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Abelian sharp permutation groups ☆

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Abstract

A permutation group G of finite degree n is a sharp irredundant group of type $\{k\}$, k a positive integer, if each non-identity element of G fixes exactly k points, $|G| = n - k$ and G has no global fixed point and no regular orbit. In this note we give a method to construct all faithful representations of finite abelian groups as sharp irredundant permutation groups of type $\{k\}$ for some positive integer k.

Keywords: Sharp permutation group; Partition

Let G be a permutation group of finite degree n and let K be a set of non-negative integers. G is called a group of type K if $K = \{\mathrm{fix}(g) \mid g \in G, g \neq 1\}$, where $\mathrm{fix}(g)$ denotes the number of fixed points of g. We say that G *is a permutation group of finite type* if G is a group of type $\{k\}$ for some non-negative integer k. Blichfeld [2] has shown that if G is a group of type K, then $|G|$ divides the product $\prod_{k \in K}(n - k)$. When equality holds, G is called *sharp permutation group of type K* (see [4]). Thus in particular a sharp group of degree n and type $\{k\}$ has order $n - k$. Note that each non-trivial permutation group with k global fixed points and an arbitrary number of regular orbits is of type $\{k\}$. Moreover a group of type $\{k\}$ which has $h \leqslant k$ global fixed points is isomorphic to a group of type $\{k - h\}$. A group is called an *irredundant* (*permutation*) *group of type* $\{k\}$ if it is a group of

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type $\{k\}$ without global fixed points and regular orbits (see [5]). A sharp irredundant group of type $\{k\}$ is a sharp group of type $\{k\}$ which is also irredundant. Note that the absence of regular orbits forces k to be positive.

Finite groups admitting a faithful representation as sharp irredundant permutation groups of finite type have been investigated in [5]. In particular it is proved that a finite abelian group can be faithfully represented as a sharp irredundant group of finite type if and only if it is a non-cyclic elementary abelian p-group, for some prime number p. In this note we show that there is a correspondence between non-trivial partitions of an elementary abelian finite p-group G and faithful representations of G as sharp irredundant group of finite type. Recall that a *partition* of a group G is a set of non-trivial subgroups of G such that each non-identity element of G belongs to exactly one of them. A partition is *non-trivial* if each component is a proper subgroup.

Let G be a group. If ρ_i, $i = 0, \dots, r$, are permutation representations of G on some sets Ω_i, we denote by $\sum_{i=0}^r \rho_i$ the permutation representation ρ of G on the disjoint union $\Omega = \dot{\bigcup}_{i=0}^r \Omega_i$ defined by the position $\omega g^\rho = \omega g^{\rho_i}$ if $\omega \in \Omega_i$, $g \in G$. Moreover we denote by ρ_X the standard permutation representation of G on the right cosets of the subgroup X.

An autoduality of G is a bijection δ of the subgroup lattice of G into itself such that for all $H, K \leqslant G$, $H \leqslant K$ if and only if $K^\delta \leqslant H^\delta$, or equivalently $H^\delta \cap K^\delta = \langle H, K\rangle^\delta$ (see [7, Chapter 8]). Note that if δ is an autoduality of a finite p-group G, then $|H| = |G : H^\delta|$ for each $H \leqslant G$.

Let $\pi = \{X_i\}_{i=0,\dots,r}$ be a finite family of (not necessarily distinct) non-trivial subgroups of a group G. We say that π is an *α-covering* of G if it has the following properties:

(i) G is the set-theoretical union of the subgroups $X_0, \dots, X_r$.
(ii) For each maximal subgroup M of G the sum $\sum_{X_i \leqslant M} |X_i|$ is a constant independent on M.

An α-covering π is *non-trivial* if X_i is a proper subgroup of G for each $0 \leqslant i \leqslant r$.

Theorem 1. *Let G be a non-cyclic elementary abelian finite p-group and let δ be an autoduality of G. A permutation representation of G is a faithful representation of G as irredundant group of finite type if and only if it is equivalent to the representation $\rho_\pi^\delta = \sum_{i=0}^r \rho_{X_i^\delta}$ for some non-trivial α-covering $\pi = \{X_i\}_{i=0,\dots,r}$ of G. Furthermore ρ_π^δ represents G as sharp irredundant group of finite type if and only if π is a non-trivial partition of G. In such a case, $G^{\rho_\pi^\delta}$ is of type $\{r\} = \{|\pi| - 1\}$.*

Proof. Let $\pi = \{X_i\}_{i=0,\dots,r}$ be a non-trivial α-covering of the elementary abelian p-group G. Let us show that ρ_π^δ is a faithful representation of G as irredundant permutation group of finite type. Since $\ker \rho_\pi^\delta = \bigcap_{i=0}^r X_i^\delta = (\prod_{i=0}^r X_i)^\delta = G^\delta = 1$, ρ_π^δ is faithful. Let $1 \neq g$ be an element of G. Then

$$\operatorname{fix}\left(g^{\rho_\pi^\delta}\right) = \sum_{g \in X_i^\delta} \left|G : X_i^\delta\right| = \sum_{X_i \leqslant \langle g\rangle^{\delta^{-1}}} |X_i|.$$

Since $\langle g\rangle^{\delta^{-1}}$ is a maximal subgroup of G, by the property (ii) of α-coverings $\mathrm{fix}(g^{\rho_\pi^\delta})$ does not depend on g. Hence ρ_π^δ represents G as a permutation group of finite type $\{\sum_{X_i\leqslant M}|X_i|\}$, where M is a maximal subgroup of G. Moreover since X_i^δ is a proper non-trivial subgroup of G for each i, the representation ρ_π^δ is irredundant.

Conversely let ρ be a faithful representation of G as irredundant permutation group of type $\{k\}$ on a set Ω, with orbits Ω_i, $i=0,\dots,r$. Then ρ is equivalent to the representation $\sum_{i=0}^r \rho_{H_i}$, where H_i is the stabilizer of a point $\omega_i\in\Omega_i$, $0\leqslant i\leqslant r$. Set $X_i=H_i^{\delta^{-1}}$ for each $i=0,\dots,r$. To prove the first part of the theorem we show that $\pi=\{X_i\}_{i=0,\dots,r}$ is a non-trivial α-covering of G. Note that H_i is a non-trivial proper subgroup of G for each $i=0,\dots,r$ and thus the X_i's are non-trivial proper subgroups as well. Hence, since an element $g\in G$ belongs to X_i if and only if $X_i^\delta\leqslant\langle g\rangle^\delta$ and $\langle g\rangle^\delta$ is a maximal subgroup of G whenever $g\neq 1$, to prove that $G=\bigcup_{i=0}^r X_i$ it is enough to show that for each maximal subgroup M of G there exists an index i, $0\leqslant i\leqslant r$, such that $X_i^\delta=H_i\leqslant M$. Suppose by contradiction that there exists a maximal subgroup M of G such that $H_i\nleqslant M$ for each $i=0,\dots,r$. Then $|M:M\cap H_i|=|G:H_i|$ for each $i=0,\dots,r$. It follows that the M-orbits of Ω coincide with the G-orbits and thus M has $r+1$ orbits as G does. Hence the Orbit-Counting Lemma [3, Theorem 2.2] applied to G and M gives $|G|(r+1-k)=|\Omega|-k=|M|(r+1-k)$. Since $M<G$ it follows that $|\Omega|=k$: a contradiction since G is non-trivial and ρ is faithful. Therefore $G=\bigcup_{i=0}^r X_i$. Moreover, for each maximal subgroup M of G we have

$$\sum_{X_i\leqslant M}|X_i|=\sum_{M^\delta\leqslant X_i^\delta}\left|G:X_i^\delta\right|=k.$$

Therefore π is a non-trivial α-covering of G.

It remains to show that ρ_π^δ represents G as a sharp irredundant group of type $\{k\}$ for some $k>0$ if and only if the α-covering $\pi=\{X_i\}_{i=0,\dots,r}$ is a non-trivial partition of G. Since $G=\bigcup_{i=0}^r X_i$, we have that π is a non-trivial partition of G if and only if $r>0$ and $\sum_{i=0}^r(|X_i|-1)=|G|-1$. Thus, since

$$\sum_{i=0}^r\bigl(|X_i|-1\bigr)=\sum_{i=0}^r\left|G:X_i^\delta\right|-(r+1)=|G|(r+1-k)+k-r-1,$$

it follows that π is a non-trivial partition if and only if $|G|=|G|(r+1-k)+k-r$, that is $r=k$ and ρ_π^δ represents G as a sharp irredundant group of type $\{r\}$. Hence to conclude the proof of the theorem we need only to show that any partition of G is an α-covering. This follows from next Lemma 2. □

Lemma 2. *Partitions of elementary abelian finite p-groups are α-coverings.*

Proof. Let G be an elementary abelian finite p-group and let π be a partition of G. Let M be a maximal subgroup of G and set $s_M=\sum_{X\in\pi,X\leqslant M}|X|$. If π is the trivial partition, then $s_M=0$ and we are done. So let us assume that π is non-trivial. Then $|G|=p^n$, $n>1$. For each $t=1,\dots,n-1$ denote by m_t the number of the components of π of order p^t

and by l_t the number of the components of π of order p^t which are contained in M. The number of subgroups of order p of M which are contained in a component $X \in \pi$ of order p^t, $1 \leqslant t \leqslant n-1$, is equal to $1 + p + \cdots + p^{t-1}$ if $X \leqslant M$ and to $1 + p + \cdots + p^{t-2}$ if $X \nleqslant M$, since in the latter case $|M \cap X| = p^{t-1}$. Then, by counting the number of subgroups of order p of M, we get

$$1 + p + \cdots + p^{n-2} = \sum_{t=1}^{n-1} l_t\bigl(1 + p + \cdots + p^{t-1}\bigr) + \sum_{t=1}^{n-1} (m_t - l_t)\bigl(1 + p + \cdots + p^{t-2}\bigr).$$

It follows that

$$\sum_{t=1}^{n-1} l_t p^{t-1} = 1 + p + \cdots + p^{n-2} - \sum_{t=1}^{n-1} m_t\bigl(1 + p + \cdots + p^{t-2}\bigr)$$

and thus

$$s_M = \sum_{t=1}^{n-1} l_t p^t = p + \cdots + p^{n-1} - \sum_{t=1}^{n-1} m_t\bigl(p + p + \cdots + p^{t-1}\bigr).$$

Therefore s_M does not depend on M and we are done. □

Note that if G is an elementary abelian finite p-group of order p^n with $n > 2$, then the set of all maximal subgroups of G is an α-covering which is not a partition, and the set of all maximal subgroups of G containing a fixed non-identity element is not an α-covering of G, even if G is the set-theoretical union of them.

The next proposition describes how the permutation representation ρ_π^δ of Theorem 1 depends on the autoduality δ.

Proposition 3. *Let G be a non-cyclic elementary abelian finite p-group and let π be a non-trivial α-covering of G. If δ and σ are autodualities of G, then there exists an automorphism φ of G such that the permutation representations of G ρ_π^δ and $\varphi\rho_\pi^\sigma$ are equivalent.*

Proof. Let $\pi = \{X_i\}_{i=0,\dots,r}$, δ, σ be as in the statement. If $|G| = p^2$, then for each non-trivial proper subgroup H of G there exists i such that $X_i = H$ and the cardinality of the set $\{i \mid X_i = H\} \subseteq \{0, \dots, r\}$ does not depend on H. Since δ and σ permute non-trivial proper subgroups of G, it is clear that ρ_π^δ and ρ_π^σ are equivalent. So let us assume that $|G| = p^n$ with $n > 2$. Then, since $\delta^{-1}\sigma$ is a autoprojectivity of G (see [7, Lemma 8.1.5.a]), by the Fundamental Theorem of Projective Geometry (see, for example, [1, p. 44]) there exists an automorphism φ of G such that $H^{\delta^{-1}\sigma} = H^\varphi$ for each $H \leqslant G$. Hence $H^\sigma = (H^\delta)^\varphi$ for each $H \leqslant G$. Denote by Ω and Σ the sets on which G acts via ρ_π^δ and ρ_π^σ respectively. For each $0 \leqslant i \leqslant r$, $x \in G$ we have

$$\bigl(X_i^\delta x\bigr)^\varphi = \bigl(X_i^\delta\bigr)^\varphi x^\varphi = X_i^\sigma x^\varphi.$$

Since φ is an automorphism of G, it follows that the position $X_i^\delta x \mapsto X_i^\sigma x^\varphi$ for each $0 \leqslant i \leqslant r$, $x \in G$, defines a bijection $\beta : \Omega \to \Sigma$; and for each $x, y \in G$, $0 \leqslant i \leqslant r$, we have $((X_i^\delta x)y)^\beta = (X_i^\delta x)^\beta y^\varphi$. Therefore ρ_π^δ and $\varphi\rho_\pi^\sigma$ are equivalent representations. □

Remark that when G is an elementary abelian finite p-group of order not less than p^3, then every autoduality δ of G is represented by a non-degenerate bilinear form f on G regarded as a vector space over the field with p elements, that is to say $H^\delta = \{x \in G \mid f(x, y) = 0 \text{ for all } y \in H\}$ for each $H \leqslant G$ (see [1, Chapter 4]).

Examples of sharp irredundant permutation finite abelian groups of finite type can be constructed, by using the tecnique of Theorem 1, once we are given an elementary abelian finite p-group G and a non-trivial partition of G. Note that the groups of Theorem 7 and Proposition 2 in [5] are actually constructed in this way. By [6, Lemma 4], if G is an elementary abelian finite p-group of order p^n, $n > 1$ and n can be written as $n = kn' + t$ where $1 \leqslant n' < n$ and $n' < t < 2n'$, then G has a partition consisting of $p^{n-n'} + p^{n-2n'} + \cdots + p^{n-kn'} + 1$ components. Therefore, by Theorem 1, G has a faithful representation as a sharp irredundant permutation group of type $\{p^{n-n'} + p^{n-2n'} + \cdots + p^{n-kn'}\}$.

Note that by [5, Theorem 1] if G is any irredundant permutation group of type $\{k\}$ on a set Ω, then G has a non-trivial partition denoted by π_Ω (called the standard partition associated to Ω) whose components are the subgroups $G_\Delta = \{g \in G \mid \omega g = \omega \text{ for each } \omega \in \Delta\}$ where $\Delta \subseteq \Omega$, $|\Delta| = k$ and $G_\Delta \neq 1$. Now assume that G is a sharp irredundant elementary abelian finite p-group of type $\{k\}$. If δ is an autoduality of G, then by Theorem 1 G has a non-trivial partition π whose components are the subgroups $S^{\delta^{-1}}$ where $S = St_G(\omega)$, $\omega \in \Omega$. We claim that $\pi_\Omega = \{Y^\delta \mid Y = \prod_{X\in\pi, X\leqslant M} X \text{ and } |G : M| = p\}$. To see this, for each element $1 \neq g \in G$, let us denote by H_g the intersection of all the point-stabilizers $St_G(\omega)$, $\omega \in \Omega$, containing g. Then we have that $\pi_\Omega = \{H_g \mid 1 \neq g \in G\}$ and

$$H_g = \bigcap_{\substack{\omega\in\Omega,\\ g\in St_G(\omega)}} St_G(\omega) = \bigcap_{\substack{X\in\pi,\\ g\in X^\delta}} X^\delta = \Bigg(\prod_{\substack{X\in\pi,\\ X\leqslant\langle g\rangle^{\delta^{-1}}}} X\Bigg)^\delta .$$

Since, when g varies in the set of non-identity elements of G, $\langle g\rangle^{\delta^{-1}}$ varies in the set of all maximal subgroups of G, the claim follows.

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