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Note

Covering symmetric semi-monotone functions

Roland Grappe, Zoltán Szigeti

G-SCOP, INPGrenoble, UJF, CNRS, 46 avenue Félix Viallet, 38031 Grenoble Cedex, France

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Abstract

We define a new set of functions called semi-monotone, a subclass of skew-supermodular functions. We show that the problem of augmenting a given graph to cover a symmetric semi-monotone function is NP-complete if all the values of the function are in $\{0, 1\}$ and we provide a minimax theorem if all the values of the function are different from 1. Our problem is equivalent to the node to area augmentation problem. Our contribution is to provide a significantly simpler and shorter proof.

Keywords: Edge-connectivity augmentation; Skew-supermodular functions

1. Introduction

In this paper we only consider loopless graphs. The global edge-connectivity augmentation problem of graphs consists of adding a minimum number of new edges to a given graph to obtain a k-edge-connected graph. The problem has been generalized in many directions, for example for directed graphs, for local edge-connectivity, for bipartite graphs, for hypergraphs, for adding stars. For a survey, we refer to [5].

Another way of generalization is to cover a function by a graph. Here we are looking for a graph so that each cut contains at least as many edges as the value of the function. We may start with the empty graph or more generally with a given graph. For symmetric supermodular functions, the problem was solved in [1]. For a larger class of functions, namely for symmetric skew-supermodular functions, the problem is already NP-complete, see in [5].

Here we propose to consider symmetric semi-monotone functions. We call a function R on V *semi-monotone* if $R(\emptyset) = R(V) = 0$ and for each set $\emptyset \neq X \neq V$, $0 \leqslant R(X) \leqslant R(X')$ either for all $\emptyset \neq X' \subseteq X$ (in this case, X is *in-monotone*) or for all $\emptyset \neq X' \subseteq V - X$ (then X is *out-monotone*). We remark that if R is symmetric, then X is out-monotone if and only if $R(X') \geqslant R(X)$ holds for all $V \neq X' \supseteq X$.

The subject of the present paper is to solve the following problem. Given a graph $G = (V, E)$ and a symmetric semi-monotone function R on V, add a minimum number $Opt(R, G)$ of new edges M to G to get a *covering* of R, that is

$$d_{G+M}(X) \geqslant R(X) \quad \text{for all } X \subseteq V, \tag{1}$$

where $d_L(X)$ denotes the number of edges in L having exactly one end-vertex in X.

E-mail address: Zoltan.Szigeti@g-scop.inpg.fr (Z. Szigeti).

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It is easy to see that symmetric semi-monotone functions are skew-supermodular, see Lemma 4. The proof of Z. Király in [5], for the NP-completeness of the skew-supermodular function covering problem, provides the NP-completeness of our problem. It shows that

Theorem 1. *Covering a symmetric semi-monotone function valued in* $\{0, 1\}$ *is* NP-*complete*.

By consequence, we suppose from now on that

$$R(X) \neq 1 \quad \text{for all } X \subseteq V. \tag{2}$$

In this case we provide a minimax theorem for the symmetric semi-monotone function covering problem, see Theorem 13.

The starting point of our research was the paper of Ishii and Hagiwara [4] on node to area augmentation. This problem can be defined as follows: Given a graph $G = (V, E)$, a family $\mathscr{W}$ of sets $W \subseteq V$ (called areas), and a requirement function $r : \mathscr{W} \to \mathbb{Z}_+$, add a minimum number of new edges to G so that the resulting graph contains $r(W)$ edge-disjoint paths from any area W to any vertex $v \notin W$. As Ishii showed in [3], our problem is equivalent to this, see also Claim 3.

In order to explain how we deal with our problem, we need a few definitions. Let $G' = (V, E')$ be a graph. The deficiency of $X \subseteq V$ is defined as follows: $q_{E'}(X) = R(X) - d_{E'}(X)$. For $Y \subseteq V$, let us define $Q_{E'}(Y) := \max\{\sum_{X\in\mathscr{X}} q_{E'}(X) : \mathscr{X} \text{ subpartition of } Y\}$. A subpartition $\mathscr{X}$ is called *optimal*, if it provides the maximum. Let $Q(G') := Q_{E'}(V)$. We mention that, by Lemma 14, $\lceil Q(G')/2\rceil$ is a lower bound for $Opt(R, G')$.

Let $K = (V + s, E' \cup F')$ be a graph where F' denotes the set of edges incident to s. We call a connected component K_i of $K - s$ such that $d_K(s, V(K_i)) = 1$ (resp. odd, $\geqslant 3$.) a *small* (resp. *odd*, *big*) component of K. A small component C contains a unique neighbour v_C of s. We will see that most of the difficulties come from the existence of a unique small component, hence we will try to get rid of them as soon as possible. We say that K *covers* R if

$$d_K(X) \geqslant R(X) \quad \text{for all } X \subseteq V \;\; (\text{equivalently } d_{F'}(X) \geqslant q_{E'}(X) \text{ for all } X \subseteq V). \tag{3}$$

Suppose that K covers R. By *splitting off* a pair su, sv of edges incident to s, we mean the operation that deletes these edges and add a new edge uv. We say that the pair or equivalently the splitting off is *admissible* if the graph after the splitting still covers R. A *complete* splitting off is a sequence of splitting off which decreases the degree of s to 0. We will use the technique of splitting off to get the minimax result.

First we extend the graph $G = (V, E)$ by adding a new vertex s and a minimum set $F_{\min}$ of new edges incident to s so that the new graph covers the function R. By Lemma 4, R is symmetric skew-supermodular, so we may apply the following general theorem of Frank [2].

Theorem 2. $|F_{\min}| = Q(G)$.

Then, if this number is odd, we add another edge incident to s as follows. If $(V + s, E + F_{\min})$ has a unique small component C: add a copy of sv_C, if it has only small components: add an edge anywhere, otherwise: add an edge not incident to a small component. The graph obtained after these operations is denoted by $H = (V + s, E + F)$ and called an *optimal extension* of $G = (V, E)$. Note that $d_H(s)$ is even, and if $Q(G)$ is odd, H has none or several small components. The reader should keep in mind that in this paper G denotes the starting graph, and H an optimal extension of G.

Finally, we will split off the edges incident to s to get the cover. The complete admissible splitting off will exist in H (in other words, the lower bound given by the deficient subpartitions can be achieved) only if H does not have a special obstacle, or equivalently, G contains no configuration, see Theorems 11 and 12. If G does contain a configuration, then an extra edge is needed, see Theorem 13.

We would like to emphasize that our approach provides a significantly simpler and shorter proof than that in [4]. This is due to the efficient tools we developed here (like Lemma 5) and to the use of allowed pairs (defined in Section 5).

2. Semi-monotone functions

We present some important properties on semi-monotone functions in this section.

Claim 3. *Covering a symmetric semi-monotone function is equivalent to solving a problem of node to area connectivity augmentation.*

Proof. *Sufficiency*: Given $\mathscr{W}$, r, the function $R_{\mathscr{W}}$ defined by $R_{\mathscr{W}}(X)=\max\{r(W): W\in\mathscr{W}, W\cap X=\emptyset \text{ or } W\subseteq X\}$ if $V\neq X\neq\emptyset$ and $R_{\mathscr{W}}(V)=R_{\mathscr{W}}(\emptyset)=0$ is symmetric semi-monotone.

Necessity: Given R symmetric semi-monotone, for all $\emptyset\neq X\subset V$, let W_X be the out-monotone set of $\{X, V-X\}$, $r(W_X)=R(X)$ and $\mathscr{W}=\{W_X, \emptyset\neq X\subset V\}$. We show that $R_{\mathscr{W}}(X)=R(X)$ for all $\emptyset\neq X\subset V$. Since $W_X\cap X=\emptyset$ or $W_X\subseteq X$, we have $R_{\mathscr{W}}(X)\geqslant r(W_X)=R(X)$. Let $W\in\{Z\subset V: Z\cap X=\emptyset \text{ or } Z\subseteq X\}$ such that $R_{\mathscr{W}}(X)=r(W)$. Then since X or $V-X$ is out-monotone, and R is symmetric, $R(X)\geqslant R(W)=r(W)=R_{\mathscr{W}}(X)$. □

A function R is called *skew-supermodular* if for all $X, Y\subset V$, $R(X)+R(Y)\leqslant\max\{R(X\cap Y)+R(X\cup Y), R(X-Y)+R(Y-X)\}$.

Lemma 4. *A symmetric semi-monotone function is skew-supermodular.*

Proof. For $X, Y\subset V$, apply that if X is out-monotone, then $R(X)\leqslant\min\{R(X\cup Y), R(Y-X)\}$, and if X is in-monotone, then $R(X)\leqslant\min\{R(X\cap Y), R(X-Y)\}$. □

For $Y_1, Y_2, Y_3\subset V$, let $Y_i^\star := Y_i-\bigcup_{j\neq i} Y_j$ $(1\leqslant i\leqslant 3)$, and $Y_4^\star := \bigcap_1^3 Y_i$.

Lemma 5. *Let R be a semi-monotone function and $Y_1, Y_2, Y_3\subset V$ with $Y_i^\star\neq\emptyset$ $(1\leqslant i\leqslant 4)$. Then there exists an index $1\leqslant j\leqslant 4$ such that $\sum_{1,i\neq j}^4 R(Y_i^\star)\geqslant\sum_1^3 R(Y_i)$.*

Proof. Apply that, $R(Y_j^\star)\geqslant R(Y_i)$ for $j=i,4$ if Y_i is in-monotone and for $j\neq i,4$ if Y_i is out-monotone. □

3. Preliminaries

Given a graph $L=(U,J)$ and $X, Y\subset U$, $d_L(X,Y)$ denotes the number of edges in J between $X-Y$ and $Y-X$, while $\overline{d}_L(X,Y)=d_L(U-X,Y)$. We will apply the following equalities.

$$d_L(X)+d_L(Y)=d_L(X\cup Y)+d_L(X\cap Y)+2d_L(X,Y), \tag{4}$$

$$d_L(X)+d_L(Y)=d_L(X-Y)+d_L(Y-X)+2\overline{d}_L(X,Y). \tag{5}$$

In Sections 3 and 4, we will deal with a graph $K=(V+s, E'+F')$ satisfying (3) and $d_K(s)$ is even and positive, where $E\subseteq E'$ and F' denotes the set of edges incident to s. Such a graph K may be obtained from H by splitting off some admissible pairs. $E'-E$ will be the set of split edges.

A set $X\subset V$ is called *tight* (resp. *dangerous*) if $2\leqslant R(X)$ and $d_K(X)=R(X)$ or equivalently $d_{F'}(X)=q_{E'}(X)$ holds (resp. $2\leqslant R(X)$ and $d_K(X)\leqslant R(X)+1$ or equivalently $d_{F'}(X)\leqslant q_{E'}(X)+1$). We say that a subpartition $\mathscr{X}$ is tight (resp. in-monotone) if each member is tight (resp. in-monotone). To clear up the notations, we may use Y for the subgraph induced by the vertex set Y. $\Gamma_K(s)$ is the set of neighbours of s in K. From now on, let $su\in F'$.

Claim 6. *Let $\emptyset\neq X, Y\subset V$.*

(6.1) *If Y is dangerous out-monotone and X is a connected component of $K-s$ with $X-Y\neq\emptyset$, then $d_K(s,X-Y)+1\geqslant d_K(s,Y)$. Moreover, if Y is tight, then the inequality is strict.*
(6.2) *Every in-monotone dangerous set Y is connected.*
(6.3) *If X and Y are both in- or out-monotone, both tight (resp. dangerous and $u\in X\cap Y$) and $X-Y\neq\emptyset\neq Y-X$, then $X-Y$, $Y-X$ are tight in-monotone, $\overline{d}_K(X,Y)=0$ (resp. $=1$).*

(6.4) *If X and Y are dangerous in-monotone, for* $A \in \{X \cap Y, X - Y, Y - X\}$, $A \cap \Gamma_K(s) \neq \emptyset$, *then* $X \cup Y$ *is connected and for all* $\emptyset \neq Z \subset X \cup Y$, $d_K(Z) \geqslant 2$.

Proof. (6.1) $R(Y)+1 \geqslant d_K(Y) \geqslant d_K(s, Y)+d_K(Y, X-Y) = d_K(s, Y)+d_K(X-Y)-d_K(s, X-Y) \geqslant d_K(s, Y)+R(Y)-d_K(s, X-Y)$. (6.2) If $\emptyset \subset X \subset Y$, then $R(Y)+R(Y) \leqslant R(X)+R(Y-X) \leqslant d_K(X)+d_K(Y-X) = d_K(Y)+2d_K(X, Y-X) \leqslant R(Y)+1+2d_K(X, Y-X)$, so $R(Y) \geqslant 2$ implies $d_K(X, Y-X) \geqslant 1$. (6.3) Suppose both are out-monotone, the other case is similar. By (5) and (1), $X-Y$, $Y-X$ are tight and $R(X-Y) = R(Y)$, $R(Y-X) = R(X)$, $\overline{d}_K(X, Y) = 0$ (resp. $=1$, for dangerous sets). Combined with X, Y are out-monotone, it concludes. (6.4) Since $X \cap Y \neq \emptyset$, and, by (6.2), X and Y are connected, so is $X \cup Y$. Let $\emptyset \neq Z \subseteq X \cup Y$. If $Z \subseteq X$, then since X is in-monotone and dangerous, $d_K(Z) \geqslant R(Z) \geqslant R(X) \geqslant 2$. Similarly, if $Z \subseteq Y$, then $d_K(Z) \geqslant 2$. Otherwise, Z intersects X and Y. By (6.3), $X-Y$ and $Y-X$ are in-monotone and tight hence connected by (6.2). So $d_K(Z) \geqslant 2$. □

Claim 7. *Suppose that* $Q(G)$ *is even. Let* $H = (V+s, E+F)$ *be an optimal extension of* $G = (V, E)$.

(7.1) *A subpartition* $\mathscr{X}$ *of V is optimal if and only if* $\mathscr{X}$ *is tight and each neighbour of s is contained in some* $X \in \mathscr{X}$.
(7.2) *Let* $\mathscr{X}$ *be an optimal subpartition of V. If* $Y \subset V$ *contains some members of* $\mathscr{X}$ *and is disjoint from the others, then* $d_F(Y) = Q_E(Y)$.

Proof. (7.1) In both directions we use that, by $Q(G)$ is even, Theorem 2 implies $Q(G) = |F| = d_F(s)$.
Sufficiency: $Q(G) = \sum_{X \in \mathscr{X}} q_E(X) \leqslant \sum_{X \in \mathscr{X}} d_F(X) \leqslant d_F(s) = Q(G)$, so we have equality everywhere.
Necessity: $Q(G) = |F| = \sum_{X \in \mathscr{X}} d_F(X) = \sum_{X \in \mathscr{X}} q_E(X)$, so $\mathscr{X}$ is optimal.
(7.2) Let $\mathscr{X}_Y$ be an optimal subpartition of Y. Then, by (7.1), $Q_E(Y) \geqslant \sum_{Y \supset X \in \mathscr{X}} q_E(X) = \sum_{Y \supset X \in \mathscr{X}} d_F(X) = d_F(Y) \geqslant \sum_{X \in \mathscr{X}_Y} d_F(X) \geqslant \sum_{X \in \mathscr{X}_Y} q_E(X) = Q_E(Y)$. □

4. Dangerous families

In this section we present a few results about dangerous families to describe the structure of the graph K for which no complete admissible splitting off exists. For a neighbour u of s and $S \subseteq \Gamma_K(s)$, we say that $\mathscr{Y}$ is a *dangerous family* covering u and S if each set in $\mathscr{Y}$ is dangerous, contains u and a vertex of S not contained in the other sets of $\mathscr{Y}$, and $S \subseteq \cup\mathscr{Y}$. A neighbour of s contained in a big component of K is called *big-neighbour*. A connected component B of $K - s$ with $d_K(B) = R(B) = 2$ is called a *boring* component of K. Let $\mathscr{B}_K$ be the family of *boring* components of K.

Lemma 8. *In the graph K, the edge su belongs to no admissible pair if and only if there is a dangerous family* $\mathscr{Y}$ *covering u and* $\Gamma_K(s)$. *In this case, K has a unique small component C. If* $u \notin C$, *then C and a unique big component D of K cover* $\Gamma_K(s)$ *and D is the union of two dangerous in-monotone sets containing u.*

Proof. The first part is obvious. We show first that $|\mathscr{Y}| \geqslant 3$. For $Y \in \mathscr{Y}$, we have $d_{F'}(V-Y) \geqslant q_{E'}(V-Y) = q_{E'}(Y) \geqslant d_{F'}(Y) - 1 = d_{F'}(s) - d_{F'}(V-Y) - 1$. Then, $d_{F'}(V-Y) \geqslant \lceil (d_{F'}(s)-1)/2 \rceil = d_{F'}(s)/2 > 0$. Thus $|\mathscr{Y}| \geqslant 2$. Suppose $\mathscr{Y} = \{Y_1, Y_2\}$. By the above inequality, $u \in Y_1 \cap Y_2$ and $\Gamma(s) \subseteq Y_1 \cup Y_2$, we have $d_{F'}(s) = d_{F'}(V-Y_1) + d_{F'}(V-Y_2) + d_{F'}(Y_1 \cap Y_2) \geqslant d_{F'}(s)/2 + d_{F'}(s)/2 + 1$, a contradiction.

Let $Y_1, Y_2, Y_3 \in \mathscr{Y}$. By Y_i dangerous, a well-known inequality on d_K, (1), Lemma 5 and $u \in \cap\mathscr{Y}$, $\sum_1^3 (R(Y_i)+1) \geqslant \sum_1^3 d_K(Y_i) \geqslant \sum_1^4 d_K(Y_i^\star) + 2d_K(Y_4^\star, s) \geqslant \sum_{1, i \neq j}^4 R(Y_i^\star) + d_K(Y_j^\star) + 2 \geqslant \sum_1^3 R(Y_i) + 3$. Then $d_K(Y_j^\star) = 1$, and, by (1) and (2), $R(Y_j^\star) = 0$. It follows that if $j = 4$, then Y_1, Y_2, Y_3 are out-monotone and $d_K(Y_4^\star) = d_K(s, Y_4^\star) = 1$, and if say $j = 3$, then Y_3 is out-monotone with $d_K(Y_3^\star) = d_K(s, Y_3^\star) = 1$ and Y_1 and Y_2 are in-monotone. Note that if $j \neq 4$, each triplet of $\mathscr{Y}$ consists of an in-monotone and two out-monotone sets, therefore $|\mathscr{Y}| = 3$.

It follows that K contains a small component C. We show that the small component is unique. In the first case ($j = 4$), by contradiction, let C' be another one. By $d_K(Y_4^\star) = 1$, $C' \cap Y_4^\star = \emptyset$. We suppose that $v_{C'} \notin Y_1$. By (6.3) and (6.2), $Y_1 - Y_i$ is connected ($2 \leqslant i \leqslant 3$), thus so is $Y_1 - Y_4^\star$. Since C' is small, $(Y_1 - Y_4^\star) \cap C' = \emptyset$. Thus $C' \cap Y_1 = \emptyset$. By Y_1 is out-monotone, $0 = R(C') \geqslant R(Y_1) \geqslant 2$, contradiction. In the second case ($j \neq 4$, e.g. $j = 3$) that is when $u \notin C$, by (6.4) and $|\mathscr{Y}| = 3$, $Y_1 \cup Y_2$ is contained in a big component D covering $\Gamma(s) - v_C$ implying that C is unique.

To prove the last statement, suppose that $u \notin C$ and $Z := D - (Y_1 \cup Y_2) \neq \emptyset$. By $d_K(Y_3^\star) = 1$, $Z \cap Y_3 = \emptyset$. By (1) and Y_3 out-monotone, $d_K(\bigcup_1^3 Y_i) \geqslant d_K(Z) + d_K(s, \bigcup_1^3 Y_i) \geqslant R(Z) + 4 \geqslant R(Y_3) + 4$. Then, by Y_i dangerous, (4), Y_1 and Y_2 in-monotone, we have $\sum_1^3 (R(Y_i) + 1) \geqslant \sum_1^3 d_K(Y_i) \geqslant d_K(Y_1) + d_K(Y_2 \cup Y_3) + d_K(Y_2 \cap Y_3) \geqslant d_K(Y_1 \cap (Y_2 \cup Y_3)) + d_K(\bigcup_1^3 Y_i) + R(Y_2) \geqslant R(Y_1) + R(Y_3) + 4 + R(Y_2)$, contradiction. □

Lemma 9. *Suppose K has a big component. Let $\mathscr{Y}$ be a dangerous family covering u and the set of big-neighbours of s. If u belongs to a small component C, then $C \subseteq \cap\mathscr{Y}$ and each $v \in \Gamma_K(s) - u$ belongs to either a boring component disjoint from $\cup\mathscr{Y}$ or a big component.*

Proof. Since u belongs to a small component, each set in $\mathscr{Y}$ is disconnected, so by (6.2), out-monotone. Suppose $\mathscr{Y} = \{Y_1\}$. $Y_1 \neq V$ so there exists a connected component X of $K - s$ not contained in Y_1. Then, since Y_1 contains all the big-neighbours of s, we have, by (6.1), $2 + 1 \geqslant d_K(s, X - Y_1) + 1 \geqslant d_K(s, Y_1) \geqslant 4$, contradiction. So $|\mathscr{Y}| \geqslant 2$, let $Y_1, Y_2 \in \mathscr{Y}$. By (6.1) applied to C and Y_i, and $u \in Y_i$, we have $C \subseteq Y_i$ for all $Y_i \in \mathscr{Y}$. Hence $C \subseteq \cap\mathscr{Y}$.

To prove the second statement, let X be a not big component of K with $X \cap (\Gamma_K(s) - u) \neq \emptyset$. Then $1 \leqslant d_K(s, X) \leqslant 2$. By (6.3), $Y_1 - Y_2$ is tight in-monotone, hence connected by (6.2), thus, since by definition $Y_1 - Y_2$ contains a big-neighbour, $(Y_1 - Y_2) \cap X = \emptyset$. By (6.3), $\overline{d}_H(Y_1, Y_2) = 1$, thus $Y_1 \cap Y_2 \cap X = \emptyset$. It follows that $Y_1 \cap X = \emptyset$. So $X \cap \cup\mathscr{Y} = \emptyset$. Then, since Y_1 is out-monotone, $2 \leqslant R(Y_1) \leqslant R(X) \leqslant d_K(X) = d_K(s, X) \leqslant 2$, so X is a boring component of K. □

We provide here a first result on complete admissible splitting off, an easy consequence of Lemma 8, which will be useful later in the general case.

Lemma 10. *If K has no odd or big component, then there is a complete admissible splitting off in K.*

Proof. After an admissible splitting, both properties are preserved, so we only have to show that there is an admissible pair. Otherwise, by Lemma 8, $K - s$ has a unique small component. This is a contradiction because in both cases the number of small components is even ($d_K(s)$ being even). □

5. Configuration and obstacle

We denote by $\mathbb{B}$ the set of in-monotone connected components B of G satisfying $R(B) = Q_E(B) = 2$. When $Q(G)$ is even, these sets will be boring components in an optimal extension.

We say that G contains a *configuration* if $Q(G)$ is even, there exist a unique connected component C of G with $Q_E(C) = 1$, and families $\mathscr{X}$ and $\mathscr{Y}$ of subsets of $V - \cup\mathbb{B}$; $\mathscr{X} \cup \mathbb{B}$ is an optimal in-monotone subpartition of G; $\mathscr{Y}$ consists of out-monotone sets Y_i, containing C, containing or disjoint from each member of $\mathscr{X}$, satisfying $Q_E(Y_i) \leqslant q_E(Y_i) + 1$, whose union covers all members of $\mathscr{X}$.

We say that an optimal extension H of G contains an *obstacle* if $Q(G)$ is even, there exists a unique small component C, it satisfies $Q_E(C) = 1$, and there exists a dangerous family $\mathscr{Y}$ covering v_C and the set of big-neighbours of s. Note that, by (6.2) and (6.1), $\mathscr{Y}$ consists of out-monotone sets containing C.

Theorem 11. *Let $H = (V + s, E + F)$ be an optimal extension of $G = (V, E)$. Then G contains a configuration if and only if H contains an obstacle.*

Proof. In both cases, by definition, $Q(G)$ is even.

Suppose G contains a configuration, then choose one with $\mathscr{X}$ and $\mathscr{Y}$ minimal. Then $q_E(X) \geqslant 1$ for all $X \in \mathscr{X}$ and each $Y_i \in \mathscr{Y}$ contains a set $X_i \in \mathscr{X}$ not contained in C. Since $\mathscr{X} \cup \mathbb{B}$ is an optimal subpartition, each $X \in \mathscr{X}$ is tight by (7.1) and in-monotone therefore connected by (6.2). Thus if $C \cap X \neq \emptyset$, $X \in \mathscr{X}$, then $X \subseteq C$. By (7.2), $d_F(C) = Q_E(C) = 1$, so C is a small component. Then, by (7.2), $2 \leqslant d_F(C) + d_F(X_i) \leqslant d_F(Y_i) = Q_E(Y_i) \leqslant q_E(Y_i) + 1$, so each Y_i is dangerous. From the definition of the configuration, their union covers all big-neighbours of s.

Suppose that H contains an obstacle. By parity, there exists a big component. Lemma 9 applies to v_C and $\mathscr{Y} = \{Y_1, \ldots, Y_k\}$, so $Y_i \subseteq V - \cup\mathscr{B}_H$. By $Q_E(C) = 1$, v_C belongs to a tight in-monotone set $X_{v_c} \subset C$. For a big-neighbour v in some Y_i, let X_v be the minimal tight in-monotone set containing v. By (6.3), for $j \neq i$, $Y_i - Y_j$ is tight in-monotone. Hence $X_v \subseteq Y_i - Y_j$, $\forall i \neq j$. Therefore $X_v \subseteq \bigcap_{j \neq i} (Y_i - Y_j) = Y_i - \bigcup_{j \neq i} Y_j$. Let $\mathscr{X} = \{X_{v_C}\} \cup \{X_v : v \text{ big-neighbour}\}$.

Clearly each Y_i contains or is disjoint from each member of $\mathscr{X} \cup \mathscr{B}_H$. By (6.3), the members of $\mathscr{X}$ are disjoint (they are also disjoint from the members of $\mathscr{B}_H$). By Lemma 9, $\mathscr{X} \cup \mathscr{B}_H$ covers $\Gamma(s)$, every $X \in \mathscr{X} \cup \mathscr{B}_H$ is tight so, by (7.1), $\mathscr{X} \cup \mathscr{B}_H$ is an optimal subpartition of V in G. By Y_i dangerous and by (7.2), $q_E(Y_i) + 1 \geqslant d_F(Y_i) = Q_E(Y_i)$. For every $B \in \mathscr{B}_H$, $C \cap B = \emptyset$ thus $\mathscr{X} \cup \mathscr{B}_H$ is in-monotone. Moreover we have $2 = R(B) = q_E(B) \leqslant Q_E(B) = d_H(B) = 2$. Therefore $\mathscr{B}_H = \mathbb{B}$. □

6. Complete admissible splitting off

Let H be an optimal extension of G. This section provides a complete admissible splitting off when H contains no obstacle. The case when H contains an obstacle is handled in Theorem 13. In Section 4, we have seen that when a big-neighbour belongs to no admissible pair, the graph can easily be described. This led us to use *allowed* pairs, that is admissible pairs su, sv with at least one of u and v being a big-neighbour.

Theorem 12. *If H contains no obstacle, then there is a complete admissible splitting off in H.*

Proof. We may assume that H has a big component, otherwise we are done by Lemma 10.

Step 1: If there exists a unique small component C of H, we prove that we can destroy C (by moving sv_C, or by splitting off an allowed pair containing sv_C). Since there is no obstacle in H, one of the following cases happens.

1. $Q(G)$ is odd. In fact this case is impossible by construction of the optimal extension.
2. $Q_E(C) \neq 1$. Then $Q_E(C) = 0$ and v_C belongs to no tight in-monotone set, so there exists a minimal tight out-monotone set X containing v_C. By (6.3), an out-monotone tight set containing v_C contains X. Since X is out-monotone and $d_H(X) = R(X) \geqslant 2$, we have $X \nsubseteq C$ hence there exists a connected component Z in $H - s$ with $X \cap Z \neq \emptyset$. Then, by (6.1), $Z \cap \Gamma_H(s) \neq \emptyset$. Let $x \in X \cap Z$. Replace sv_C by sx, the new graph still satisfies (1) and has no small component.
3. There is an allowed pair containing sv_C. Split it off.

Let H' be the graph obtained after Step 1 (eventually, $H' = H$).

Step 2: H' has none or several small components. Split off allowed pairs as long as possible. If there is no big component anymore, then, by Lemma 10, find a complete admissible splitting off. Otherwise, Lemma 8 applied for a big-neighbour u implies that the new graph H'' has a unique small component C and a unique big component D (which is in fact odd as well). If H' contains no small component then C contains a split edge ab which is not a bridge. We show that this is also true if H' contains several small components. Let $X \neq C$ be a small component of H'. Since C is unique in H'', sv_X has been split off previously, (let's say with sy). Note that the new edge yv_X is a bridge in H''. Hence, by Lemma 8 and (6.4), yv_X is not in D. So it is in $H'' - D$. Since the splittings were allowed, it follows that C contains a split edge and the last one ab is not a bridge.

Let us unsplit ab that is replace the edge ab by sa and sb. Then there is no small component anymore. Therefore by Lemma 8 there exists an admissible pair $\{su, sv\}$. Since D is the union of two dangerous sets containing u in H'' and also in the graph after the unsplitting, su belongs to no admissible pair su, sx with $x \in D$, so necessarily $v \in C$. After splitting this pair, the new graph has no odd component, so Lemma 10 provides a complete admissible splitting off. □

7. Augmentation

By applying the above splitting result we can solve the augmentation problem.

Theorem 13. *Let $G = (V, E)$ be a graph and R a symmetric semi-monotone function on V. If G contains no configuration, then $Opt(R, G) = \lceil Q(G)/2 \rceil$, otherwise $Opt(R, G) = \lceil Q(G)/2 \rceil + 1$.*

Proof. The following lemmas prove the theorem. □

Lemma 14. *$Opt(R, G) \geqslant \lceil Q(G)/2 \rceil$. If G contains a configuration, then the inequality is strict.*

Proof. For a minimum set M of edges such that $G+M$ satisfies (1), since for any edge f, $Q_{E+f}(V) \geqslant Q(G)-2$, we have $0 \geqslant Q_{E+M}(V) \geqslant Q(G)-2|M|$. Now suppose G contains a configuration and equality holds. Let H be the extension of G from which we can obtain $G+M$ by a complete admissible splitting off. By the minimality of M, H is an optimal extension of G. Since G contains a configuration, by Theorem 11, H contains an obstacle. Then sv_C belongs to one of the admissible pairs, say $\{su, sv_C\}$. Since sv_C belongs to no allowed pair, by Lemma 9, u belongs to a boring set B. Split off $\{su, sv_C\}$, denote by H' the new graph. Note that H' is an optimal extension of $G+uv_C$. Note that $Y_i' = Y_i \cup B$ is dangerous in H' because $R(Y_i \cup B)+1 \geqslant R(Y_i)+1 \geqslant d_H(Y_i)+d_H(B)-2 \geqslant d_H(Y_i \cup B)-2 = d_{H'}(Y_i \cup B)$ and, by (6.2), it is also out-monotone. $C' = C \cup B$ has a unique neighbour $v_{C'}$ of s and $1 = d_{H'}(C') \geqslant Q_{E+uv_C}(C') \geqslant Q_{E+uv_C}(B) \geqslant R(B) - d_{H'-s}(B) = 1$. Then $v_{C'}, C', Y_1', \dots Y_k'$ form an obstacle in H', and $|\mathscr{B}_{H'}| = |\mathscr{B}_H| - 1$. Repeating this operation, we may assume $\mathscr{B}_H = \emptyset$. Then sv_C belongs to no admissible pair, contradiction. □

Lemma 15. *$Opt(R, G) \leqslant \lceil Q(G)/2 \rceil + 1$. If G contains no configuration, then the inequality is strict.*

Proof. Let H be an optimal extension of G. By Theorem 2, $|F| = 2\lceil Q(G)/2 \rceil$. If G contains no configuration, then, by Theorem 11, H contains no obstacle and hence, by Theorem 12, there exists a complete admissible splitting off, and the strict inequality follows. Otherwise, we split off admissible pairs as long as possible. In the new graph, by Lemma 8, there exist a unique small and a unique big component, C and D. We add an edge between C and D. Since there is no odd component anymore, by Lemma 10, we have a complete admissible splitting off and the inequality follows. □

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