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Periodic Solutions of Nonautonomous Stage-Structured Cooperative System

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Abstract—By using the continuation theorem of coincidence degree theory, the existence of a positive periodic solution for the nonautonomous stage-structured cooperative system is established.

Keywords—Stage-structured and cooperative system, Periodic solution, Continuation theorem, Topological degree theory.

1. INTRODUCTION

In a natural world, there exist many individuals of species which experience two stages in the lifetime, i.e., immature stage and mature stage, for example, animal and amphibian. Therefore, to make the models more practical, species are usually considered by dividing the individuals into two stages. Recently, there exist many papers [1–3] in the literature which investigate some stage-structured predator-prey systems, however, the papers which investigate stage-structured cooperative systems are scarce. In this paper, we study stage-structured nonautonomous cooperative system of two species. Consider the following model:

$$\begin{aligned} x_1'(t) &= \alpha(t)x_2(t) - r_1(t)x_1(t) - \beta(t)x_1(t) - \eta_1(t)x_1^2(t), \\ x_2'(t) &= \beta(t)x_1(t) - r_2(t)x_2(t) - \eta_2(t)x_2^2(t) + b(t)x_2(t)y(t), \\ y'(t) &= y(t)[R(t) - a(t)y(t) + c(t)x_2(t)], \end{aligned} \tag{1}$$

where $x_1(t)$ denotes the density of immaturity of species X at time t, $x_2(t)$ denotes the density of maturity of species X at time t, $y(t)$ denotes the density of species Y at time t, $r_1(t)$ is the death rate of the immature of species X, and $r_2(t)$ is the death rate of the mature of species X. $\alpha(t)$ is the birth rate of species X, and $\beta(t)$ is the change rate of species X from the immature to mature, which is directly proportional to the density of the immature. Our purpose in the paper is, by using the continuation theorem which was proposed in [4] by Gaines and Mawhin, to establish the existence of at least one positive w-periodic solution of system (1). For the work

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concerning the existence of periodic solutions of delay differential equations which was done by using coincidence degree theory, we refer to [5–7] and references cited therein.

2. EXISTENCE OF A POSITIVE PERIODIC SOLUTION

In this section, by using Mawhin's continuation theorem we shall show the existence of at least one positive periodic solution of system (1). To do so, we need to make some preparations.

Let X, Y be real Banach space, let $L : \operatorname{Dom} L \subset X \to Y$ be a Fredholm mapping of index zero, and let $P : X \to X$, $Q : Y \to Y$ be continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$, and $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. Denote by L_p the restriction of L to $\operatorname{Dom} L \cap \operatorname{Ker} P$, $K_p : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ the inverse (to L_p), and $J : \operatorname{Im} Q \to \operatorname{Ker} L$ an isomorphism of $\operatorname{Im} Q$ onto $\operatorname{Ker} L$.

For convenience, we introduce Mawhin's continuation theorem [4, p. 40] as follows.

LEMMA 1. *Let $\Omega \subset X$ be an open bounded set and let $N : X \to Y$ be a continuous operator which is L-compact on $\bar{\Omega}$ (i.e., $QN : \bar{\Omega} \to Y$ and $K_p(I - Q)N : \bar{\Omega} \to Y$ are compact). Assume the following.*

(i) *For each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$.*
(ii) *For each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$.*
(iii) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \operatorname{Dom} L$.

In what follows, we use the following notation:

$$\bar{f} = \frac{1}{w} \int_0^w f(t)\, dt, \qquad f^l = \min_{t \in [0,w]} |f(t)|, \qquad f^u = \max_{t \in [0,w]} |f(t)|,$$

where f is a periodic continuous function with period $w > 0$.

In system (1), we always assume the following.

ASSUMPTION H1. *$\alpha(t)$, $r_1(t)$, $r_2(t)$, $\beta(t)$, $\eta_1(t)$, $\eta_2(t)$, $b(t)$, $R(t)$, $a(t)$, $c(t)$ are positive periodic continuous functions with period $w > 0$.*

Now we state our fundamental theorem about the existence of a positive w-periodic solution of system (1).

THEOREM 2. *In addition to Assumption H1, we assume the following:*

(i) $b^l \bar{R} > \bar{r}_2 a^u$;
(ii) $\eta_2^l a^l > b^u c^u$.

Then system (1) has at least one positive w-periodic solution.

PROOF. Consider the system

$$\begin{aligned}
\frac{dy_1}{dt} &= \alpha(t) e^{y_2(t) - y_1(t)} - r_1(t) - \beta(t) - \eta_1(t) e^{y_1(t)}, \\
\frac{dy_2}{dt} &= \beta(t) e^{y_1(t) - y_2(t)} - r_2(t) - \eta_2(t) e^{y_2(t)} + b(t) e^{y_3(t)}, \\
\frac{dy_3}{dt} &= R(t) - a(t) e^{y_3(t)} + c(t) e^{y_2(t)},
\end{aligned} \tag{2}$$

where $\alpha(t)$, $r_1(t)$, $r_2(t)$, $\beta(t)$, $\eta_1(t)$, $\eta_2(t)$, $b(t)$, $R(t)$, $a(t)$, $c(t)$ are the same as those in Assumption H1. It is easy to see that if system (2) has a w-periodic solution $(y_1^*(t), y_2^*(t), y_3^*(t))^\top$, then $(e^{y_1^*(t)}, e^{y_2^*(t)}, e^{y_3^*(t)})^\top$ is a positive w-periodic solution of system (1). Therefore, for (1) to have at least one positive w-periodic solution, it is sufficient that (2) has at least one w-periodic solution. In order to apply Lemma 1 to system (2), we take

$$X = Y = \left\{ (y_1(t), y_2(t), y_3(t))^\top \in C\left(R, R^3\right) : y_i(t + w) = y_i(t),\ i = 1, 2, 3 \right\}$$

and

$$\left\|(y_1(t),y_2(t),y_3(t))^\top\right\| = \max_{t\in[0,w]}|y_1(t)| + \max_{t\in[0,w]}|y_2(t)| + \max_{t\in[0,w]}|y_3(t)|,$$

here $|\cdot|$ denotes the Euclidean norm in R. With the norm $\|\cdot\|$, X is a Banach space. Set $L : \operatorname{Dom} L \cap X$, $L(y_1(t),y_2(t),y_3(t))^\top = (y_1'(t),y_2'(t),y_3'(t))^\top$, where

$$\operatorname{Dom} L = \left\{(y_1(t),y_2(t),y_3(t))^\top \in C^1\left(R,R^3\right)\right\}, \qquad N : X \to X,$$

and

$$N\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \alpha(t)e^{y_2(t)-y_1(t)} - r_1(t) - \beta(t) - \eta_1(t)e^{y_1(t)} \\ \beta(t)e^{y_1(t)-y_2(t)} - r_2(t) - \eta_2(t)e^{y_2(t)} + b(t)e^{y_3(t)} \\ R(t) - a(t)e^{y_3(t)} + c(t)e^{y_2(t)} \end{bmatrix}.$$

Define two projectors P and Q as

$$P\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = Q\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{w}\int_0^w y_1(t)\,dt \\ \frac{1}{w}\int_0^w y_2(t)\,dt \\ \frac{1}{w}\int_0^w y_3(t)\,dt \end{bmatrix}, \qquad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in X.$$

Clearly, $\operatorname{Ker} L = R^3$, $\operatorname{Im} L = \{(y_1,y_2,y_3)^\top \in X : \int_0^w y_i(t)\,dt = 0,\ i = 1,2,3\}$ is closed in X and $\operatorname{dimKer} L = \operatorname{codim} \operatorname{Im} L = 3$. Hence, L is a Fredholm mapping of index zero. Furthermore, through an easy computation, we can find that the inverse K_p of L_p has the form $K_p : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$,

$$K_p\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \int_0^t y_1(s)\,ds - \frac{1}{w}\int_0^w\int_0^\eta y_1(t)\,dt\,d\eta \\ \int_0^t y_2(s)\,ds - \frac{1}{w}\int_0^w\int_0^\eta y_2(t)\,dt\,d\eta \\ \int_0^t y_3(s)\,ds - \frac{1}{w}\int_0^w\int_0^\eta y_3(t)\,dt\,d\eta \end{bmatrix}.$$

Obviously, we can prove that QN and $K_p(I-Q)N$ are continuous by Lebesgue theorem and that $QN(\bar\Omega)$, $K_p(I-Q)N(\bar\Omega)$ are relatively compact for any open bounded set $\Omega \subset X$ by the Arzela-Ascoli theorem. Therefore, N is L-compact on $\bar\Omega$ for any open bounded set $\Omega \subset X$. Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have

$$\begin{aligned} \frac{dy_1}{dt} &= \lambda\left[\alpha(t)e^{y_2(t)-y_1(t)} - r_1(t) - \beta(t) - \eta_1(t)e^{y_1(t)}\right], \\ \frac{dy_2}{dt} &= \lambda\left[\beta(t)e^{y_1(t)-y_2(t)} - r_2(t) - \eta_2(t)e^{y_2(t)} + b(t)e^{y_3(t)}\right], \\ \frac{dy_3}{dt} &= \lambda\left[R(t) - a(t)e^{y_3(t)} + c(t)e^{y_2(t)}\right]. \end{aligned} \tag{3}$$

Suppose that $(y_1(t),y_2(t),y_3(t))^\top \in X$ is a solution of system (3) for a certain $\lambda \in (0,1)$. By integrating (3) over the interval $[0,w]$, we obtain

$$\int_0^w \alpha(t)e^{y_2(t)-y_1(t)}\,dt = \int_0^w \eta_1(t)e^{y_1(t)}\,dt + \int_0^w (r_1(t)+\beta(t))\,dt, \tag{4}$$

$$\int_0^w \beta(t)e^{y_1(t)-y_2(t)}\,dt + \int_0^w b(t)e^{y_3(t)}\,dt = \int_0^w r_2(t)\,dt + \int_0^w \eta_2(t)e^{y_2(t)}\,dt, \tag{5}$$

and

$$\int_0^w c(t)e^{y_2(t)}\,dt + \int_0^w R(t)\,dt = \int_0^w a(t)e^{y_3(t)}\,dt. \tag{6}$$

From (3)–(6), it follows that

$$\begin{aligned}\int_0^w |y_1'(t)|\,dt &\le \int_0^w \alpha(t)e^{y_2(t)-y_1(t)}\,dt + \int_0^w (r_1(t)+\beta(t))\,dt + \int_0^w \eta_1(t)e^{y_1(t)}\,dt \\ &= 2\overline{(r_1+\beta)}w + 2\int_0^w \eta_1(t)e^{y_1(t)}\,dt,\end{aligned} \tag{7}$$

$$\begin{aligned}\int_0^w |y_2'(t)|\,dt &\le \int_0^w \beta(t)e^{y_1(t)-y_2(t)}\,dt + \int_0^w b(t)e^{y_3(t)}\,dt + \int_0^w r_2(t)\,dt + \int_0^w \eta_2(t)e^{y_2(t)}\,dt \\ &= 2\bar{r}_2 w + 2\int_0^w \eta_2(t)e^{y_2(t)}\,dt,\end{aligned} \tag{8}$$

and

$$\int_0^w |y_3'(t)|\,dt \le \int_0^w c(t)e^{y_2(t)}\,dt + \int_0^w R(t)\,dt + \int_0^w a(t)e^{y_3(t)}\,dt = 2\int_0^w a(t)e^{y_3(t)}\,dt. \tag{9}$$

Choose $t_i \in [0,w]$, $i=1,2,3$, such that

$$y_i(t_i) = \max_{t\in[0,w]} y_i(t), \qquad i=1,2,3.$$

Then it is clear that

$$y_i'(t_i) = 0, \qquad i=1,2,3.$$

From this and (3), we have

$$\alpha(t_1)e^{y_2(t_1)-y_1(t_1)} - r_1(t_1) - \beta(t_1) - \eta_1(t_1)e^{y_1(t_1)} = 0, \tag{10}$$

$$\beta(t_2)e^{y_1(t_2)-y_2(t_2)} - r_2(t_2) - \eta_2(t_2)e^{y_2(t_2)} + b(t_2)e^{y_3(t_2)} = 0, \tag{11}$$

$$R(t_3) - a(t_3)e^{y_3(t_3)} + c(t_3)e^{y_2(t_3)} = 0. \tag{12}$$

Equation (10) gives

$$\eta_1(t_1)e^{2y_1(t_1)} = \alpha(t_1)e^{y_2(t_1)} - (r_1(t_1)+\beta(t_1))e^{y_1(t_1)},$$

which implies that

$$\eta_1^l e^{2y_1(t_1)} < \alpha^u e^{y_2(t_2)}. \tag{13}$$

Equation (11) gives

$$\eta_2^l e^{2y_2(t_2)} = \beta(t_2)e^{y_1(t_2)} + b(t_2)e^{y_3(t_2)+y_2(t_2)} - r_2(t_2)e^{y_2(t_2)},$$

which implies that

$$\eta_2^l e^{2y_2(t_2)} < \beta^u e^{y_1(t_1)} + b^u e^{y_3(t_3)+y_2(t_2)}. \tag{14}$$

Combining (14) and (12) gives

$$\begin{aligned}\eta_2^l e^{2y_2(t_2)} &< \beta^u e^{y_1(t_1)} + \frac{b^u}{a^l}\left[R(t_3)e^{y_2(t_2)} + c(t_3)e^{2y_2(t_2)}\right] \\ &< \beta^u e^{y_1(t_1)} + \frac{b^u}{a^l}\left[R^u e^{y_2(t_2)} + c^u e^{2y_2(t_2)}\right],\end{aligned}$$

that is,

$$\left(\eta_2^l a^l - b^u c^u\right) e^{2y_2(t_2)} < \beta^u a^l e^{y_1(t_1)} + b^u R^u e^{y_2(t_2)}. \tag{15}$$

Substituting (13) into (15), we have

$$\left(\eta_2^l a^l - b^u c^u\right) e^{2y_2(t_2)} < \beta^u a^l \sqrt{\frac{\alpha^u}{\eta_1^l}} e^{y_2(t_2)/2} + b^u R^u e^{y_2(t_2)},$$

from which it follows that there exists a positive constant ρ_2 such that

$$e^{y_2(t_2)} < \rho_2. \tag{16}$$

From (13) and (16), it follows that there exists a positive constant ρ_1 such that

$$e^{y_1(t_1)} < \rho_1. \tag{17}$$

From (12) and (16), it follows that there exists a positive constant ρ_3 such that

$$e^{y_3(t_3)} < \rho_3. \tag{18}$$

Equation (6) implies that

$$\textstyle\int_0^w e^{y_3(t)}\, dt > \dfrac{w\bar{R}}{a^u}. \tag{19}$$

Equation (5) implies that

$$\eta_2^u \int_0^w e^{y_2(t)}\, dt > \int_0^w \eta_2(t) e^{y_2(t)}\, dt > \int_0^w b(t) e^{y_3(t)}\, dt - w\bar{r}_2 > b^l \int_0^w e^{y_3(t)}\, dt - w\bar{r}_2.$$

Thus,

$$\int_0^w e^{y_2(t)}\, dt > \frac{\left(b^l \bar{R} - \bar{r}_2 a^u\right) w}{\eta_2^u a^u} > 0. \tag{20}$$

Multiplying the first equation of system (3) by $e^{y_1(t)}$ and integrating over $[0, w]$ gives

$$\int_0^w \eta_1(t) e^{2y_1(t)}\, dt + \int_0^w (r_1(t) + \beta(t)) e^{y_1(t)}\, dt = \int_0^w \alpha(t) e^{y_2(t)}\, dt,$$

from which it follows that there exists a point $t_1^* \in [0, w]$ such that

$$w\left[\eta_1\left(t_1^*\right) e^{2y_1(t_1^*)} + \left(r_1\left(t_1^*\right) + \beta\left(t_1^*\right)\right) e^{y_1(t_1^*)}\right] > \alpha^l \int_0^w e^{y_2(t)}\, dt. \tag{21}$$

Substituting (20) into (21) gives

$$\eta_1^u e^{2y_1(t_1^*)} + (r_1 + \beta)^u e^{y_1(t_1^*)} > \frac{\alpha^l \left(b^l \bar{R} - \bar{r}_2 a^u\right)}{\eta_2^u a^u}.$$

Thus,

$$2\eta_1^u e^{y_1(t_1^*)} > \sqrt{\frac{\left[(r_1 + \beta)^u\right]^2 \eta_2^u a^u + 4\eta_1^u \alpha^l \left(b^l \bar{R} - \bar{r}_2 a^u\right)}{\eta_2^u a^u}} - (r_1 + \beta)^u > 0. \tag{22}$$

From (7)–(9) and (17)–(19), we have

$$\int_0^w \left|y_1'(t)\right|\, dt \le 2\overline{(r_1 + \beta)} w + 2\eta_1^u \rho_1 w \stackrel{\text{def}}{=} d_1, \tag{23}$$

$$\int_0^w \left|y_2'(t)\right| dt \le 2\bar{r}_2 w + 2\eta_2^u \rho_2 w \stackrel{\text{def}}{=} d_2, \tag{24}$$

and

$$\int_0^w |y_3'(t)|\, dt \le 2\alpha^u \rho_3 w \stackrel{\text{def}}{=} d_3. \tag{25}$$

From (19), (20), and (22), it follows that there exist three constants ρ_1^*, ρ_2^*, and ρ_3^* and three points $\xi_i \in [0,w]$, $i = 1,2,3$, such that

$$y_i(\xi_i) > -\rho_i^*, \qquad i = 1,2,3. \tag{26}$$

Since for all $t \in [0,w]$,

$$y_i(t) = y_i(\xi_i) - \int_{\xi_i}^{t} y_i'(s)\, ds, \qquad i = 1,2,3,$$

from (23)–(25) and (26), it follows that for $i = 1,2,3$,

$$y_i(t) > -\rho_i^* - \int_0^w |y_i'(s)|\, ds > -\rho_i^* - d_i. \tag{27}$$

From (16)–(18) and (27), we can obtain

$$|y_i(t)| \le \max\{|\ln \rho_i|, \rho_i^* + d_i\} \stackrel{\text{def}}{=} R_i, \qquad i = 1,2,3.$$

Clearly, R_i ($i = 1,2,3$) are independent of λ. Denote $M = R_1 + R_2 + R_3 + R_0$; here, R_0 is taken sufficiently large such that each solution $(\alpha^*, \beta^*, \gamma^*)^\top$ of the following system:

$$\begin{aligned} \bar{\alpha} e^{\beta-\alpha} - \bar{r}_2 - \bar{\beta} - \bar{\eta}_1 e^{\alpha} &= 0, \\ \bar{\beta} e^{\alpha-\beta} - \bar{r}_2 - \bar{\eta}_2 e^{\beta} + \bar{b} e^{\gamma} &= 0, \\ \bar{R} - \bar{a} e^{\gamma} + \bar{c} e^{\beta} &= 0 \end{aligned} \tag{28}$$

satisfies $\|(\alpha^*, \beta^*, \gamma^*)^\top\| = |\alpha^*| + |\beta^*| + |\gamma^*| < M$, provided that system (28) has a solution or a number of solutions. Now we take $\Omega = \{(y_1(t), y_2(t), y_3(t))^\top \in X : \|(y_1, y_2, y_3)^\top\| < M\}$. This satisfies Condition (i) of Lemma 1. When $(y_1, y_2, y_3)^\top \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^3$, $(y_1, y_2, y_3)^\top$ is a constant vector in R^3 with $|y_1| + |y_2| + |y_3| = M$. If system (28) has a solution or a number of solutions, then

$$QN \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \bar{\alpha} e^{y_2 - y_1} - \bar{r}_2 - \bar{\beta} - \bar{\eta}_1 e^{y_1} \\ \bar{\beta} e^{y_1 - y_2} - \bar{r}_2 - \bar{\eta}_2 e^{y_2} + \bar{b} e^{y_3} \\ \bar{R} - \bar{a} e^{y_3} + \bar{c} e^{y_2} \end{bmatrix} \ne 0.$$

If system (28) does not have a solution, then naturally

$$QN \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \ne \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This proves that Condition (ii) of Lemma 1 is satisfied. Finally, we will prove that Condition (iii) of Lemma 1 is satisfied. To this end, we define $\phi : \operatorname{Dom} L \times [0,1] \to X$ by

$$\phi(y_1, y_2, y_3, \mu) = \begin{bmatrix} \bar{\alpha} e^{y_2 - y_1} - \bar{\eta}_1 e^{y_1} \\ -\bar{r}_2 - \bar{\eta}_2 e^{y_2} + \bar{b} e^{y_3} \\ \bar{R} - \bar{a} e^{y_3} \end{bmatrix} + \mu \begin{bmatrix} -\bar{r}_2 - \bar{\beta} \\ \bar{\beta} e^{y_1 - y_2} \\ \bar{c} e^{y_2} \end{bmatrix},$$

where $\mu \in [0,1]$ is a parameter. When $(y_1, y_2, y_3)^\top \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^3$, $(y_1, y_2, y_3)^\top$ is a constant vector in R^3 with $|y_1| + |y_2| + |y_3| = M$. We will show that when $(y_1, y_2, y_3)^\top \in$

$\partial\Omega \cap \operatorname{Ker} L$, $\phi(y_1, y_2, y_3, \mu) \neq 0$. If the conclusion is not true, then constant vector $(y_1, y_2, y_3)^\top$ with $|y_1| + |y_2| + |y_3| = M$ satisfies $\phi(y_1, y_2, y_3, \mu) = 0$. From

$$\begin{aligned} \bar{\alpha} e^{y_2 - y_1} - \bar{\eta}_1 e^{y_1} + \mu\left(-\bar{r}_2 - \bar{\beta}\right) &= 0, \\ -\bar{r}_2 - \bar{\eta}_2 e^{y_2} + \bar{b} e^{y_3} + \mu \bar{\beta} e^{y_1 - y_2} &= 0, \\ \bar{R} - \bar{a} e^{y_3} + \mu \bar{c} e^{y_2} &= 0, \end{aligned}$$

and following the argument of (16)–(18) and (26), we obtain

$$|y_i| < \max\{|\ln \rho_i|, \rho_i^*\}, \qquad i = 1, 2, 3.$$

Thus,

$$\sum_{i=1}^{3} |y_i| < \sum_{i=1}^{3} \max\{|\ln \rho_i|, \rho_i^*\} < M,$$

which contradicts the fact that $|y_1| + |y_2| + |y_3| = M$. Therefore, according to topological degree theory, we have

$$\begin{aligned} &\deg\left(JQN(y_1, y_2, y_3)^\top, \Omega \cap \operatorname{Ker} L, (0,0,0)^\top\right) \\ &= \deg\left(\phi(y_1, y_2, y_3, 1), \Omega \cap \operatorname{Ker} L, (0,0,0)^\top\right) \\ &= \deg\left(\phi(y_1, y_2, y_3, 0), \Omega \cap \operatorname{Ker} L, (0,0,0)^\top\right) \\ &= \deg\left(\left(\bar{\alpha} e^{y_2 - y_1} - \bar{\eta}_1 e^{y_1}, -\bar{r}_2 - \bar{\eta}_2 e^{y_2} + \bar{b} e^{y_3}, \bar{R} - \bar{a} e^{y_3}\right)^\top, \Omega \cap \operatorname{Ker} L, (0,0,0)^\top\right). \end{aligned}$$

Because of Condition (i) of Theorem 1, then the system of algebraic equations

$$\begin{aligned} \frac{\bar{\alpha} y}{x} - \bar{\eta}_1 x &= 0, \\ -\bar{r}_2 - \bar{\eta}_2 y + \bar{b} z &= 0, \\ \bar{R} - \bar{a} z &= 0 \end{aligned}$$

has a unique solution $(x^*, y^*, z^*)^\top$ which satisfies $x^* > 0$, $y^* > 0$, and $z^* > 0$, and thus,

$$\begin{aligned} &\deg\left(\left(\bar{\alpha} e^{y_2 - y_1} - \bar{\eta}_1 e^{y_1}, -\bar{r}_2 - \bar{\eta}_2 e^{y_2} + \bar{b} e^{y_3}, \bar{R} - \bar{a} e^{y_3}\right)^\top, \Omega \cap \operatorname{Ker} L, (0,0,0)^\top\right) \\ &= \operatorname{sign} \begin{vmatrix} -\bar{\eta}_1 x^* - \dfrac{\bar{\alpha} y^*}{x^*} & \dfrac{\bar{\alpha} y^*}{x^*} & 0 \\ 0 & -\bar{\eta}_2 y^* & \bar{b} z^* \\ 0 & 0 & -\bar{a} z^* \end{vmatrix} \\ &= -\operatorname{sign}\left[\left(\bar{\eta}_1 x^* + \frac{\bar{\alpha} y^*}{x^*}\right) \bar{\eta}_2 \bar{a} y^* z^*\right] = -1. \end{aligned}$$

Consequently,

$$\deg\left(JQN(y_1, y_2, y_3)^\top, \Omega \cap \operatorname{Ker} L, (0,0,0)^\top\right) \neq 0.$$

This shows that Condition (iii) of Lemma 1 is satisfied. By now Ω verifies all the requirements of Lemma 1 and then system (2) has at least one w-periodic solution. This completes the proof of Theorem 2.

3. AN EXAMPLE

We give a specific numerical example to test the validity of our result.

Consider the following nonautonomous stage-structured cooperative system:

$$
\begin{aligned}
x_1' &= (2+\cos t)x_2 - \left(3+\frac{\sin t}{2}\right)x_1 - \left(\frac{5}{2}+\sin t\right)x_1 - (10+\cos t)x_1^2, \\
x_2' &= \left(\frac{5}{2}+\sin t\right)x_1 - (2+\cos t)x_2 - (12+2\sin t)x_2^2 + (6+\sin t)x_2 y, \\
y' &= y\left(1+\frac{\cos t}{2} - \frac{4+\sin t}{2}y + \frac{3+\sin t}{2}x_2\right).
\end{aligned}
\tag{29}
$$

In Theorem 2, $b(t) = 6+\sin t$, $R(t) = 1+(\cos t)/2$, $a(t) = (4+\sin t)/2$, $r_2(t) = 2+\cos t$, $\eta_2(t) = 12+2\sin t$, $c(t) = (3+\sin t)/2$. It is easy to verify $b^l\bar{R} > \bar{r}_2 a^u$ and $\eta_2^l a^l > b^u c^u$. By Theorem 2, system (29) has at least one 2π-periodic solution.

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