collapsible nor peripheral relative to β. Conversely, If β' is reducible, each reducing curve of ϕ must enclose either at most one of or all of the punctures corresponding to β, because β is pseudo-Anosov. Hence β' is either collapsible or peripheral relative to β. □

Below, we conclude this paper by presenting some examples.

Example 6.2. Under the basis $\mathcal{E}_{3,2} = \{(2,0), (1,1), (0,2)\}$, the representation $\zeta_{3,2}$ of B_3 is given by the matrices (cf. the matrices of $\xi_{3,2}$ from [25])

$$\zeta_{3,2}(\sigma_1) = \sigma_1 \cdot \begin{pmatrix} -A_{14}A_{15}\sigma_4 & -A_{14}A_{15} & A_{14}A_{15} \\ 0 & -A_{15} & A_{15}(1-\sigma_4) \\ 0 & 0 & 1 \end{pmatrix},$$

$$\zeta_{3,2}(\sigma_1^{-1}) = \begin{pmatrix} -\sigma_4^{-1}A_{15}^{-1}A_{14}^{-1} & \sigma_4^{-1}A_{15}^{-1} & 1 \\ 0 & -A_{15}^{-1} & 1-\sigma_4 \\ 0 & 0 & 1 \end{pmatrix} \cdot \sigma_1^{-1},$$

$$\zeta_{3,2}(\sigma_2) = \sigma_2 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 1-\sigma_4 & -A_{24} & 0 \\ 1 & -A_{24} & -A_{24}A_{25}\sigma_4 \end{pmatrix},$$

$$\zeta_{3,2}(\sigma_2^{-1}) = \begin{pmatrix} 1 & 0 & 0 \\ A_{24}^{-1}(1-\sigma_4) & -A_{24}^{-1} & 0 \\ A_{25}^{-1}A_{24}^{-1} & \sigma_4^{-1}A_{25}^{-1}A_{24}^{-1} & -\sigma_4^{-1}A_{25}^{-1}A_{24}^{-1} \end{pmatrix} \cdot \sigma_2^{-1}.$$

For the reader's convenience, we illustrate by figures how to obtain the equality

$$(1,1) \cdot \zeta_{3,2}(\sigma_1) = -\sigma_1 A_{15} \cdot (1,1) + \sigma_1 A_{15}(1-\sigma_4) \cdot (0,2).$$

See Fig. 7. The set $V_{(1,1)}$ consists of those points $[x_1, x_2]$ with $x_1, x_2 \in Y_3$ positioned as in the top left figure. Note that the set $V_{(1,1)} \cap \phi_1^{-1}(V_{(2,0)})$ is empty; $V_{(1,1)} \cap \phi_1^{-1}(V_{(1,1)})$ has one component, illustrated by the top right figure; and $V_{(1,1)} \cap \phi_1^{-1}(V_{(0,2)})$ has two components, illustrated by the bottom two figures. From the last three figures, one reads out A_{15}, A_{15} and $A_{15}\sigma_4$, respectively. Together with the contribution of the signs, they are assembled to give the above equality.

Example 6.3. Under the basis $\mathcal{E}_{n,1}$, the representation $\zeta_{n,1}$ of B_n is given by the matrices

$$\zeta_{n,1}(\sigma_i) = \sigma_i \cdot \begin{pmatrix} I_{i-2} & & & & \\ & 1 & 0 & 0 & \\ & A_{i,n+1} & -A_{i,n+1} & 1 & \\ & 0 & 0 & 1 & \\ & & & & I_{n-i-2} \end{pmatrix}.$$

Note that if we replace σ_i by 1 and replace $A_{i,n+1}$ by a number a, the representation specializes to the reduced Burau representation (Ref. [3])

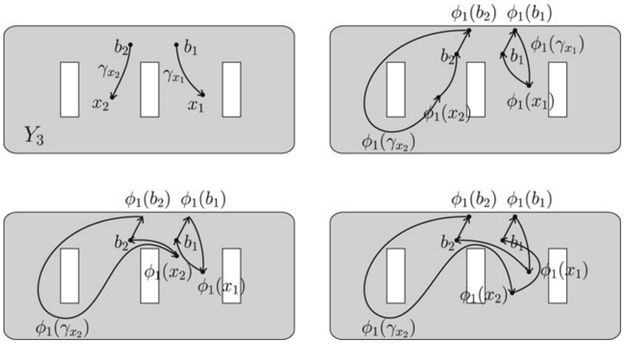


Fig. 7. Figures for computing $(1, 1) \cdot \zeta_{3,2}(\sigma_1)$.

$$\sigma_i \mapsto \begin{pmatrix} I_{i-2} & & & & \\ & 1 & 0 & 0 & \\ & a & -a & 1 & \\ & 0 & 0 & 1 & \\ & & & & I_{n-i-2} \end{pmatrix}.$$

Example 6.4. For the simplest pseudo-Anosov braid $\beta = \sigma_1\sigma_2^{-1}$, we have

$$\begin{aligned} \mathrm{tr}_{B_5}\zeta_{3,2}(\beta) &= \mathrm{tr}_{B_5}\zeta_{3,2}(\sigma_1)\zeta_{3,2}(\sigma_2^{-1}) \\ &= [\sigma_1 \cdot (-A_{14}A_{15}\sigma_4 - A_{14}A_{15}A_{24}^{-1}(1-\sigma_4) + A_{14}A_{15}A_{25}^{-1}A_{24}^{-1} + A_{15}A_{24}^{-1} \\ &\quad + A_{15}(1-\sigma_4)\sigma_4^{-1}A_{25}^{-1}A_{24}^{-1} - \sigma_4^{-1}A_{25}^{-1}A_{24}^{-1}) \cdot \sigma_2^{-1}] \\ &= [\beta - \beta A_{35}^{-1}A_{34}^{-1}\sigma_4^{-1} - \beta A_{14}A_{15}\sigma_4 - \beta A_{34}^{-1} - \beta A_{15} \\ &\quad + \beta A_{34}^{-1}\sigma_4^{-1} + \beta A_{15}\sigma_4 + \beta A_{15}A_{34}^{-1}]. \end{aligned}$$

See Fig. 8, in which the collapsible or peripheral strands are depicted as dotted lines. Clearly, the first five braids in the figure are reducible. An algorithmic test shows the last three are pseudo-Anosov. It follows from Proposition 6.1 that precisely the last three terms in $\mathrm{tr}_{B_5}\zeta_{3,2}(\beta)$ are neither collapsible nor peripheral. Therefore, up to conjugacy, there are a total of three 5-strand forced extensions of β: $\beta A_{34}^{-1}\sigma_4^{-1}$, $\beta A_{15}\sigma_4$ and $\beta A_{15}A_{34}^{-1}$.

Example 6.5. Suppose $\beta = \sigma_1 \cdots \sigma_{n_1}\sigma_{n_1+1}^{-1} \cdots \sigma_{n_1+n_2}^{-1} \in B_n$ where $n_1, n_2 \geq 2$ and $n = n_1 + n_2 + 1$. For $2 \leq m \leq \min(n_1, n_2)$. We then have

$$\mathrm{tr}_{B_{n+1}}\zeta_{n,1}(\beta^m) = [\beta^m - (\beta A_{1,n+1})^m - (\beta A_{n,n+1}^{-1})^m],$$
$$\mathrm{tr}_{B_{n+m}}\zeta_{n,m}(\beta) = [\beta(1 - A_{1,n+2})(1 - A_{n,n+1}^{-1})].$$

Either of the above formulae implies that the (pseudo-Anosov) cyclic braid β forces no m-strand cyclic braid (see [13, Theorem 7] for the case $n_2 = m = 2$). This contrasts sharply to Guaschi's theorem [12] which asserts that a pseudo-Anosov braid on three or four strands forces at least one m-strand cyclic braid for every $m \geq 1$.

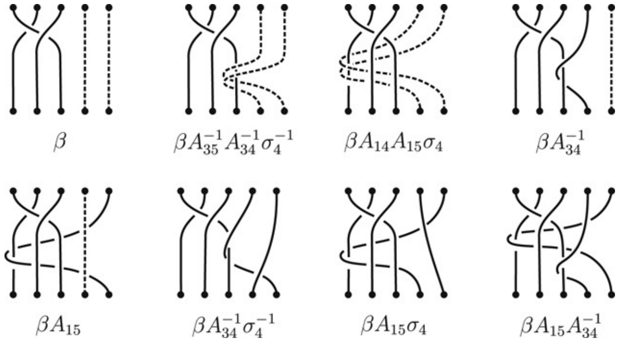


Fig. 8. The braids appearing in $\mathrm{tr}_{B_5}\zeta_{3,2}(\beta)$ for $\beta = \sigma_1\sigma_2^{-1}$.

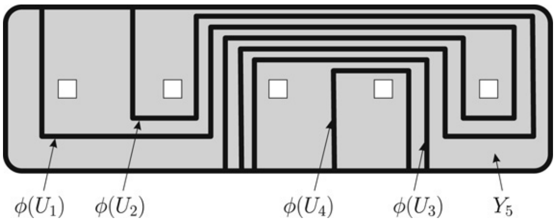


Fig. 9. The map $\phi : Y_5 \to Y_5$ representing $\beta = \sigma_1\sigma_2\sigma_3^{-1}\sigma_4^{-1}$.

Without loss of generality, we sketch the computation of the above formulae for $n_1 = n_2 = m = 2$. First, we translate the braid $\beta = \sigma_1\sigma_2\sigma_3^{-1}\sigma_4^{-1} \in B_5$ into the self embedding $\phi : Y_5 \to Y_5$ depicted in Fig. 9.

Keep the notations of Section 4. There are a total of 11 components in $V \cap \phi^{-1}(V)$, so there should be the same number of nonzero terms in the matrix $\zeta_{5,1}(\beta)$ under the standard basis $\mathcal{E}_{5,1}$

$$\zeta_{5,1}(\beta) = \beta \cdot \begin{pmatrix} 0 & A_{1,6}A_{2,6}(-1 + A_{5,6}^{-1}) & 0 & A_{1,6}A_{2,6}A_{5,6}^{-1} \\ 1 & A_{2,6}(-1 + A_{5,6}^{-1}) & 0 & A_{2,6}A_{5,6}^{-1} \\ 0 & A_{5,6}^{-1} & 0 & -A_{5,6}^{-1} \\ 0 & 0 & A_{5,6}^{-1} & -A_{5,6}^{-1} \end{pmatrix}.$$

Since the matrix is almost upper triangular (this is quite evident for larger n_1, n_2), the following trace can be computed without much difficulty.

$$\begin{aligned} \mathrm{tr}_{B_6}\zeta_{5,1}(\beta^2) &= [\beta A_{1,6}A_{2,6}(-1 + A_{5,6}^{-1})\beta + \beta^2 A_{1,6}A_{2,6}(-1 + A_{5,6}^{-1}) \\ &\quad + (\beta A_{2,6}(-1 + A_{5,6}^{-1}))^2 - (\beta A_{5,6}^{-1})^2 - (\beta A_{5,6}^{-1})^2 + (\beta A_{5,6}^{-1})^2] \\ &= [\beta^2 - (\beta A_{1,6})^2 - (\beta A_{5,6}^{-1})^2]. \end{aligned}$$

In the equality, we used the identities $\beta A_{1,6} = A_{2,6}\beta$ and $[\beta A_{2,6}A_{5,6}^{-1}] = [\beta]$.

Next, we compute the second formula. The matrix $\zeta_{5,2}(\beta)$ is a 10×10 one, but we are only concerned with its diagonal part. Notice that there are totally 10 components in $\bigcup_{\mu \in \mathcal{E}_{5,2}} V_\mu \cap \phi^{-1}(V_\mu)$. The trace is computed as

$$\begin{aligned}\operatorname{tr}_{B_7}\zeta_{5,2}(\beta) = [&-\beta\sigma_6 A_{1,6}A_{2,6}(-1+A_{5,6}^{-1}) - \beta A_{2,6}A_{2,7}\sigma_6 \\ &- \beta A_{2,6}A_{2,7}A_{5,6}^{-1}(1-\sigma_6) + \beta A_{2,6}A_{2,7}(A_{5,6}A_{5,7})^{-1} \\ &- \beta A_{2,7}(-1+A_{5,7}^{-1})A_{5,6}^{-1} + \beta(A_{5,6}A_{5,7})^{-1}\sigma_6^{-1} - \beta(A_{5,6}A_{5,7})^{-1}\sigma_6^{-1}].\end{aligned}$$

On the right hand side, the six terms containing $\sigma_6^{\pm 1}$ cancel pairwise, and we get

$$\begin{aligned}&= [-\beta A_{2,6}A_{2,7}A_{5,6}^{-1} + \beta A_{2,6}A_{2,7}(A_{5,6}A_{5,7})^{-1} - \beta A_{2,7}(-1+A_{5,7}^{-1})A_{5,6}^{-1}] \\ &= [\beta(1-A_{1,7})(1-A_{5,6}^{-1})].\end{aligned}$$

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