On the complexity of recognizing tough graphs

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Abstract
We consider the relationship between the minimum degree \( \delta \) of a graph and the complexity of recognizing if a graph is \( t \)-tough. Let \( t \geq 1 \) be a rational number. We first show that if \( \delta(G) \geq \frac{tn}{t+1} \), then \( G \) is \( t \)-tough. On the other hand, for any fixed \( \varepsilon > 0 \), we show that it is NP-hard to determine if \( G \) is \( t \)-tough, even for the class of graphs with \( \delta(G) \geq \frac{t}{t+1} n + \varepsilon n \). In particular, for any fixed \( \varepsilon < 1/2 \), it is NP-hard to recognize \( t \)-tough graphs within the class of graphs \( G \) with \( \delta(G) \geq cn \).

1. Introduction
We consider only graphs without loops or multiple edges. Our terminology will be standard except as indicated; a good reference for any undefined terms is [3]. We use \( V(G), \alpha(G) \) and \( \omega(G) \) to denote the vertex set, independence number and the number of components in a graph \( G \), respectively.

Chvátal [4] introduced the notion of tough graphs. Let \( t \) be any positive real number. A graph \( G \) is \( t \)-tough if \( t \alpha(G - X) \leq |X| \) for all \( X \subseteq V(G) \) with \( \alpha(G - X) > 1 \). The interest in tough graphs stems primarily from the fact that all hamiltonian graphs are \( 1 \)-tough. It is still an open problem to determine if there exists a positive constant \( t_0 \) such that all \( t_0 \)-tough graphs are hamiltonian [4]. It is now known [8] that for any

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fixed $\epsilon > 0$ there exists an infinite family of $(2 - \epsilon)$-tough nonhamiltonian graphs having no 2-factor. It is also known [2] that if $G$ is a 2-tough graph on $n$ vertices and the degree sum of any three independent vertices is at least $n$, then $G$ is hamiltonian.

Recently it was shown [1] that the following problem is NP-hard, thus answering a question that had been open for some time [5, 6, 10]. Let $t$ be any positive rational number.

**NOT **$t$-TOUGH

INSTANCE: An undirected graph $G$.

QUESTION: Does there exist $X \subseteq V(G)$ with $\omega(G - X) > 1$ such that $t\omega(G - X) > |X|$?

**Theorem 1.1.** NOT $t$-TOUGH is NP-hard.

The proof for $t = 1$ is accomplished by reducing the following problem, which is known [9] to be NP-complete for any fixed $\beta$, $0 < \beta < 1$:

**INDEPENDENT $\beta$-MAJORITY**

INSTANCE: An undirected graph $G$ on $n$ vertices.

QUESTION: Is $\alpha(G) \geq \beta n$?

Define $\omega(r)$ to be the class of all graphs $G$ with $\delta(G) \geq rn$, where $n = |V(G)|$. It was noted in [1] that for $t = 1$ the proof of Theorem 1.1 could be modified to show that for any fixed $\epsilon > 0$, it is NP-hard to recognize 1-tough graphs in the restricted class $\omega(1/3 - \epsilon)$. In this note we improve this result by showing that for any fixed $\epsilon > 0$, it is NP-hard to recognize 1-tough graphs in the restricted class $\omega(1/2 - \epsilon)$. In fact, we prove the following two results for any rational number $t > 1$.

**Theorem 1.2.** Let $G$ be a graph in $\omega(t/(t + 1))$. Then $G$ is $t$-tough.

**Theorem 1.3.** For any fixed $\epsilon > 0$ it is NP-hard to recognize $t$-tough graphs in $\omega(t/(t + 1) - \epsilon)$.

Note that if $t = 1$, then Theorem 1.2 is an easy consequence of the following well-known theorem of Dirac [7].

**Theorem 1.4.** Let $G$ be a graph on $n \geq 3$ vertices with $\delta \geq n/2$. Then $G$ is hamiltonian.

2. Proofs

**Proof of Theorem 1.2.** If $t = 1$ the result follows from Theorem 1.4, and hence we assume $t > 1$. Let $G$ be a graph on $n$ vertices for which the theorem fails. Hence
δ ≥ tn/(t + 1) and there exists $X \subseteq V(G)$ such that $tw(G-X) > |X|$. Let $Z \subseteq V(G)$ be the vertex set of a component of $G-X$ having the fewest number of vertices. Let $x=|X|$ and $z=|Z|$. Then $n > x + xz/t$ or $z < t(n-x)/x$. Hence if $w \in V(Z)$, $d(w) < x - 1 + t(n-x)/x = x - (t+1)/x$. Thus, to derive the contradiction that $δ < tn/(t + 1)$, it suffices to show

$$\frac{tn}{x + (x - (t + 1))} \leq \frac{tn}{t + 1}.$$ (1)

To do this we first establish

$$x \geq t + 1. \tag{2}$$

Suppose otherwise, i.e. $x < t + 1$. Since $ω(G-X) > 1$, $z < (n-x)/2$. Thus if $w \in V(Z)$, $d(w) \leq -1 + x + (n-x)/2 = -1 + (n+x)/2$. Hence, $-1 + (n+t+1)/2 > -1 + (n+x)/2 \geq d(w) \geq tn/(t + 1)$. This leads to $t^2 - 1 > n(t-1)$ and since $t > 1$ we conclude $t+1 > n$. But then $δ \geq tn/(t+1) > (n-1)n/n = n-1$, which is impossible. This proves (2).

To establish (1) note that (1) is equivalent to $tn/(t+1) - tn/x \geq x -(t+1)$ or

$$\frac{tn(x - (t + 1))}{x(t + 1)} \geq x(t + 1). \tag{3}$$

If $x - (t + 1) = 0$, then (3) follows immediately. Otherwise, (2) implies $x -(t + 1) > 0$ and now (3) is equivalent to

$$tn \geq x(t + 1). \tag{4}$$

However (4) follows easily since $t(n-x) > tw(G-X) > x$. This completes the proof. \( \square \)

To prove Theorem 1.3 we first require a proof of Theorem 1.1 for $t \geq 1$ that differs from the proof given in [1]. We again reduce INDEPENDENT $β$-MAJORITY to NOT $t$-TOUGH.

Alternate proof of Theorem 1.1 for $t \geq 1$. Let $t = a/b \geq 1$ for positive integers $a$ and $b$, and fix $β$, where $0 < β < 1$. Let $G$ be a graph with vertex set $\{v_1, \ldots, v_n\}$ and let $k = \lfloor β n \rfloor$. Construct $G'$ from $G$ as follows. First add to $G$ a set $A = \{w_1, \ldots, w_n\}$ of independent vertices, and join $v_i$ to $w_i$ for $i = 1, 2, \ldots, n$. Then add another set $T$ of $br$ independent vertices to $G$, where $r \geq 1$ is an integer. Now add a set $B$ of $\lceil (t-1)n \rceil + k-1+ar$ vertices which induces a complete graph, and join each vertex of $B$ to every vertex of $V(G) \cup A \cup T$. It suffices to show that $x(G) \geq k$ if and only if $G'$ is not $t$-tough.

First suppose that $G$ contains an independent set $I$ with $|I| = k$. Define $X = V(G') \setminus (V(G) \setminus I) \cup B$. Then $ω(G'-X') = n + br$ and $|X'| = n - k + \lceil (t-1)n \rceil + k - 1 + ar = \lceil tn \rceil + 1 + ar$. Thus $tw(G'-X') = an/b + ar > \lceil an/b \rceil + 1 + ar = |X'|$ and $G'$ is not $t$-tough.

Conversely, suppose $G'$ is not $t$-tough. Then there exists $X' \subseteq V(G')$ with $ω(G'-X') > 1$ such that $tw(G'-X') > |X'|$. Clearly $B \subseteq X'$. We may also assume
$X' \cap (A \cup T) = \emptyset$; otherwise, $tw(G' - (X' - (A \cup T))) \geq tw(G' - X') > |X'| > |X' - (A \cup T)|$
and we could use $X' - (A \cup T)$ instead of $X'$. Let $X = X' \cap V(G)$, $x = |X|$ and $x' = |X'|$.

**Claim 2.1.** $n \geq x + k$.

Suppose $x > n - k$. Then $x' = x + |B| > n - k + \left\lceil (t - 1)n \right\rceil + k - 1 + ar = \left\lceil tn \right\rceil - 1 + ar$. Clearly $tw(G' - X') \leq t(n + br)$. Thus $tw(G' - X') \leq tn + ar \leq x'$. This contradiction establishes Claim 2.1.

**Claim 2.2.** $\omega(G - X) \geq k$.

Clearly $x' = x + \left\lceil (t - 1)n \right\rceil + k - 1 + ar$ and $\omega(G' - X') = \omega(G - X) + x + br$. Since $tw(G' - X') > x'$, we have

$$t(\omega(G - X) + x + br) > x + (t - 1)n + k - 1 + ar.$$

Since $n \geq x + k$ by Claim 2.2, we conclude

$$tw(G - X) > (t - 1)(n - x) + k - 1 + ar$$

$$= (t - 1)(n - x) + k - 1$$

$$= tk + (t - 1)(n - x - k) - 1$$

$$\geq tk - 1.$$

So $\omega(G - X) > k - 1/t$, and thus $\omega(G - X) \geq k$. This proves Claim 2.2.

Since it is possible to form an independent set in $G$ by choosing one vertex from each component of $G - X$, we conclude $\omega(G) \geq k$. □

**Proof of Theorem 1.3.** Given $\varepsilon > 0$ and $t = a/b \geq 1$, choose $\gamma$ such that $0 < \gamma < 1$, and then choose $r$ sufficiently large such that

$$\frac{(t - \gamma)n + ar}{(t + 2 - \gamma)n + 1 + (a + b)r} > \frac{t}{t + 1} - \varepsilon.$$  

(5)

Let $\beta = 1 - \gamma$. The reduction described in the alternate proof of Theorem 1.1 yields a graph $G'$ with $|V(G')| = 2n + br + \left\lceil (t - 1)n \right\rceil + \left\lceil \beta n \right\rceil - 1 + ar \leq (t + \gamma)n + 1 + (a + b)r$ and $\delta(G') = \left\lceil (t - 1)n \right\rceil + \left\lceil \beta n \right\rceil + ar \geq (t - \gamma)n + ar$. By (5) it follows that $G' \in \Omega(t/(t + 1) - \varepsilon)$. This establishes that it is NP-hard to recognize $t$-tough graphs in this class. □

**References**

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