Discrete Mathematics 120 (1993) 211–214
North-Holland

Communication

A characterization of intersection graphs of the maximal rectangles of a polyomino

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Communicated by C. Berge
Received 2 February 1993

Abstract

The interior of an orthogonal polygon drawn on a regular grid of the plane defines a set of cells (or squares) called a polyomino. We prove that the intersection graph of the maximal rectangles contained in a polyomino is slightly triangulated or has a star cutset.

1. Introduction

Shearer proved in [1] that the intersection graph of the maximal rectangles of a polyomino is perfect. This problem was posed in [3]. It seems difficult to deduce a characterization for this class of graphs with Shearer's proof. We would like to find simple reasons why these graphs are perfect. Here we present a new proof of Shearer's theorem, which uses a recent result on minimal imperfect graphs [2].

As usual in Perfect Graph Theory, the subgraphs we are interested in are those which are induced by a subset of vertices. For an introduction to perfect graph Theory see [4] or [5]. We write $H \subseteq G$ to represent the fact that H is an induced subgraph of G. Let P_k denote the chordless path with k vertices and C_k the chordless cycle with k vertices. Let $\bar{G}$ denote the complement of the graph G. We say that a vertex x is *loose* if its neighbourhood is P_4-free. In [2] we proved the following.

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Lemma 1. *A minimal imperfect graph G without chordless cycle of length 5 or more has no loose vertex.*

A graph is called *slightly triangulated* if it contains no chordless cycle with 5 or more vertices, and if every induced subgraph has a loose vertex. The lemma above implies that these graphs are perfect. They generalize triangulated graphs. We first consider the cycles of the intersection graph $G(P)$ of the maximal rectangles of a polyomino P, and show that the length of a chordless cycle in $G(P)$ is less than 5. Then we establish that every induced subgraph of G has a loose vertex or a star cutset. At this point we have the following.

Theorem 1. *The intersection graph $G(P)$ of the maximal rectangles of a polyomino P has no chordless cycle of length 5 or more, and every induced subgraph contains a loose vertex or has a star cutset.*

Hence $G(P)$ is perfect. In the following G stands for the intersection graph $G(P)$ of the maximal rectangles of a polyomino P. The graph G is derived from P as follows. Let the vertices of G be the maximal rectangles contained in P. Let two vertices of G be joined by an edge iff the rectangles have a non-empty intersection.

2. Cycles of $G(P)$

A (geometric) vertex of a maximal rectangle of P will be called a *corner* to avoid confusion with the vertices of the graph. The *interior* of a rectangle is the interior in the usual Euclidean topological sense. Let x and y be two maximal rectangles of P. Since the rectangles are maximal we have the following.

Proposition 1. *A corner of x is in the interior of y if and only if a corner of y is in the interior of x.*

The intersection of x and y is called a *cross* if no corner of a rectangle is contained in the interior of the other. Otherwise the intersection is called a *step.*

Let $C=(r_1,\dots,r_k)$, $k\geqslant 5$, be a chordless cycle of G. Let a_i be a cell of $r_i\cap r_{i+1}$. We note $W_{i,j}$ the polygonal line from a_i to a_j through the segments $[a_i, a_{i+1}], [a_{i+1}, a_{i+2}], \dots, [a_{j-1}, a_j]$. We are counting modulo k. We assume that C is chosen with a minimum number of step intersections. The smallest simply connected set containing C is an orthogonal polygon for which the rectangles of C are maximal too. We can thus assume that P is this polyomino. Suppose that there is an i such that $r_i\cap r_{i+1}$ is a step. Then $r_{i-1}\cap r_i$ and $r_{i+1}\cap r_{i+2}$ are crosses. Otherwise, if for example $r_{i-1}\cap r_i$ is a step then one of the rectangles of the sequence $(r_{i+2},\dots,r_{i-2})$ meets r_i. (Here we use the simple connectivity of P and the maximality of the rectangles). This would contradict that C is chordless. Therefore we can assume that $r_{i-1}\cap r_i$ and

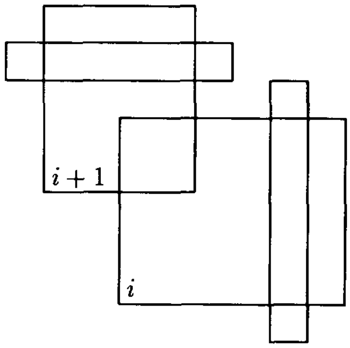

Fig. 1.

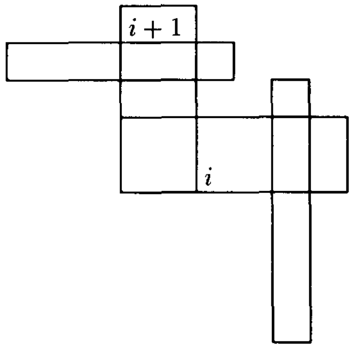

Fig. 2.

$r_{i+1} \cap r_{i+2}$ are crosses. The sides of r_i crossed by r_{i-1} and the sides of r_{i+1} crossed by r_{i+2} cannot be parallel. Otherwise $W_{i+1,i}$ would contradict the maximality of r_i or r_{i+1}. Therefore these sides are perpendicular. (See Fig. 1.) We can shrink the rectangles r_i and r_{i+1} so that their intersection become a cross. (See Fig. 2.) These rectangles are still maximal in the polyomino generated by the shrinked rectangles and the other rectangles of C. We have decreased the number of step intersections of C. In fact we can assume that all intersections are crosses. So, the cells a_i can be chosen so that the a_i are exactly the vertices of an orthogonal polygon. It is a well-known result that an orthogonal polygon admits a reflex vertex (its interior angle is equal to $3\pi/2$) as soon as the number of its vertices is greater than 4. Let a_i be this vertex. It is easy to check that r_i cannot be maximal. Hence we have proved the following.

Theorem 2. *The length of a chordless cycle in G is at most* 4.

3. Star cutset or loose vertex

Let L be a maximal rectangle of the polyomino P with the lowest top row. Let l be the top row of L. It is easy to see that the following holds.

Proposition 2. *Every rectangle which meets L meets also l.*

Now we shall show that if the neighbourhood of L in G contains a P_4 then G contains a star cutset. Let T_1, T_2, T_3, T_4 be a P_4 in the neighbourhood of L. Without loss of generality we assume that these rectangles are numbered so that the top row of T_3 is higher than the top row of T_2. Then the low row of T_2 is lower than the low row of T_3. Otherwise T_2 would meet T_4. (See Fig. 3.) The set of vertices $T_3 \cup (\Gamma_G(T_3) \setminus T_4)$ is a star cutset deconnecting T_1 and T_4. Because if a chain of rectangles goes from T_1 to T_4 without meeting T_3 then the top left corner of T_3 would

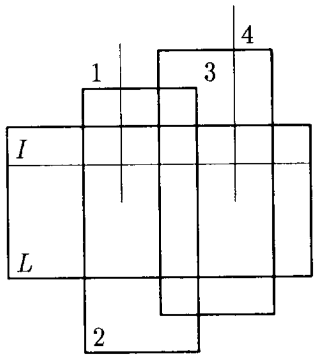

Fig. 3.

be in the interior of P, contradicting the maximality of T_3. This ends the proof of Theorem 1.

References

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