Dynamic effects of time delay on a coupled FitzHugh–Nagumo neural system

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Abstract In this paper, a coupled FitzHugh–Nagumo (FHN) neural system with time delay has been proposed and its stability and Hopf bifurcation are researched. Specifically, the stability of equilibrium point is analyzed by employing the corresponding characteristic equation. Sufficient conditions for existence of Hopf bifurcation are obtained. The results show that the FHN neural system exhibits the parameter regions involved the delay-independence stability and delay dependence stability. Increase of time delay can induce the stability switches between resting state and periodic activity. Furthermore, the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions are determined by the normal form theory and the center manifold theorem for functional differential equations. Finally, some numerical simulations are carried out for illustrating the theoretical results.

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1. Introduction

In recent years, there is growing interest in applying nonlinear dynamics to Neural network model such as Hopfield/Cohen–Grossberg neuron networks [1–6], the Hodgkin-Huxley (H–H) models [7], FitzHugh–Nagumo (FHN) neurons [8–11], or Kuramoto models [12]. The FHN neuronal model is one of the most popular mathematical models which can exhibit a hard oscillation, separatrix loops, as well as bifurcations of equilibria under suitable values of the parameters. It was derived as a simplified model of H–H equation by FitzHugh [8] and Nagumo [9]. The FHN model can describe physiological phenomena similar to those corresponding to the H–H model. A complete topological and qualitative investigation of the FHN equation with a cubic nonlinearity has been done by Bautin [13] and a rich variety of nonlinear phenomena are observed. In recent years, the FHN model is commonly used to study neural firings because of its simplicity. The dynamics of the coupled FHN neurons with a symmetric and nonsymmetric coupling was studied in [14–16], in which rich bifurcation behavior for equilibria and limit cycles was observed. Recently, for bidirectional coupling case between two oscillators, Anderson Hoff et al. [17], have studied the following model:

\[
\begin{align*}
\dot{x}_1 &= c(y_1 + x_1 - \frac{1}{3}x_1^3) + \gamma(x_1 - x_2), \\
\dot{y}_1 &= -\frac{1}{3}(x_1 - a + by_1), \\
\dot{x}_2 &= c(y_2 + x_2 - \frac{1}{3}x_2^3) + \gamma(x_2 - x_1), \\
\dot{y}_2 &= -\frac{1}{3}(x_2 - a + by_2),
\end{align*}
\]

(1)
where $x_i$ and $y_i$, $i = 1, 2$, represent the voltage across the cell membrane, and the recovery state of the resting membrane of a neuron, respectively. On the other hand, $a, b$ and $c$ are parameters, and $y$ is the coupling strength between the network elements.

In fact, considerable time delays are ubiquitous in all the biological processes. Moreover, time delays play an important role in the system dynamics and cannot be ignored in the modeling [18–21]. Particularly, it is known that time delays always occur in the signal transmission for real neurons. In [17], these FHN neurons with time-delay coupling have been numerically investigated involving the effects of the four parameters on bifurcations and synchronization. In [22], Gan et al. give the following model by adding Gaussian noise and delay $\tau$ to the FHN equations to study its stochastic resonance phenomenon:

\[
\begin{align*}
\rho \dot{x}_i &= x_i - \frac{1}{2}x_i - y_i + \sigma \xi_i(t) + D \sum_{j=1}^{\infty} e^{\beta j}[x_j(t-\tau) - x_i(t)], \\
\dot{y}_i &= x_i + a, (i = 1, 2 \ldots L),
\end{align*}
\]

(2)

Because the time delay in the system (1) is inevitable due to the finite propagating speed in the signal transmission between the neurons [23]. As a special case of systems (1) and (2), we now rewrite system (1) as the following form:

\[
\begin{align*}
\dot{x}_i &= c(x_i + x_1 \frac{1}{2} x_1^2) + \gamma(x_1 - x_2(t - \tau)), \\
\dot{y}_i &= -\frac{1}{6} (x_i - a + b y_i), \\
\dot{x}_2 &= c(x_2 + x_2 \frac{1}{2} x_2^2) + \gamma(x_2 - x_1(t - \tau)), \\
\dot{y}_2 &= -\frac{1}{6} (x_2 - a + b y_2),
\end{align*}
\]

(3)

where all of the variables have the same meaning of system (1). $\tau > 0$ represents the time delay in signal transmission.

Although Gan et al. studied the stochastic neuronal dynamics of the system (2), the stability and Hopf bifurcation of system (2) have not been further researched. Thus, in the present paper, we discuss the stability and the local Hopf bifurcation of system (3) continuously. It has been proved that time delay can drive the system to occur sustained oscillations by Hopf bifurcation analysis. The results show that the FHN neural system exhibits the parameter regions involved the delay-independence stability and delay dependence stability. Increase of time delay can induce the stability switches between resting state and periodic activity. Therefore, this study might be helpful to the comprehension of Neural network system. We would like to mention that the normal form method has been applied effectively in the study of singularities of vector fields and in bifurcation analysis [24–26]. And recently there are several articles [27–29] on the zero singularity in delayed differential equations based on the normal form method and center manifold theorem introduced by Faria [30] and Hale [31], respectively.

The paper is organized as follows. In Section 2, by analyzing the characteristic equation of the linearized system of system (3) at the unique equilibrium, it is found that under suitable conditions on the parameters the unique equilibrium is asymptotically stable when $\tau$ is less than a certain critical value and unstable when $\tau$ is greater than this critical value. Meanwhile, according to the Hopf bifurcation theorem for functional differential equations (FDEs) [31,32], we find that the system can also undergo a Hopf bifurcation at the unique equilibrium when the delay crosses through a sequence of critical values. And a family of nonconstant periodic solutions is emerged. In Section 3, to determine the direction of the Hopf bifurcations and the stability of bifurcated periodic solutions occurring through Hopf bifurcations, an explicit algorithm is given by applying the normal form theory and the center manifold reduction for FDEs developed by Hassard, Kazarinoff and Wan [33]. To verify our theoretical results, some numerical simulations are also included in Section 4.

2. Stability of the equilibrium and local Hopf bifurcations

Obviously, the equilibrium $(x_1, y_1, x_2, y_2)$ of system (3) for $\tau = 0$ satisfies

\[
\begin{align*}
\begin{cases}
 c(y_i + x_i - \frac{1}{2} x_i^2) &= 0, \\
 -\frac{1}{6} (x_i - a + b y_i) &= 0.
\end{cases}
\end{align*}
\]

(4)

Therefore, we have $b x_i^3 + 3(1-b)x_i - 3a = 0$ and $y_i = \frac{1}{2} (-3x_i + x_i^2) = \frac{2a+c}{6b}$. $P_0(x_1, y_1, x_2, y_2)$ is a unique positive equilibrium when the condition (H1) $a > \sqrt[3]{3}$ and $0 < b < 1$ holds. Throughout this section, we always assume that the condition (H1) holds.

In the following, we focus on the existence of local Hopf bifurcation at equilibrium $P_0(x_1, y_1, x_2, y_2)$ of system (3). Let $x_i(t) = x_i(t) - x_i$, $y_i(t) = y_i(t) - y_i$, $x_2(t) = x_2(t) - x_2$, $y_2(t) = y_2(t) - y_2$, and still denote $x_i(t)$, $y_i(t)$, $x_2(t)$, $y_2(t)$ by $x_i(t)$, $y_i(t)$, $x_2(t)$, $y_2(t)$. The system (3) is equivalent to the following system:

\[
\begin{align*}
\dot{x}_1 &= M x_1(t) + N y_1(t) + Q x_2(t - \tau) + F x_i^2(t) + K x_2^2(t), \\
\dot{y}_1 &= D x_1(t) + E y_1(t), \\
\dot{x}_2 &= M x_2(t) + N y_2(t) + Q x_1(t - \tau) + F x_i^2(t) + K x_2^2(t), \\
\dot{y}_2 &= D x_2(t) + E y_2(t),
\end{align*}
\]

(5)

where

\[
\begin{align*}
M &= c(1 - x_2^2) + \gamma, \quad N = c, \quad Q = -\gamma, \quad F = -\frac{1}{3} c, \\
K &= -c x_i, \quad D = -\frac{1}{c}, \quad E = -\frac{b}{c},
\end{align*}
\]

and the unique equilibrium $(x_1, y_1, x_2, y_2)$ of system (3) is transformed into the zero equilibrium $(0, 0, 0, 0)$ of system (5). It is easy to see that characteristic equation of the linearized system of system (5) at the zero equilibrium $(0, 0, 0, 0)$ is

\[
[(\lambda - E)(\lambda - M) - DN]^2 - ([\lambda - E]Qe^{-2\xi})^2 = 0.
\]

(6)

From (6) we have

\[
\begin{align*}
\lambda^2 + a_1 \lambda + a_2 + (b_1 \lambda + b_2)e^{-2\xi} &= 0, \\
\lambda^2 + a_1 \lambda + a_2 - (b_1 \lambda + b_2)e^{-2\xi} &= 0,
\end{align*}
\]

(7)

where

\[
\begin{align*}
a_1 &= -M - E, \quad a_2 = ME - ND, \\
b_1 &= Q, \quad b_2 = -EQ.
\end{align*}
\]

It is well known that the stability of the zero equilibrium $(0, 0, 0, 0)$ of system (5) is determined by the real parts of the roots of Eq. (7). If all roots of Eq. (7) locate the left-half complex plane, then the zero equilibrium $(0, 0, 0, 0)$ of system (5) is asymptotically stable. If Eq. (7) has a root with positive real
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part, then the zero solution is unstable. Therefore, to study the stability of the zero equilibrium \((0, 0, 0, 0)\) of system \((5)\), an important problem is to investigate the distribution of roots in the complex plane of the characteristic Eq. (7).

For Eq. (7), according to the Routh–Hurwitz criterion, we have the following result.

**Lemma 2.1.** When \(\tau = 0\) and \(a_k(k = 1, 2)\) satisfy the condition

\[
\begin{align*}
& a_1 + b_1 > 0, a_2 + b_2 > 0, \\
& a_1 - b_1 > 0, a_2 - b_2 > 0,
\end{align*}
\]

then all roots of Eq. (7) have negative real parts, and hence the zero equilibrium \((0, 0, 0, 0)\) of system \((5)\) is asymptotically stable.

Next, we consider the effects of a positive delay \(\tau\) on the stability of the zero equilibrium \((0, 0, 0, 0)\) of system \((5)\). Since the roots of the characteristic Eq. (7) depend continuously on \(\tau\), a change of \(\tau\) must lead to change of the roots of Eq. (7). If there is a critical value of \(\tau\) such that a certain root of Eq. (7) has zero real part, then at this critical value the stability of the zero equilibrium \((0, 0, 0, 0)\) of system \((5)\) will switch, and under certain conditions a family of small amplitude periodic solutions can bifurcate from the zero equilibrium \((0, 0, 0, 0)\); that is, a Hopf bifurcation occurs at the zero equilibrium \((0, 0, 0, 0)\).

Now, we look for conditions under which the characteristic Eq. (7) has a pair of purely imaginary roots.

Clearly, \(i\omega^+(\omega^+ > 0)\) is a root of the first Eq. (7) if and only if \(\omega^+\) satisfies the following equation:

\[-(\omega^+)^2 + a_1i\omega^+ + a_2 + (i\omega^+ b_1 + b_2)(\cos\omega^+\tau - \sin\omega^+\tau) = 0.\]

Separating the real and imaginary parts of the above equation yields the following equations:

\[
\begin{align*}
& (\omega^+)^2 - a_2 = b_2\cos\omega^+\tau + b_1\omega^+\sin\omega^+\tau, \\
& a_1\omega^+ = b_2\sin\omega^+\tau - b_1\omega^+\cos\omega^+\tau.
\end{align*}
\]

Adding up the squares of the corresponding sides of the above equations yields the following algebra equation with respect to \(\omega^+\):

\[
(\omega^+)^4 + (a_2^2 - 2a_2 - b_1^2)(\omega^+)^2 + a_2^2 - b_1^2 = 0.
\]

Let \(z = (\omega^+)^2\), then Eq. (9) can be denoted simply as the following equation:

\[z^2 + p_1z + r_1 = 0,\]

where

\[
\begin{align*}
p_1 &= a_2^2 - 2a_2 - b_1^2, \\
r_1 &= a_2^2 - b_1^2.
\end{align*}
\]

Let

\[h_1(z) = z^2 + p_1z + r_1.\]

It is easy to see that if the condition

\[(H3) \quad p_1 > 0 \text{ and } r_1 > 0\]

holds, then (9) has no positive roots. Hence, all roots of the first equation of (7) have negative real parts when \(\tau \in (0, \infty)\) under the condition (H3).

If

\[(H4) \quad r_1 < 0\]

holds, then (9) has a unique positive root \(\omega^+_0 = \sqrt{z_0}\). Substituting \((\omega^+_0)^2\) into (8), we obtain

\[
\cos\omega^+_0\tau = \frac{(b_2 - a_1b_1)(\omega^+_0)^2 - a_2b_2}{b_2^2 + b_1^2(\omega^+_0)^2}
\]

and

\[
\tau^j = \frac{1}{\omega^+_0} \left[ \arccos \left( \frac{(b_2 - a_1b_1)(\omega^+_0)^2 - a_2b_2}{b_2^2 + b_1^2(\omega^+_0)^2} \right) + 2j\pi \right],
\]

\[j = 0, 1, 2, \ldots.\]

If

\[(H5) \quad p_1 < 0, r_1 > 0 \quad \text{and} \quad (p_1)^2 > 4r_1\]

holds, then (9) has two positive roots \(\omega^+_1 = \sqrt{z_1}\) and \(\omega^+_2 = \sqrt{z_2}\).

Substituting \((\omega^+_k)^2\) into (8), we obtain

\[
\tau^k_{2+} = \frac{1}{\omega^+_k} \left[ \arccos \left( \frac{(b_2 - a_1b_1)(\omega^+_k)^2 - a_2b_2}{b_2^2 + b_1^2(\omega^+_k)^2} \right) + 2k\pi \right],
\]

\[k = 0, 1, 2, \ldots.\]

Similarly, \(i\omega^- (\omega^- > 0)\) is a root of the second Eq. (7) if and only if \(\omega^-\) satisfies the following equation:

\[-(\omega^-)^2 + a_1i\omega^- + a_2 - (i\omega^- b_1 + b_2)(\cos\omega^-\tau - \sin\omega^-\tau) = 0.\]

Separating the real and imaginary parts of the above equation yields the following equations

\[
\begin{align*}
& -(\omega^-)^2 + a_2 = b_2\cos\omega^-\tau + b_1\omega^-\sin\omega^-\tau, \\
& a_1\omega^- = -b_2\sin\omega^-\tau + b_1\omega^-\cos\omega^-\tau.
\end{align*}
\]

Adding up the squares of the corresponding sides of the above equations yields the following algebra equation with respect to \(\omega^-\):

\[
(\omega^-)^4 + (a_2^2 - 2a_2 - b_1^2)(\omega^-)^2 + a_2^2 - b_1^2 = 0.
\]

Let \(z = (\omega^-)^2\), then Eq. (16) can be denoted simply as the following equation

\[z^2 + p_2z + r_2 = 0,\]

where

\[
\begin{align*}
p_2 &= a_2^2 - 2a_2 - b_1^2, \\
r_2 &= a_2^2 - b_1^2.
\end{align*}
\]

Let

\[h_2(z) = z^2 + p_2z + r_2.\]

Obviously, \(p_2 = p_2; \quad r_2 = r_2.\) Thus, separating (11) and (18), we have

\[
\begin{align*}
h_1(z) &\equiv h_2(z), \\
z^+ &\equiv z^-, \\
w^+ &\equiv w^-. 
\end{align*}
\]
\[ \cos \omega_0^* \tau = \frac{(a_1 b_1 - b_2)(\omega_0^* \tau)^2 + a_2 b_2}{b_2^* + b_1^*(\omega_0^* \tau)^2} = -\cos \omega_0^* \tau \] (20)

and

\[ \tau_j = \frac{1}{\omega_0^*} \left[ \arccos \left( \frac{(a_1 b_1 - b_2)(\omega_0^* \tau)^2 + a_2 b_2}{b_2^* + b_1^*(\omega_0^* \tau)^2} \right) + 2\pi j \right], \]

\[ j = 0, 1, 2, \ldots \] (21)

\[ \tau_k^\pm = \frac{1}{\omega_{i_k}} \left[ \arccos \left( \frac{(a_1 b_1 - b_2)(\omega_{i_k} \tau)^2 + a_2 b_2}{b_2^* + b_1^*(\omega_{i_k} \tau)^2} \right) + 2k\pi \right], \]

\[ k = 0, 1, 2, \ldots \] (22)

For convenience, we rewrite \( \omega_0^* \) and \( \omega_j \) as \( \omega_0^* \) and \( \omega_{i_j} \), \( \omega_{j_2} \) as \( \omega_0^* \) and \( \omega_{i_j} \), \( \omega_{j_2} \), then we have \( \pm i\omega_0^* \) that are a pair of purely imaginary roots of (7) with \( \tau = \tau_j^\pm, j = 0, 1, 2, \ldots \)

Rewrite \( \tau^* \) and \( \tau^- \) as \( \omega_0^* \), \( h_1(z) \) and \( h_2(z) \) as \( h(z) \).

For (H4), let

\[ \{ \tau_j^\pm \}_{j=0}^{\infty} = \{ \tau_j \}_{j=0}^{\infty}, \]

such that

\[ \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_j < \ldots \]

where

\[ \tau_0 = \min \{ \tau_0^+, \tau_0^- \}. \]

For (H5), let

\[ \{ \tau_k^\pm \}_{k=0}^{\infty} \cup \{ \tau_k^\pm \}_{k=0}^{\infty} = \{ \tau_k \}_{k=0}^{\infty} \]

such that

\[ \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots \]

where

\[ \tau_0 = \min \{ \tau_0^+, \tau_0^-, \tau_0^+, \tau_0^- \} \]

Let \( \lambda(\tau) = x(\tau) + i\omega(\tau) \) be a root of (7) near \( \tau = \tau_j \)

and \( x(\tau_j) = \omega_0, \omega_j(\tau_j) = \omega_0, j = 0, 1, 2, \ldots \)

Lemma 2.2. If \( (2a_0 b_2 - a_1 b_1 \omega_0^2) \cos \omega_0 \tau_j^+ + (2b_1 \omega_0^3 - a_2 b_2 \omega_0) \sin \omega_0 \tau_j^+ - b_2^* \omega_0^2 > 0 \) is satisfied,

\[ \left[ \frac{d(\Re(\lambda))}{dt} \right]_{t=\tau_j} > 0 \] for \( j = 0, 1, 2, \ldots \)

holds.

In fact, when \( \tau = \tau_j^+ \), substituting \( \lambda(\tau) \) into the left hand side of the first equation of (7) and taking derivative with respect to \( \tau \), we have

\[ \left( \frac{d\lambda}{dt} \right)^{-1} = \frac{(2\lambda + a_0)e^{i\lambda} - b_1 \lambda^2 - b_2 \tau + b_1}{b_2^2 \lambda^2 + b_2^2 \lambda} \]

which leads to

\[ \frac{d(\Re(\lambda(\tau)))}{dt} \bigg|_{\tau=\tau_j} = \Re \left\{ \frac{(2a_0 b_2 - a_1 b_1 \omega_0^2) \cos \omega_0 \tau_j^+ + (2b_1 \omega_0^3 - a_2 b_2 \omega_0) \sin \omega_0 \tau_j^+ - b_2^* \omega_0^2}{b_2^* \omega_0^2 + b_2^* \omega_0^2} \right\} \]

Lemma 2.2 is proved.

Similarly, we can obtain

\[ \frac{d(\Re(\lambda))}{dt} \bigg|_{t=\tau_j} > 0, \quad \frac{d(\Re(\lambda))}{dt} \bigg|_{t=\tau_k} > 0. \]

Lemma 2.3. For the transcendental equation

\[ p(\lambda, e^{-j_k \omega}, \ldots, e^{-j_0 \omega}) \]

\[ = 2^2 + p_0(\lambda) e^{-j_0 \omega} + \ldots + p_{2n}(\lambda) e^{-j_0 \omega} \]

\[ + p_1^{(1)}(\lambda) e^{-j_1 \omega} + \ldots + p_{2n-1}(\lambda) e^{-j_1 \omega} + \ldots + p_1^{(m)}(\lambda) e^{-j_1 \omega} \]

\[ + p_{m-1}(\lambda) e^{-j_1 \omega} + \ldots + p_{2n-1}(\lambda) e^{-j_1 \omega} + p_{2n}(\lambda) e^{-j_1 \omega} = 0 \]

as \( \tau_1, \tau_2, \ldots, \tau_m \) vary, the sum of orders of the zeros of \( p(\lambda, e^{-j_k \omega}, \ldots, e^{-j_0 \omega}) \) in the open right half plane can change, and only a zero appears on z or crosses the imaginary axis.

According to above analysis and the Corollary 2.4 in Ruan and Wei [34] we have the following results:

Theorem 2.4. For system (3), assume that (H1) and (H2) are satisfied. Then the following conclusions hold:

(i) If (H3) holds, then the equilibrium \( P_0(x_0, y_0, x_1, y_1) \) of system (3) is asymptotically stable for all \( \tau \geq 0 \).

(ii) If (H4) holds, there exists \( \delta_j > 0 \) such that \( 0 < \delta_j \leq \tau_j \). The equilibrium \( P_0(x_0, y_0, x_1, y_1) \) of system (3) is asymptotically stable for \( \delta_j < \tau < \tau_j \) and unstable for \( \delta_{j+1} > \tau > \tau_j \). Furthermore, system (3) undergoes a supercritical Hopf bifurcation at the equilibrium \( P_0(x_0, y_0, x_1, y_1) \) when \( \tau = \tau_j \).

(iii) If (H5) holds, there exists \( \sigma_j > 0 \) such that \( 0 < \sigma_j \leq \tau_k \). The equilibrium \( P_0(x_0, y_0, x_1, y_1) \) of system (3) is asymptotically stable for \( \sigma_j < \tau < \tau_k \) and unstable for \( \sigma_{j+1} > \tau > \tau_j \). Furthermore, system (3) undergoes a supercritical Hopf bifurcation at the equilibrium \( P_0(x_0, y_0, x_1, y_1) \) when \( \tau = \tau_k \).

Without loss of generality, here we only discuss the case of

\[ \left[ \frac{d(\Re(\lambda))}{dt} \right]_{t=\tau_j} > 0 \] for \( j = 0, 1, 2, \ldots \), subcritical Hopf bifurcation will occur instead of supercritical Hopf bifurcation. It will not go into details here.
3. Direction and stability of the Hopf bifurcation

In the previous section, we studied mainly the stability of the unique equilibrium \( p_0 \) of system (3) and the existence of Hopf bifurcations at \( p_0 \).

In this section, we shall study the properties of the Hopf bifurcations obtained by Theorem 2.4 and the stability of bifurcated periodic solutions occurring through Hopf bifurcations by using the normal form theory and the center manifold reduction for retarded functional differential equations (RFDEs) due to Hassard et al. [33]. Throughout this section, we always assume that system (3) undergoes Hopf bifurcation at the equilibrium \( p_0 \) for \( \tau = \tau_r \), and then \( \pm i\omega_0 \) are corresponding purely imaginary roots of the characteristic equation at the equilibrium \( p_0 \).

Without loss of generality, we assume that \( \tau \in (0, \tau_0) \). Let \( x_r(t, s) = x_r(t)(t = 1, 2, 3, 4) \) and \( \tau = \tau_r + \mu \), \( x(t) = (x_1(t), y_1(t), y_2(t), y_2(t))^T \) where \( \tau \) is defined by (23) and \( \mu \in \mathbb{R} \), drop the bar for simplicity of notation. Then system (4) can be rewritten as a system of RFDEs in \( C([-1, 0], \mathbb{R}^4) \) of the form

\[
\begin{align*}
\dot{x}(t) &= (\tau_r + \mu)[Mx(t) + Ny(t) + Qx(t-1) + Fx_1(t) + Kc_2(t)], \\
\dot{y}(t) &= (\tau_r + \mu)[Dx(t) + Ey(t)], \\
\dot{y}_2(t) &= (\tau_r + \mu)[Mx(t) + Ny(t) + Qx(t-1) + Fx_1(t) + Kc_2(t)], \\
\dot{x}_2(t) &= (\tau_r + \mu)[Dx(t) + Ey(t)].
\end{align*}
\]  

Defining the linear operator \( L(\mu) : C \to \mathbb{R}^4 \) and the nonlinear operator \( f(\cdot, \mu) : C \to \mathbb{R}^4 \) by

\[
L(\mu) = (\tau_r + \mu) \begin{pmatrix} M & N & 0 & 0 \\ D & E & 0 & 0 \\ 0 & 0 & M & N \\ 0 & 0 & D & E \end{pmatrix},
\]

\[
f(\cdot, \mu) = (\tau_r + \mu) \begin{pmatrix} 0 & 0 & Q & 0 \\ 0 & 0 & 0 & 0 \\ Q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

and

\[
f(\mu, \phi) = (\tau_r + \mu) \begin{pmatrix} F\phi_1 \alpha + K\phi_2 \beta \\ F\phi_2 \alpha + K\phi_2 \beta \\ 0 \\ 0 \end{pmatrix}
\]

respectively, where \( \phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in C \). By the Riesz representation theorem, there exists a \( 4 \times 4 \) matrix function \( \eta(\theta, \mu) \), \(-1 \leq \theta \leq 0\), whose elements are of bounded variation such that \( L(\mu) = \int_{-1}^{0} \eta(\theta, \mu) d\eta(\theta, \mu) \) for \( \phi \in C([-1, 0], \mathbb{R}^4) \). In fact, we can choose

\[
\eta(\theta, \mu) = (\tau_r + \mu) \begin{pmatrix} M & N & 0 & 0 \\ D & E & 0 & 0 \\ 0 & 0 & M & N \\ 0 & 0 & D & E \end{pmatrix}, \quad \eta_{-1} = \begin{pmatrix} 0 & 0 & Q & 0 \\ 0 & 0 & 0 & 0 \\ Q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

For \( \phi \in C^1([-1, 0], \mathbb{R}^4) \), define

\[
A(\mu)\phi = \begin{pmatrix} \frac{d\phi(\theta)}{d\theta} \\ \int_0^\theta \tau \eta(\mu, \theta)\phi(\theta) \\ 0 \end{pmatrix}, \quad \theta \in (-1, 0),
\]

and

\[
R(\mu)\phi = \begin{pmatrix} 0 \\ \int_0^\theta \eta(\mu, \theta) \\ 0 \end{pmatrix}, \quad \theta \in (-1, 0).
\]

Then system (27) is equivalent to

\[
\dot{x}_r = A(\mu)x_r + R(\mu)x_r,
\]

where \( x_r(\theta) = x(t + \theta) \).

**Remark 3.1.** Here, it should be pointed out that (34) is the normal form of original system (3).

For \( \psi \in C^1([-1, 1], \mathbb{R}^4) \), define

\[
A^*\psi = \begin{pmatrix} -\frac{d\psi(\theta)}{d\theta} \\ \int_{-1}^{0} \tau \eta(\mu, \theta)\psi(-\theta) \\ 0 \\ \int_{-1}^{0} \eta(\mu, \theta)\psi(\theta) \\ 0 \end{pmatrix}, \quad s \in (-1, 1),
\]

and a bilinear inner product

\[
\langle\psi(s), \phi(t)\rangle = \overline{\psi(\theta)}\phi(\theta) - \int_{-1}^{0} \int_{-1}^{0} \overline{\psi(s)}(\theta)\phi(t)\eta(\theta, \mu) d\theta dt,
\]

where \( \eta(\theta) = \eta(\theta, 0) \). Then \( A(\mu) \) and \( A^*(\mu) \) are adjoint operators. In addition, from Section 2, we know that \( \pm i\omega_0 \tau_r \) are eigenvalues of \( A(\mu) \). Thus, they are also eigenvalues of \( A^*(\mu) \). Let \( q(\theta) \) be the eigenvector of \( A(\mu) \) corresponding to \( i\omega_0 \tau_r \) and \( q^*(s) \) is the eigenvector of \( A^*(\mu) \) corresponding to \(-i\omega_0 \tau_r \).

Let \( q(\theta) = (1, \nu_1, \nu_2, \nu_3)e^{i\omega_0 \tau \theta} \) and \( q^*(s) = (1, \nu_1^*, \nu_2^*, \nu_3^*)e^{i\omega_0 \tau^* s} \). From the above discussion, it is easy to know that

\[
A(\mu)q(\theta) = i\omega_0 \tau \eta(\mu)q(\theta) \text{ and } A^*(\mu)q^*(s) = -i\omega_0 \tau q^*(s),
\]

that is

\[
\begin{pmatrix} M & N & 0 & 0 \\ D & E & 0 & 0 \\ 0 & 0 & M & N \\ 0 & 0 & D & E \end{pmatrix} q(\theta) + \begin{pmatrix} 0 & 0 & Q & 0 \\ 0 & 0 & 0 & 0 \\ Q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} q(-\tau) = i\omega_0 \tau q(\theta)
\]

and

\[
\begin{pmatrix} M & D & 0 & 0 \\ N & E & 0 & 0 \\ 0 & 0 & M & D \\ 0 & 0 & N & E \end{pmatrix} q^*(s) + \begin{pmatrix} 0 & 0 & Q & 0 \\ 0 & 0 & 0 & 0 \\ Q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} q^*(-\tau) = -i\omega_0 \tau q^*(s)
\]

Thus, we can easily obtain

\[
q(\theta) = \begin{pmatrix} 1, \frac{D}{i\omega_0 - E}, -\omega_0^2 - iE\omega_0 + iM\omega_0 + ME - ND \\ -\omega_0^2 - iE\omega_0 + iM\omega_0 + ME - ND \end{pmatrix} e^{i\omega_0 \tau \theta}
\]

\[
N(-\omega_0^2 + iE\omega_0 + iM\omega_0 + ME - ND) e^{i\omega_0 \tau \theta},
\]

\[
q^*(s) = G(1, \frac{N}{-i\omega_0 - E}, -\omega_0^2 + iE\omega_0 - iM\omega_0 + ME - ND)
\]

\[
(-\omega_0^2 + iE\omega_0 - iM\omega_0 + ME - ND) e^{i\omega_0 \tau^* s},
\]
since

\[ (q'(s),q(0)) = \mathcal{G}(0)(q(0)) - \int_{0}^{s} \int_{0}^{1} \mathcal{G}'(\xi,t) d\eta(0) q(\xi) d\xi, \]

\[ = \mathcal{G}(0)(q(0)) - \int_{0}^{1} \mathcal{G}(1,1,2,\mathbb{T}) e^{-\mathcal{G}(1,2,\mathbb{T})} d\eta(0) q(0), \]

\[ = \mathcal{G}(0)(q(0)) - \mathcal{G}(0)(q(0)) + \mathcal{G}(0)(\mathbb{T}) \tau e^{-\mathcal{G}(0)(\mathbb{T})}, \]

\[ = \mathcal{G}(1,1,2,\mathbb{T}) + \mathcal{G}(1,2,\mathbb{T}) e^{-\mathcal{G}(0)(\mathbb{T})}. \]

We may choose

\[ \mathcal{G} = \frac{1}{(1 + \mathbb{T}_1 \mathbb{T}_2 + \mathbb{T}_2 \mathbb{T}_3 + \mathbb{T}_3 \mathbb{T}_1) + (\mathcal{Q} \mathbb{Q} + \mathcal{Q} \mathbb{Q}) \tau e^{-\mathcal{G}(0)(\mathbb{T})}}, \]

\[ G = \frac{1}{(1 + \mathbb{T}_1 \mathbb{T}_2 + \mathbb{T}_2 \mathbb{T}_3 + \mathbb{T}_3 \mathbb{T}_1) + (\mathcal{Q} \mathbb{Q} + \mathcal{Q} \mathbb{Q}) \tau e^{-\mathcal{G}(0)(\mathbb{T})}}, \]

which assures that \((q'(s), q(0)) = 1\).

By using the same notions as in [33], we first compute the coordinates to describe the center manifold \(C_0\) at \(\mu = 0\). Let \(x_i\) be the solution of (26) when \(\mu = 0\). Define \(z(t) = (q', x_i), \quad W(t, \theta) = x_i(\theta) - 2\Re\{z(t)q(\theta)\}. \tag{38} \)

**Remark 3.2.** Actually, (38) is the center manifold of system (3). On the center manifold \(C_0\), we have \(W(t, \theta) = W(z(t), \tau(t), \theta)\) where

\[ W(z, \tau, \theta) = W_{20}(\theta) z^2 + W_{11}(\theta) z \tau + W_{02}(\theta) \tau^2 \]

\[ + W_0(\theta) \zeta^2 + \ldots, \tag{39} \]

\(z\) and \(\tau\) are local coordinates for center manifold \(C_0\) in the direction of \(q'\) and \(q\). Note that \(W\) is real if \(x_i\) is real. We consider only real solution. For solution \(x_i \in C_0\) of (27), since \(\mu = 0\),

\[ z(t) = i \omega_0 \tau z + \tau^0 q(0) f(0, W(z, \tau, \theta)) + 2\Re\{z(t)q(\theta)\} \]

\[ = i \omega_0 \tau z + \tau^0 f(0), \tag{40} \]

that is,

\[ z(t) = i \omega_0 \tau z(t) + g(z, \tau), \tag{41} \]

where

\[ g(z, \tau) = g_{20} z^2 + g_{11} z \tau + g_{02} \tau^2 + g_{21} \tau^2 + \ldots. \tag{42} \]

Then it follows from (38) that

\[ x_i = W(t, \theta) + 2\Re\{z(t)q(\theta)\} \]

\[ = W_{20}(\theta) z^2 + W_{11}(\theta) z \tau + W_{02}(\theta) \tau^2 \]

\[ + (1, v_1, v_2, v_3) e^{i \omega_0 \tau} z + (1, v_1, v_2, v_3) e^{-i \omega_0 \tau} \tau + \ldots. \tag{43} \]

It follows together with (30) that

\[ g(z, \tau) = \tau q(0)f(0, z, \tau) \]

\[ = \tau Q(0) \frac{F(W(1)^{0}(0) + z + \tau^{2}) + K(W(1)^{0}(0) + z + \tau^{2})}{0} \]

\[ = \tau Q(0) \frac{F(W(1)^{0}(0) + v_1 + \tau^{2}) + K(W(1)^{0}(0) + v_1 + \tau^{2})}{0} \]

\[ = \tau Q(0) \frac{F(W(1)^{0}(0) + v_1 + \tau^{2}) + K(W(1)^{0}(0) + v_1 + \tau^{2})}{0} \]

\[ = \tau Q(0) \frac{F(W(1)^{0}(0) + v_1 + \tau^{2}) + K(W(1)^{0}(0) + v_1 + \tau^{2})}{0} \]

\[ = \tau Q(0) \frac{F(W(1)^{0}(0) + v_1 + \tau^{2}) + K(W(1)^{0}(0) + v_1 + \tau^{2})}{0} \]

\[ = \tau Q(0) \frac{F(W(1)^{0}(0) + v_1 + \tau^{2}) + K(W(1)^{0}(0) + v_1 + \tau^{2})}{0} \]

\[ = \tau Q(0) \frac{F(W(1)^{0}(0) + v_1 + \tau^{2}) + K(W(1)^{0}(0) + v_1 + \tau^{2})}{0} \]

\[ = \tau Q(0) \frac{F(W(1)^{0}(0) + v_1 + \tau^{2}) + K(W(1)^{0}(0) + v_1 + \tau^{2})}{0} \]

\[ = \tau Q(0) \frac{F(W(1)^{0}(0) + v_1 + \tau^{2}) + K(W(1)^{0}(0) + v_1 + \tau^{2})}{0} \]

\[ = \tau Q(0) \frac{F(W(1)^{0}(0) + v_1 + \tau^{2}) + K(W(1)^{0}(0) + v_1 + \tau^{2})}{0} \]

Comparing the coefficients with (42), we obtain

\[ g_{20} = 2 i \omega_0 (K + \tau^2 q(0)), \]

\[ g_{11} = 2 i \omega_0 (K + \tau q(0)), \]

\[ g_{02} = 2 i \omega_0 q(0), \]

\[ g_{21} = 2 i \omega_0 (K W_{11}^{(0)}(0) + K W_{11}^{(0)}(0) + 2 K W_{11}^{(0)}(0) + 2 K W_{11}^{(0)}(0) + 2 K W_{11}^{(0)}(0) + 5 F + 3 F (\tau_1^2 q(0) + \ldots)). \tag{43} \]

Since there are \(W_{20}(0)\) and \(W_{11}(0)\) in \(g_{21}\), we need to compute them.

From (34) and (38), we have

\[ \tilde{W} = \tilde{x}_i - \tilde{q} = -\frac{\tilde{z} q}{\bar{q}} \]

\[ = \begin{cases} \tilde{A} W - 29 \Re\{\tilde{Q}(0) f(0, q(\theta))\}, & \theta \in [-1, 0) \\ \tilde{A} W - 29 \Re\{\tilde{Q}(0) f(0, q(\theta))\} + \bar{f}_0, & \theta = 0 \end{cases} \tag{44} \]

\[ = \tilde{A} W + H(z, \tau, \theta), \]

where

\[ H(z, \tau, \theta) = H_{20}(\theta) z^2 + H_{11}(\theta) z \tau + H_{02}(\theta) \tau^2 + \ldots. \tag{45} \]

Substituting the corresponding series into (44) and comparing the coefficients, we obtain

\[ (A - 2 i \omega_0 \tau) W_{20}(\theta) = -H_{20}(\theta), \]

\[ A W_{11}(\theta) = -H_{11}(\theta), \ldots. \tag{46} \]

From (44), we know that for \(\theta \in [-1, 0)\),

\[ H(z, \tau, \theta) = -\tilde{Q}(0) f(0, q(\theta) - q(0)) \bar{q}(\theta) \]

\[ = -g(z, \tau) q(\theta) - \tilde{g}(z, \tau) \bar{q}(\theta). \tag{47} \]

Comparing the coefficients with (45) gives that

\[ H_{20}(\theta) = -g_{20} q(\theta) - \bar{g}_{02} \bar{q}(\theta) \tag{48} \]

and

\[ H_{11}(\theta) = -g_{11} q(\theta) - \bar{g}_{11} \bar{q}(\theta) \tag{49} \]

From (46) and (48), we get

\[ \tilde{W}_{20}(\theta) = 2 i \omega_0 \tau W_{20}(\theta) + g_{20} q(\theta) + \bar{g}_{02} \bar{q}(\theta). \]

Note that \(q(\theta) = q(0)e^{i \omega_0 \tau}, \theta\); hence, we obtain

\[ W_{20}(\theta) = \frac{g_{20} \omega_0 \tau}{\omega_0} q(0)e^{i \omega_0 \tau} + \frac{g_{02}}{3 \omega_0 \tau} \bar{q}(0) e^{-i \omega_0 \tau} + E_{1} e^{i \omega_0 \tau}. \tag{50} \]

Similarly, from (46) and (49), we have

\[ \tilde{W}_{11}(\theta) = g_{11} q(\theta) + \bar{g}_{11} \bar{q}(\theta). \]
and

\[
W_{11}(\theta) = -\frac{i g_{21}}{\omega_{0} \tau_{f}} q(\theta) e^{\omega_{0} \tau_{f} \theta} + \frac{i g_{21}}{\omega_{0} \tau_{f}} \eta(0) e^{-\omega_{0} \tau_{f} \theta} + E_2. 
\]  
(51)

In what follows, we shall seek appropriate \( E_1 \) and \( E_2 \) in (50) and (51), respectively. It follows from the definition of \( A \) and (46) that

\[
\int_{-1}^{0} d\eta(\theta) W_{20}(\theta) = 2i \omega_{0} \tau_{f} W_{20}(\theta) - H_{20}(\theta) 
\]  
(52)

and

\[
\int_{-1}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(0) 
\]  
(53)

where \( \eta(\theta) = \eta(0, \theta) \). From (44), we have

\[
H_{20}(0) = -g_{20} q(0) - \frac{g_{20}}{c_{0}} \eta(0) + 2\tau \begin{pmatrix} K \\ 0 \\ K \tau_{f}^2 \\ 0 \end{pmatrix} 
\]  
(54)

and

\[
H_{11}(0) = -g_{31} q(0) - \frac{g_{31}}{c_{0}} \eta(0) + 2\tau \begin{pmatrix} K \\ 0 \\ K \tau_{f}^2 \\ 0 \end{pmatrix} 
\]  
(55)

Substituting (50) and (54) into (52), we obtain

\[
(2i \omega_{0} \tau_{f} I - \int_{-1}^{0} e^{2\omega_{0} \tau_{f} \theta} d\eta(\theta)) E_1 = 2\tau \begin{pmatrix} K \\ 0 \\ K \tau_{f}^2 \\ 0 \end{pmatrix}. 
\]  
(56)

From the definition of \( A \), we have

\[
\int_{-1}^{0} e^{2\omega_{0} \tau_{f} \theta} d\eta(\theta) = A(\mu) e^{2\omega_{0} \tau_{f} \theta} = L_{\mu} (e^{2\omega_{0} \tau_{f} \theta}).
\]

Therefore, when \( \mu = 0 \), we have

\[
\int_{-1}^{0} e^{2\omega_{0} \tau_{f} \theta} d\eta(\theta) = \tau \begin{pmatrix} M \\ N \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \tau \begin{pmatrix} Q \\ 0 \end{pmatrix} e^{-2\omega_{0} \tau_{f} \theta}. 
\]

Therefore,

\[
\begin{pmatrix} \lambda - M & -D \\ -N & \lambda - E \end{pmatrix} - \begin{pmatrix} 0 \\ -Q e^{-2\omega_{0} \tau_{f} \theta} \end{pmatrix} \begin{pmatrix} K \\ 0 \\ K \tau_{f}^2 \\ 0 \end{pmatrix} E_1 = 2\tau \begin{pmatrix} K \\ 0 \\ K \tau_{f}^2 \\ 0 \end{pmatrix}.
\]  
(57)

where \( \lambda = i \omega_{0} \). Similarly, substituting (51) and (55) into (53), we get

\[
\int_{-1}^{0} d\eta(\theta) E_2 = -2\begin{pmatrix} K \\ 0 \\ K \tau_{f}^2 \\ 0 \end{pmatrix}. 
\]  
(58)

It follows from (50), (51), (57) and (58) that \( g_{21} \) can be expressed. Thus, we can compute the following values:

\[
c_0(0) = \frac{i}{2c_{0} \omega_{0} \tau_{f}} (g_{21} g_{20} - 2|g_{20}|^2 - \frac{|g_{20}|^2}{3} + \frac{g_{21}^2}{2}),
\]

\[
\mu_2 = -\frac{\Re \{c_0(0)\}}{\Re \{\lambda_0(\tau_{f})\}},
\]

\[
\beta_2 = 2\Re \{c_0(0)\},
\]

\[
T_2 = -\frac{\Im \{c_0(0)\} + \mu_2 \Im \{\lambda_0(\tau_{f})\}}{\Im \{c_0(0)\}}.
\]

which determine the quantities of bifurcating periodic solutions at the critical value \( \tau_{f} \). Specifically, \( \mu_2 \) determines the directions of the Hopf bifurcation. If \( \mu_2 > 0(\mu_2 < 0) \), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for \( \tau > \tau_{f}(\tau < \tau_{f}) \). \( \beta_2 \) determines the stability of the bifurcating periodic solutions. The bifurcating periodic solutions in the center manifold are stable (unstable) if \( \beta_2 < 0(\beta_2 > 0) \). \( T_2 \) determines the period of the bifurcating periodic solutions. The period increases (decreases) if \( T_2 > 0(T_2 < 0) \). Further, it follows from Theorem 2.2 and (59) that the following results about the direction of the Hopf bifurcation hold.

Remark 3.3. The appearance or disappearance of a periodic orbit through a local change in the stability properties of a steady point is known as the Hopf bifurcation. There are two kinds of Hopf bifurcation. The bifurcation is called supercritical if the bifurcated periodic solutions are stable and subcritical if they are unstable. \( \mu_2 \) determines the directions of the Hopf bifurcation: if \( \mu_2 > 0(\mu_2 < 0) \), then the Hopf bifurcation is supercritical (subcritical).

Theorem 3.1. Suppose that (H1), (H2), and (H4) (H5) hold. If \( \Re \{c_0(0)\} < 0(\Re \{c_0(0)\} > 0) \), then the system (5) can undergo a supercritical (subcritical) Hopf bifurcation at the equilibrium \( P_0(x, y, z) \) when \( \tau \) crosses through the critical values \( \tau = \tau_{f} \). In addition, the bifurcated periodic solutions occurring through Hopf bifurcations are orbitally asymptotically stable on the center manifold if \( \Re \{c_0(0)\} < 0 \) and unstable if \( \Re \{c_0(0)\} > 0 \).

4. Numerical simulations

In this section, we give some numerical simulations for a special case of system (3) to support our analytical results obtained in Sections 2 and 3. We take system (3) with the coefficients \( a = 1.95, b = 0.17, c = 0.35, \gamma = 1.3 \) for example; that is,

\[
\begin{align*}
\dot{x}_1 &= 0.35(y_1 + x_1 - \frac{1}{3} x_1^3) + 1.3(x_1 - x_3(t - \tau)), \\
\dot{y}_1 &= -\frac{1}{35}(x_1 - 1.95 + 0.17y_1), \\
\dot{x}_2 &= 0.35(y_2 + x_2 - \frac{1}{3} x_2^3) + 1.3(x_2 - x_1(t - \tau)), \\
\dot{y}_2 &= -\frac{1}{35}(x_2 - 1.95 + 0.17y_2).
\end{align*}
\]  
(60)

The above set of parameters are chosen by using method of parameter estimation under the biochemical constraints [35–37]. Obviously, \( a = 1.95 > \sqrt{3} \) and \( 0 < b = 0.17 < 1 \), the condition (H1) holds. Therefore system (3) has a unique positive equilibrium \( P_0(1.88911, 0.358548, 1.88911, 0.358548) \).

Under the set of parameter values, the conditions (H2), (H5) and Lemma 2.2 are satisfied. By computing, we
may obtain that a unique positive equilibrium $\omega_0 = 1.79323$, $\tau_k = 0.763578 + 3.50384k$, ($k = 0, 1, 2, ...$). From Lemma 2.1, we know that the transversal condition is satisfied. Thus the equilibrium $P_0(1.88911, 0.358148, 1.88911, 0.358148)$ of system (60) is asymptotically stable when $\tau = 0$ (see Fig. 1). On the other hand, the conditions (H1), (H2) and (H5) hold. From the second point in Theorem 2.4, we know that the transversal condition is satisfied. Therefore, from Theorem 3.1, we know that system (60) can undergo a supercritical Hopf bifurcation at the positive equilibrium $P_0$ when $\tau = \tau_0 = 0.763578$ and the bifurcated periodic solution occurring from the Hopf bifurcation is orbitally asymptotically stable on the center manifold. To summarize, the positive equilibrium point $P_0(1.88911, 0.358148, 1.88911, 0.358148)$ of the system without delay is asymptotically stable. When delay is less than critical values, the positive equilibrium $P_0(1.88911, 0.358148, 1.88911, 0.358148)$ is asymptotically stable, as shown in Fig. 2. And unstable when delay is greater than critical values and can undergo a Hopf bifurcation that occurs at the positive equilibrium as shown in Fig.3.

![Figure 1](image1.png)  
**Figure 1** The numerical approximations of system (60) when $\tau = 0$. The positive equilibrium $P_0(1.88911, 0.358148, 1.88911, 0.358148)$ is asymptotically stable.

![Figure 2](image2.png)  
**Figure 2** The numerical approximations of system (60) when $\tau = 0.72 < \tau_0 = 0.763578$. The positive equilibrium $P_0(1.88911, 0.358148, 1.88911, 0.358148)$ is asymptotically stable.
5. Conclusions

In this paper, a coupled FHN neural system with time delay has been proposed and dynamic behaviors are analyzed by using Hopf bifurcation technique. Specifically, the stability of equilibrium point is analyzed via corresponding characteristic equation. Sufficient conditions for existence of Hopf bifurcation are obtained. Moreover, we proved that some families of periodic solutions appear when the delay $\tau$ passes through some certain critical values and meanwhile the equilibrium point will lose its stability and Hopf bifurcation will occur. Finally, numerical simulations are given to support the theoretical results.

These results show that the FHN neural system exhibits the parameter regions involved the delay-independence stability and delay dependence stability. Increase of time delay can induce the stability switches between resting state and periodic activity. In other words, the speed of signal transmission between neurons can largely affect the property of neural network. Changing the speed can lead to the neuron switch its state between quiescent state and oscillation. This might provide some clues or helps to the fields of neuroscience and biomedicine.

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