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A Generalization of a Lemma of Bihari and Applications to Pointwise Estimates for Integral Equations

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1. The most commonly used generalization of the Bellman-Gronwall inequality is due to I. Bihari [1]. See also [2] where related integral inequalities are discussed and more references given. The integral equation associated with Bihari's inequality is nonlinear in the function of interest (see Section 2), but linear as a function of the integral involved. Recent special uses of integral inequalities in which the associated integral equation is nonlinear in the integral have been considered in papers by D. Willett [3] and H.E. Gollwitzer [4].

In Section 2 of this paper, Bihari's inequality is generalized. Some results of Muldowney and Wong in [5] are improved, while it is shown that in some situations results of Willett and Gollwitzer are considerable sharpened.

Results in terms of maximal solutions and uniqueness theorems are anticipated in a later paper. Fred Brauer considered such extensions of Bihari's inequality in [6]. S C. Chu and F. T. Metcalf in [7] produced an explicit maximal solution for an integral inequality involving a general Volterra integral.

2. In this section we generalize the following lemma of Bihari [1]:

LEMMA (Bihari). *Let $x(t)$, $k(t)$ be positive, continous functions in $c \leqslant t \leqslant d$, and let a, b be nonnegative constants; further let $g(u)$ be a positive nondecreasing function for $u \geqslant 0$. Then the inequality*

$$x(t) \leqslant a + b \int_c^t k(s)\, g(x(s))\, ds, \qquad c \leqslant t \leqslant d \tag{1}$$

implies the inequality

$$x(t) \leqslant \Omega^{-1} \left\{ \Omega(a) + b \int_c^t k(s)\, ds \right\}, \qquad c \leqslant t \leqslant d' \leqslant d \tag{2}$$

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where

$$\Omega(u) = \int_\epsilon^u \frac{ds}{g(s)} \qquad (\epsilon > 0, u > 0) \tag{3}$$

and d' is defined so that $\Omega(a) + b \int_c^t k(s)\, ds$ lies within the domain of definition of $\Omega^{-1}(u)$, for $c \leqslant t \leqslant d'$. We prove the following:

THEOREM. *Let $x(t)$, $a(t)$, $b(t)$ be positive functions of t, bounded in $c \leqslant t \leqslant d$, let $k(t, s)$ be nonnegative, bounded on the triangular region $s \leqslant t \leqslant d$, $c \leqslant s \leqslant d$; assume further that $x(t)$ is measurable and $k(t, s)$ is a measurable function of s for each t with $c \leqslant s \leqslant d$. Let $f(u)$, $g(u)$ be positive functions for $u \geqslant 0$, with f strictly increasing and g nondecreasing. Then defining*

$$A(t) = \sup_{c\leqslant s\leqslant t} a(s), \qquad B(t) = \sup_{c\leqslant s\leqslant t} b(s),$$
$$K(t, s) = \sup_{s\leqslant\sigma\leqslant t} k(\sigma, s), \tag{4}$$

the inequality

$$f(x(t)) \leqslant a(t) + b(t) \int_c^t k(t, s)\, g(x(s))\, ds, \qquad c \leqslant t \leqslant d \tag{5}$$

implies the inequality

$$x(t) \leqslant f^{-1}\left[\Omega^{-1}\left\{\Omega(A(t)) + B(t) \int_c^t K(t, s)\, ds\right\}\right], \qquad c \leqslant t \leqslant d' \leqslant d \tag{6}$$

where

$$\Omega(u) = \int_\epsilon^u \frac{dw}{g(f^{-1}(w))} \qquad (\epsilon > 0, u > 0) \tag{7}$$

and

$$d' = \max\left[c \leqslant \tau \leqslant d : \Omega(A(\tau)) + B(\tau) \int_c^\tau K(\tau, s)\, ds \leqslant \Omega(f(\infty))\right]. \tag{8}$$

Proof. Let $c \leqslant t \leqslant T \leqslant d'$. It will be apparent from the proof that d' is defined in such a way that the right-hand side of (6) is meaningful. Then (5) implies that

$$f(x(t)) \leqslant A(T) + B(T) \int_c^t K(T, s)\, g(x(s))\, ds. \tag{9}$$

Denoting the right-hand side of (9) by $V(T, t)$, this is equivalent to

$$f(x(t)) \leqslant V(T, t).$$

By the monotonicity of g, f, therefore,

$$g(x(t)) \leqslant g(f^{-1}(V(T, t))). \tag{10}$$

Differentiating with respect to t and using (10), yields

$$\frac{\frac{\partial V}{\partial t}(T, t)}{g(f^{-1}(V(T, t)))} \leqslant B(T)\, K(T, t), \qquad \text{a.e.}$$

Integrating from $t = c$ to $t = T$, we have

$$\Omega(V(T, T)) \leqslant \Omega(V(T, c)) + B(T) \int_c^T K(T, s)\, ds,$$

$$V(T, T) \leqslant \Omega^{-1} \left\{ \Omega(V(T, c)) + B(T) \int_c^T K(T, s)\, ds \right\},$$

$$f(x(T)) \leqslant V(T, T) \leqslant \Omega^{-1} \left\{ \Omega(A(T)) + B(T) \int_c^T K(T, s)\, ds \right\},$$

which is (6) with a change of dummy variable.

Note if $a(t)$, $b(t)$ are nondecreasing functions of t, and $k(t, s)$ is a nondecreasing function of t for each s with $c \leqslant s \leqslant d$, the conclusion (6) of the theorem can be replaced by

$$x(t) \leqslant f^{-1} \left[\Omega^{-1} \left\{ \Omega(a(t)) + b(t) \int_c^t k(t, s)\, ds \right\} \right], \qquad c \leqslant t \leqslant d' \leqslant d,$$

with appropriate modifications if only some of these functions are nondecreasing.

3. We use the theorem proved to obtain pointwise estimates in certain special cases.

COROLLARY 1. *If in the theorem,* $f(u) = u^p$, $g(u) = u^q$, *where* $p, q > 0$, *then the inequality* (5) *implies the inequalities*

$$\text{(i)} \qquad x(t) \leqslant \left[(A(t))^{1-(q/p)} + \left(1 - \frac{q}{p}\right) B(t) \int_c^t K(t, s)\, ds \right]^{1/(p-q)},$$

for all t *with* $c \leqslant t \leqslant d'$, *if* $p \neq q$ *(in the case* $q < p$; $p \geqslant 1$, d' *is equal to* d*).*

$$\text{(ii)} \qquad x(t) \leqslant \left[A(t) \exp \left(B(t) \int_c^t K(t, s)\, ds \right) \right]^{1/p}$$

for all t *with* $c \leqslant t \leqslant d$, *if* $p = q$.

We omit the proof and remark that the inequalities in the corollary have been obtained in [5] for $p = 1$, $0 \leqslant q \leqslant 1$ and $K(t, s) = t - s$.

COROLLARY 2. *Let $x(t)$, $a(t)$, $b(t)$, $k(t, s)$ satisfy the hypotheses of the theorem. Let $p \geqslant z$ be a positive integer. Then the inequality*

$$x(t) \leqslant a(t) + b(t) \left(\int_c^t k(t, s)\, x^p(s)\, ds \right)^{1/p}, \qquad c \leqslant t \leqslant d \tag{11}$$

implies the inequality

$$x(t) \leqslant A(t)\, e(p-1) \exp\left(\frac{1}{p} B^p(t) \int_c^t K(t, s)\, ds \right), \qquad c \leqslant t \leqslant d. \tag{12}$$

Proof. The inequality (11) implies

$$x(t) - A(T) \leqslant b(t) \left(\int_c^t k(t, s)\, x^p(s)\, ds \right)^{1/p}, \qquad c \leqslant t \leqslant T \leqslant d.$$

Thus

$$y^p(t) \leqslant b^p(t) \int_c^t k(t, s)\, x^p(s)\, ds, \qquad c \leqslant t \leqslant T \leqslant d, \tag{13}$$

where

$$y(t) = \begin{cases} x(t) - A(T), & \text{if} \quad x(t) - A(T) \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

Applying the Theorem with $f(u) = u^p$, $g(u) = (u + A(T))^p$, and noting that $d' = T$ in this case, we have that inequality (13) implies

$$\Omega(y^p(t)) \leqslant \Omega(0) + B^p(t) \int_c^t K(t, s)\, ds, \qquad c \leqslant t \leqslant T \tag{14}$$

where

$$\Omega(u) = \int_0^u \frac{dw}{(w^{1/p} + A(T))^p}.$$

Thus

$$\Omega(y^p(t)) = \int_0^{y^p(t)} \frac{dw}{(w^{1/p} + A(T))^p} = p \int_0^y \frac{z^{p-1}\, dz}{(z + A(T))^p}.$$

Denoting

$$\int_0^y \frac{z^{p-1}\, dz}{(z + A(T))^p}$$

by I_p, it follows, on integrating by parts,

$$I_p = I_{p-1} - \frac{y^{p-1}}{(p-1)(y + A(T))^{p-1}}.$$

Inductively, therefore, we have

$$\Omega(y^p(t)) = p \left[\log\left(\frac{y + A(T)}{A(T)} \right) - \sum_{m=1}^{p-1} \frac{y^m}{m(y + A(T))^m} \right]. \tag{15}$$

From (14) and (15), we have

$$\log \frac{y(t) + A(T)}{A(T)} - \sum_{m=1}^{p-1} \frac{1}{m} \leqslant \frac{1}{p} B^p(t) \int_c^t K(t, s)\, ds, \qquad c \leqslant t \leqslant T.$$

Thus

$$y(t) + A(T) \leqslant A(T)\, e(p-1) \exp\left[\frac{1}{p} B^p(t) \int_c^t K(t, s)\, ds\right], \qquad c \leqslant t \leqslant T.$$

Hence, taking $t = T$, and using the fact that $x(T) \leqslant y(T) + A(T)$, we have the desired result.

It is of interest to compare our estimate with those of Willett [3] and Gollwitzer [4]. If we take $a(t) \equiv \alpha$, $b(t) \equiv 1$, $k(t, s) \equiv 1$, $c = 0$, we obtain from Corollary 2 that inequality (12) reduces to

$$x(t) \leqslant \alpha e(p-1) \exp\left(\frac{t}{p}\right), \tag{16}$$

whereas the estimate of Willett yields

$$x(t) \leqslant \alpha\left[1 + (e^t - 1) \sum_{k=0}^{p-1} (1 - e^{-t})^{k/p}\right], \tag{17}$$

and that of Gollwitzer yields

$$x(t) \leqslant \alpha[1 + \exp(2^{p-1}t - 1)^{1/p}]. \tag{18}$$

Although (17) and (18) are better estimates than (16) for small values of t, for large values of t, (16) is considerably better than either of (17) or (18).

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