Lower Bounds for the $cd$-index of Odd-dimensional Simplicial Manifolds

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We obtain lower bounds on the coefficients of the $cd$-index of any $(2k - 1)$-dimensional simplicial manifold (or, more generally, any Eulerian Buchsbaum complex) $\Delta$. These bounds imply that many of the coefficients of the $cd$-index of such $\Delta$ are positive and that

\[
(-1)^l \tilde{\chi}(\text{Skel}_l(\Delta)) > 1 + \left\{ \begin{array}{ll}
(\frac{2k-1}{k-1})\beta_{k-1} - 1 & l = 2k - 3, 2k - 2 \\
\sum_{i=0}^{\min\{k-1,l\}} (\frac{1}{2})^i \beta_i & 0 \leq l \leq 2k - 4,
\end{array} \right.
\]

where $\tilde{\chi}$ denotes the reduced Euler characteristic and $\beta_0, \beta_1, \ldots$ are reduced Betti numbers of $\Delta$.

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1. Introduction

The $cd$-index $\Phi_P(c, d)$ is a noncommutative polynomial in variables $c$ and $d$ which encodes the flag $f$-vector of an Eulerian poset $P$. The $cd$-index was first defined by J. Fine (see [3], Proposition 2.). The intriguing property of the $cd$-index due to Stanley [14] is that it is nonnegative (i.e., all its coefficients are nonnegative) for all (face posets of) simplicial spheres, and, more generally, for all Gorenstein* simplicial posets (that is, Eulerian and Cohen–Macaulay posets). (Another large class of complexes with nonnegative $cd$-index is the class of S-shellable spheres, which includes, for example, all convex polytopes, see [4, 14].)

The aim of this paper is to extend Stanley’s theorem in the following way: our main result is that many of the coefficients of the $cd$-index of an odd-dimensional simplicial manifold (i.e., a simplicial complex whose geometric realization is a manifold) are positive. (By Poincaré’s duality theorem every odd-dimensional simplicial manifold is a Eulerian complex, and therefore its $cd$-index is well defined.) Even more generally, the same result holds for all odd-dimensional simplicial complexes which are Eulerian and Buchsbaum (over a field of characteristic 0).

For a simplicial $(n - 1)$-dimensional complex $\Delta$, define $f_i(\Delta)$ as the number of $i$-dimensional faces of $\Delta$. Set $f_{-1} = 1$. If $\Delta$ is a Eulerian complex, then the numbers

\[
\tilde{\chi}_i := (-1)^i \tilde{\chi}(\text{Skel}_i(\Delta)) = \sum_{j=-1}^{i} (-1)^{i-j} f_j \quad (0 \leq i \leq n - 2)
\]

are (almost) equal to certain coefficients of the $cd$-index of $\Delta$. The application of our main result, then, shows that $\tilde{\chi}_i > 1$ ($0 \leq i < \dim(\Delta)$) if $\Delta$ is an odd-dimensional simplicial manifold. This result answers negatively the question posed by Stanley: whether there exists an odd-dimensional simplicial manifold $\Delta$ with $\tilde{\chi}_i(\Delta) \leq 0$ for some $i < \dim(\Delta)$ (private communication). The proof of the above results uses $\Phi^\prime_p(c, d)$-polynomials, which appear from the $cd$-indexes of semisuspended shelling components of an $n$-dimensional simplex, and the crucial fact (due to Stanley) that

- $\Phi^\prime_p$-polynomials are nonnegative; and

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• the cd-index of every \((n - 1)\)-dimensional simplicial Eulerian complex \(\Delta\) can be calculated using these polynomials and certain linear combinations of \(f\)-numbers of \(\Delta\) (called \(h\)-numbers) as follows

\[
\Phi_{\Delta} = \sum_{i=0}^{n-1} h_i \Phi_i^n.
\] (1)

The paper is divided into six sections. In Section 2 we review the basic definitions on the cd-index and formally state our results concerning positivity of coefficients of the cd-index. In Section 3 we discuss the necessary background on Buchsbaum complexes which leads to the lower bounds on their \(h\)-numbers. In Section 4 we obtain simple recursion formulas for \(\Phi_i^n(c, d)\)-polynomials (see Lemmas 4.1 and 4.2). Many other recurrences for \(\Phi_i^n\) are given in [6, 7, 9]. We also state similar recursion formulas for the cd-indexes of semisuspended cubical shelling components in Section 6. These recursion formulas, (1), and the lower bounds on the \(h\)-numbers yield a simple inductive proof of our main result, which appears in Section 5.

2. BASIC DEFINITIONS AND STATEMENTS OF RESULTS

An Eulerian poset is a finite graded poset with \(\hat{0}\) and \(\hat{1}\) such that every interval with more than one element has the same number of elements of odd rank as of even rank.

The flag \(f\)-vector of a Eulerian poset \(P\) counts the number of chains of \(P\) whose elements have specified ranks. More precisely, let \(P\) be a Eulerian poset of rank \(n + 1\). Given a set \(S = \{i_1 < i_2 < \cdots < i_k\} \subset [n]\), denote by \(\alpha_P(S)\) the number of chains \(\hat{0} < x_1 < x_2 < \cdots < x_k < \hat{1}\) in \(P\) satisfying \(\text{rk}(x_j) = i_j\) for \(j = 1, \ldots, k\). The function \(\alpha_P\) is called the flag \(f\)-vector of \(P\).

The complete set of linear relations satisfied by the flag \(f\)-vectors of all Eulerian posets was found by Bayer and Billera [2]. Fine (see [3, 14]), then, realized that these relations allow encoding of the flag \(f\)-vector of any Eulerian poset \(P\) via a certain noncommutative polynomial \(\Phi_P(c, d)\) called the cd-index of \(P\). This polynomial is defined as follows. Fix a graded poset \(P\) of rank \(n + 1\) with \(\hat{0}\) and \(\hat{1}\). For every \(S \subset [n]\), define a noncommutative monomial \(u_S = u_1 u_2 \ldots u_n\) in the variables \(a\) and \(b\) by

\[
u_i = \begin{cases} a & \text{if } a \notin S \\ b & \text{if } i \in S. \end{cases}
\]

Let

\[
\Upsilon_p(a, b) = \sum_{S \subset [n]} \alpha_P(S) u_S,
\]

\[
\Psi_P(a, b) = \Upsilon_p(a - b, b).
\]

Fine’s formulation of Bayer–Billera’s result is: if \(P\) is a Eulerian poset, then \(\Psi_P(a, b)\) can be written as a (unique) polynomial \(\Phi_P(c, d)\) in \(c = a + b\) and \(d = ab + ba\). Setting \(\text{deg}(c) = 1\) and \(\text{deg}(d) = 2\) turns \(\Phi_P(c, d)\) into a homogeneous polynomial of degree \(n\) (with integer coefficients).

When \(\Delta\) is a simplicial or, more generally, a regular CW complex, whose face poset with a \(\hat{1}\) adjoined \((P(\Delta))\), is Eulerian, (in the following we refer to such complex as a Eulerian complex), we define the cd-index of \(\Delta\) by \(\Phi_{\Delta}(c, d) = \Phi_{P(\Delta)}(c, d)\).

A simplicial complex \(\Delta\) is a Buchsbaum complex (over a field \(k\)) if for all \(p \in X = |\Delta|\) and \(i < \text{dim} \Delta\), \(H_i(X, X - p, k) = 0\) (where \(X = |\Delta|\) is a geometric realization of \(\Delta\) and
Let $\Delta$ be a $(2k - 1)$-dimensional simplicial complex which is Eulerian and Buchsbaum (over a field of characteristic 0). Our main result is that many of the coefficients of the $cd$-index of such $\Delta$ are positive. More precisely, we have (in the following, we denote by $[w]\Phi$ a coefficient of monomial $w$ in a polynomial $\Phi$)

**Theorem 2.1.** Let $\Delta$ be a $(2k - 1)$-dimensional simplicial complex which is Eulerian and Buchsbaum (over a field $k$, char $k = 0$), and let $\beta_0, \beta_1, \ldots, \beta_{2k-1}$ be its reduced Betti numbers (over $k$). Then

$$[wde^l]\Phi_\Delta(c,d) > \begin{cases} \sum_{j=0}^{2k-1} \beta_k \beta_{k-1} (2k-2-i) j \beta_j \Phi_\Delta(c,d) & \text{if } i = 0, 1 \\ \sum_{j=0}^{2k-1} \beta_k \beta_{k-1} (2k-2-i) j \beta_j \Phi_\Delta(c,d) & \text{if } i \geq 2 \end{cases}$$

whenever

1. $1 \leq i \leq 2k - 3$ is odd and $w$ is any monomial of degree $2k - i - 2$; or
2. $0 \leq i \leq 2k - 2$ is even and $w = e^{2k-i-2}$.

In particular, all these coefficients are positive.

It follows easily from [14, Proposition 1.3] or [4, Proposition 7.1] that for a Eulerian $(n - 1)$-dimensional complex $\Delta$

$$(-1)^l \tilde{\chi}(\text{Skel}_l(\Delta)) = 1 + [e^l d^n e^{l-2}]\Phi_\Delta(c,d) \quad \text{for } l = 0, 1, \ldots, n - 2,$$

where $\tilde{\chi}$ is a reduced Euler characteristic. Hence we obtain as a corollary to Theorem 2.1 the following result.

**Theorem 2.2.** For any $(2k - 1)$-dimensional Eulerian Buchsbaum (over a field of characteristic 0) complex $\Delta$

$$(-1)^l \tilde{\chi}(\text{Skel}_l(\Delta)) > 1 + \begin{cases} \sum_{j=0}^{2k-1} \beta_k \beta_{k-1} (2k-2-i) j \beta_j \Phi_\Delta(c,d) & \text{if } l = 2k - 3, 2k - 2 \\ \sum_{j=0}^{2k-1} \beta_k \beta_{k-1} (2k-2-i) j \beta_j \Phi_\Delta(c,d) & \text{if } 0 \leq l \leq 2k - 4. \end{cases}$$

3. **Preliminaries on Buchsbaum Complexes**

In this Section we review the necessary background on Buchsbaum complexes which leads to the lower bounds on their $h$-numbers. (For all undefined terminology the reader is referred to Chapter 2 of Stanley’s book [16].)

Let $\Delta$ be a finite simplicial Buchsbaum complex over (an infinite) field $k$. Define $f_i = f_i(\Delta)$ to be the number of $i$-dimensional faces of $\Delta$ ($i = 0, 1, \ldots, n - 1$). Set $f_{-1} = 1$. The vector $(f_{-1}, f_0, \ldots, f_{2k-1})$ is called the $f$-vector of $\Delta$. The $h$-vector of $\Delta$ is a vector $h(\Delta) = (h_0, h_1, \ldots, h_n)$ such that $\sum_{i=0}^{n} h_i x^{n-i} = \sum_{i=0}^{n} f_{i-1} (x - 1)^{n-i}$. In particular, $h_0(\Delta) = 1$.

Denote by $R_\Delta$ the Stanley–Reisner ring of $\Delta$ over $k$. Choose $y_1, \ldots, y_n$ to be a linear system of parameters of $R_\Delta$ and define

$$h_i^1 = \dim[R_\Delta/(y_1, \ldots, y_n)]_i \quad \text{(for } i = 0, 1, \ldots, n)$$

as the dimension of the $i$th homogeneous component of $R_\Delta/(y_1, \ldots, y_n)$. 

Schenzel’s theorem [13] provides connections between the \( h \)- and \( h' \)-numbers of a Buchsbaum complex \( \Delta \):

\[
h_i(\Delta) = h_i'(\Delta) = \binom{n}{i} \sum_{j=0}^{i-2} (-1)^{i-j} \beta_j \quad (i = 1, 2, \ldots, n),
\]

where \( \beta_j := \dim_k \bar{H}_j(\Delta) \) \((j = 0, \ldots, n - 1)\) are reduced Betti numbers of \( \Delta \) (over \( k \)).

Clearly, \( h_i' \geq 0 \), since \( h' \)-numbers are defined as dimensions of certain vector spaces. However, if \( \text{char } k = 0 \), then [12, Theorem 1.7] implies stronger inequalities, namely

\[
h_i'(\Delta) - \binom{n-1}{i} \beta_{i-1} \geq 0 \quad \text{for } i = 1, 2, \ldots, n - 1.
\]

These inequalities together with (2) yield the following lower bounds on the \( h \)-numbers of a Buchsbaum (over \( k \), \( \text{char } k = 0 \)) complex \( \Delta \):

\[
h_i(\Delta) \geq \binom{n-1}{i} \beta_{i-1} - \binom{n}{i} \sum_{j=0}^{i-2} (-1)^{i-j} \beta_j \quad \text{for } i = 1, 2, \ldots, n - 1.
\]

**Remark.** Although the assumption \( \text{char } k = 0 \) was not mentioned in [12, Theorem 1.7], the proof of this theorem uses symmetric algebraic shifting/generic initial ideal techniques (see [1], [8, Ch. 15], [10]). In particular, it relies on the fact that the generic initial ideal of the Stanley–Reisner ideal is strongly stable, the fact known so far in the case of characteristic 0 only. In order to try to eliminate this assumption we need either to show that symmetric shifting works for (ideals generated by) squarefree monomials over any characteristic, or to prove the analogues of facts from commutative algebra on Buchsbaum rings for the exterior version of Stanley–Reisner ring [5].

4. **Recursion Formulas for \( \Phi_i^n \)-Polynomials**

In this section we derive recursion formulas for \( \Phi_i^n \)-polynomials.

For a CW complex \( \Gamma \) whose underlying space \( |\Gamma| \) is an \( n \)-dimensional ball, we define the *semisuspension* of \( \Gamma \), \( \Gamma' \), to be a CW complex obtained from \( \Gamma \) by attaching a single new \( n \)-cell \( \tau \) along the boundary of \( \Gamma \) (that is \( \partial \tau = \partial \Gamma \)).

Let \( \Lambda^n \) be the boundary complex of an \( n \)-simplex. Denote its facets by \( \sigma_0, \sigma_1, \ldots, \sigma_n \).

Define an \((n - 1)\)-dimensional CW complex \( \Lambda_i^n \) \((i = 0, \ldots, n - 1)\) as a semisuspension of a simplicial complex generated by \( \sigma_0, \sigma_1, \ldots, \sigma_i \). In particular, \( \Lambda_{n-1}^n = \Lambda^n \).

Following [14], define \( \Phi_i^n = \Phi_{\Lambda_i^n} - \Phi_{\Lambda_i^{n-1}} \) (with \( \Phi_0^n = \Phi_{\Lambda_0^n} \) and \( \Phi_n^n = 0 \)). For example, \( \Phi_0^1 = c, \Phi_0^2 = c^2, \) and \( \Phi_1^2 = d \).

To derive recursion formulas for \( \Phi_i^n \)-polynomials, consider the boundary complex, \( \Sigma_i \cong \Lambda_i^{n-1} \), of simplex \( \sigma_i \) \((1 \leq i \leq n - 1)\) and the following complexes:

- a simplicial \((n - 2)\)-dimensional complex \( \Gamma_i \), generated by all facets of \( \sigma_i \) contained in \( \text{cl} (\sigma_0 \cup \ldots \cup \sigma_{i-1}) \) (there are \( i \) such facets), the boundary complex of \( \Gamma_i, \partial \Gamma_i \), and the semisuspension of \( \Gamma_i, \Gamma_i' \); and
- a simplicial \((n - 2)\)-dimensional complex \( \tilde{\Gamma}_i \), generated by all facets of \( \sigma_i \) which are not contained in \( \text{cl} (\sigma_0 \cup \ldots \cup \sigma_{i-1}) \) (there are \( n - i \) such facets), the boundary complex of \( \tilde{\Gamma}_i, \partial \tilde{\Gamma}_i \), and the semisuspension of \( \tilde{\Gamma}_i, \tilde{\Gamma}_i' \).
Note that $\partial \Gamma_1 = \partial \tilde{\Gamma}_1$, $\Gamma'_1 \cong \Lambda''_{n-1}$, and $\tilde{\Gamma}'_1 \cong \Lambda_{n-1}$.

It follows from [4, Lemma 4.2] that

$$\hat{\Phi}^n_i = \Phi_{\Sigma_i} c - \Phi_{\Gamma'_i} c + \Phi_{\partial \Gamma'_i} d.$$  \hspace{1cm} (4)

Comparing the coefficients of monomial $wc$ on both sides of (4) and recalling that $\Sigma_i \cong \Lambda^{n-1}$ and $\Gamma_i \cong \Lambda''_{n-1}$, we obtain

**LEMMA 4.1.** $[wc] \hat{\Phi}^n_i = [wc] \Phi_{\Lambda''_{n-1}} - [wc] \Phi_{\Lambda''_{n-1}} = [wc] \sum_{j=1}^{n-1} \hat{\Phi}^n_j$ for all $1 \leq i \leq n - 1$ and all monomials $w$, $\deg(w) = n - 1$. (Where $[wc]$ denotes the coefficient of monomial $w$ in polynomial $\Phi$.)

By [14, Lemma 1.1 and Eqn. (21)]

$$[wc^2] \hat{\Phi}^n_i = [wc] \Phi_{\Gamma'_i} - [wc] \Phi_{\partial \Gamma'_i} (1 \leq i \leq n - 1),$$

and so

$$[wc] \Phi_{\partial \Gamma'_i} = [wc] \Phi_{\partial \Gamma'_i} = [wc] \Phi_{\Gamma'_i} - [wc^2] \hat{\Phi}^n_i = [wc] \left( \Phi_{\Lambda''_{n-1}} - \sum_{j=1}^{n-1} \hat{\Phi}^n_j \right)$$

$$= [wc] \left( \sum_{j=0}^{n-i-1} \hat{\Phi}^n_j - \sum_{j=i}^{n-1} \hat{\Phi}^n_j \right),$$  \hspace{1cm} (5)

where the penultimate step is by Lemma 4.1.

Substituting (5) in (4) and comparing the coefficients of monomial $wd$ on both sides of (4), we infer

**LEMMA 4.2.**

$$[wd] \hat{\Phi}^n_i = [wc] \left( \sum_{j=0}^{n-i-1} \hat{\Phi}^n_j - \sum_{j=i}^{n-1} \hat{\Phi}^n_j \right) = [wc] \left( \sum_{j=0}^{n-i-1} \hat{\Phi}^n_j - \sum_{j=n-i}^{n-1} \hat{\Phi}^n_j \right)$$

for all $1 \leq i \leq n - 1$ and all monomials $w$, $\deg(w) = n - 2$.

In particular, $[wd] \hat{\Phi}^n_i = [wd] \hat{\Phi}^n_{n-i}$ for all $1 \leq i \leq n - 1$.

**REMARK 4.3.** (1) Since [14, Lemma 1.1] implies that $\hat{\Phi}^n_0 = \Phi_{\Lambda''_{n-1}} c = \sum_{j=0}^{n-1} \hat{\Phi}^n_j c$, the statements of Lemmas 4.1 and 4.2 hold for $i = 0$ as well.

(2) It follows from Lemmas 4.1 and 4.2 that

$$[wc] \hat{\Phi}^n_i - [wc] \hat{\Phi}^n_{i+1} = [wc] \hat{\Phi}^n_{i-1}, \quad \text{and}$$

$$[w'd] \hat{\Phi}^n_i - [w'd] \hat{\Phi}^n_{i+1} = [w'd] \left( \hat{\Phi}^n_{i-1} - \hat{\Phi}^n_i \right).$$

As a result, we obtain (by induction on $n$) that

$$[wc] \hat{\Phi}^n_0 \geq [wc] \hat{\Phi}^n_1 \geq \cdots \geq [wc] \hat{\Phi}^n_{n-1} = 0, \quad \text{for any } w, \deg(w) = n - 1;$$

$$0 = [w'd] \hat{\Phi}^n_0 \leq [w'd] \hat{\Phi}^n_1 \leq \cdots \leq [w'd] \hat{\Phi}^n_{n/2}, \quad \text{for any } w', \deg(w') = n - 2.$$
5. Positivity of Many Coefficients

The goal of this section is to obtain the lower bounds on the cd-index of odd-dimensional Eulerian Buchsbaum complexes and to prove Theorem 2.1.

Let \( \Delta \) be a \((2k-1)\)-dimensional Eulerian Buchsbaum (over \( k, \text{char} \, k = 0 \)) complex and let \((h_0, h_1, \ldots, h_{2k})\) be its \( h \)-vector. Since the \( h \)-vector of every Eulerian \((2k-1)\)-dimensional simplicial complex satisfies the Dehn–Sommerville relations (proved by Klee [11]), namely, \( h_i = h_{2k-i} \) for \( i = 0, 1, \ldots, 2k \), it follows from (1) that

\[
\Phi_\Delta(c, d) = \sum_{i=0}^{k-1} h_i(\hat{\Phi}_i^{2k} + \hat{\Phi}_{2k-i}^{2k}) + h_k \hat{\Phi}_k^{2k}.
\] (6)

Substituting lower bounds (3) on the \( h \)-numbers of \( \Delta \) in (6) and using nonnegativity of \( \hat{\Phi}_i^{2k} \) polynomials (see Remark 4.3, part 2) we obtain the following lower bound on \( \Phi_\Delta \)

\[
\Phi_\Delta \geq \binom{2k-1}{k} \hat{\Phi}_k^{2k} \beta_{k-1} + \sum_{j=0}^{k-2} B_j^k \beta_j,
\] (7)

where

\[
B_j^k = \binom{2k-1}{j+1} \hat{\Phi}_{j+1}^{2k} - \sum_{i=j+2}^{2k-j-2} (-1)^{i-j} \binom{2k}{i} \hat{\Phi}_i^{2k} + \binom{2k-1}{2k-j-2} \hat{\Phi}_{2k-j-1}^{2k}.
\] (8)

To complete the proof of Theorem 2.1 it remains to estimate the coefficients of \( \beta_0, \ldots, \beta_{k-1} \) in (7). This will be done by simple induction argument using the following three lemmas (the first two of them will serve as the base of induction and the third one as the induction step).

**Lemma 5.1.** \([wdc]B_j^k = 0\) for any \( 0 \leq j \leq k-2 \) and for any monomial \( w \).

**Proof.** It follows from Lemma 4.1 that

\[
[wdc] (\hat{\Phi}_i^{2k} - \hat{\Phi}_i^{2k+1}) = [wd] \hat{\Phi}_i^{2k-1}.
\]

Substituting this in (8) and using the fact that \([wd] \hat{\Phi}_i^{2k-1} = [wd] \hat{\Phi}_{2k-1}^{2k-1-i} \) (see Lemma 4.2), we infer the proof of the Lemma. \( \square \)

**Lemma 5.2.** \([c^{2k-2}d]B_j^k = 0\) for \( 0 \leq j \leq k-2 \).

**Proof.** Note that for any Eulerian poset \( P \) of rank \( n+1 \), \([c^n] \Phi_P = 1 \). Therefore, it follows from (4) that \([c^{2k-2}d] \hat{\Phi}_i^{2k} = 1 \) for \( 1 \leq i \leq 2k-2 \), and so

\[
[c^{2k-2}d] (\hat{\Phi}_i^{2k} - \hat{\Phi}_i^{2k+1}) = 0 \quad \text{for any} \ 1 \leq i \leq 2k-2,
\]

which together with (8) implies the lemma. \( \square \)

**Lemma 5.3.** Suppose that for a fixed monomial \( w \), \([w]B_j^{k-1} \geq 0\) for all \( j \leq k-3 \). Then

\[
[w c^2] B_j^k \geq \binom{2k-2}{j} [w] \hat{\Phi}_j^{2k-2} \quad \text{for all} \ 0 \leq j \leq k-2.
\]

In particular, \([w c^2] B_j^k \geq 0\) for \( 0 \leq j \leq k-2 \).
PROOF. Applying Lemma 4.1 twice, we obtain from (8) that
\[ [w^2c^3]\Phi^j_k = [w] \sum_{i=j+1}^{2k-j-3} (-1)^i (-1)^{j-1} \binom{2k-1}{i} \sum_{i=1}^{2k-j-3} \tilde{\Phi}^i_k \quad \text{for } j \leq k-2. \] (9)

In particular,
\[ [w^2c^3]B^{k}_{k-2} = \binom{2k-1}{k-1} [w] \tilde{\Phi}^2_{k-1} \geq \binom{2k-2}{k-2} [w] \tilde{\Phi}^2_{k-1}, \]
which proves the lemma in the case \( j = k-2 \). If \( j \leq k-3 \) then we can rewrite (9), using (8) once more, as follows
\[ [w^2c^3]B^{j}_{j} = [w] \left( B^{j}_{j} - \binom{2k-2}{j} \sum_{i=j+1}^{2k-j-3} \tilde{\Phi}^i_k \right), \]
thus showing that
\[ [w^2c^3]B^{j}_{j} \geq \binom{2k-2}{j} \tilde{\Phi}^2_{j+1} \quad \text{for all } j \leq k-3. \]

PROOF OF THEOREM 2.1. Lemmas 5.1 and 5.3 and Remark 4.3 (part 3) imply
\[ [wdc^4]B^{j}_{j} \geq \begin{cases} 0 & \text{if } i = 1 \\ \binom{2k-2}{j} [wdc^4] \Phi^2_{j+1} \Phi^2_{j+1} & \text{if } i > 2 \end{cases} \]
for all \( k \), all odd \( i \), all monomials \( w \) of appropriate degree and all \( 0 \leq j \leq k-2 \). This together with (7) and Remark 4.3 (part 3) applied to \( \Phi^k_k \) yield
\[ [wdc^4] \Phi_\Delta(c, d) \geq \begin{cases} 0 \sum_{i=0}^{2k-2} \beta_{i-1} & \text{if } i = 1 \\ \binom{2k-1}{k-1} \beta_{k-1} \sum_{i=0}^{2k-2-i} \beta_{j+1} & \text{if } i > 2, \text{is odd} \end{cases} \]
If equality is attained in one of these inequalities, then (6) and Remark 4.3 (part 3) imply that \( h_1(\Delta) \) attains the lower bound (3). That is, \( h_1 = 0 \), or, equivalently, \( f_0(\Delta) = 2k \), which is impossible, since the unique \( (2k-1) \)-dimensional complex having \( 2k \) vertices is a \( (2k-1) \)-simplex, and it is not a Eulerian complex. Thus, all the above inequalities are strict, establishing the first part of the theorem.

The proof of the second part is exactly the same, using Lemma 5.2 instead of Lemma 5.1. \( \square \)

REMARK. If, in addition to being Buchsbaum, \( \Delta \) is an orientable homology manifold (over \( k \), \( \text{char } k = 0 \)), then \( \beta_{2k-1} = 1 \), and so it follows from Schenzel’s theorem (2) that \( h'_{2k}(\Delta) = h_{2k}(\Delta) = 1 \) (where \( h' \)-numbers are defined as in Section 3). Hence \( h'_{i} \geq 1 \) for all \( 0 \leq i \leq 2k \).

Theorem 1.7 of [12] together with (2) yield the following lower bounds on the \( h \)-numbers of \( \Delta \) (compare with (3)):
\[ h_i(\Delta) \geq 1 + \binom{n-1}{j} \beta_{j-1} - \binom{n}{i} \sum_{j=0}^{i-2} (-1)^{i-j} \beta_j \quad \text{for } i = 1, 2, \ldots, n-1. \] (10)

Repeating the proof of Theorem 2.1, but using (10) instead of (3), we obtain that for any \( (2k-1) \)-dimensional orientable homology manifold \( \Delta \) the coefficients \([wdc^4] \Phi_\Delta(c, d)\) (where \( i \) is
odd and \( w \) is any monomial, or \( i \) is even and \( w = c^{2k-2-i} \) are minimized by the corresponding coefficients of the \( cd \)-index of the boundary complex of a \( 2k \)-simplex. In particular, it follows that among all \((2k-1)\)-dimensional orientable homology manifolds, the boundary complex of a \( 2k \)-simplex minimizes \((-1)^l \tilde{\chi}(\text{Skel}_l(\Delta))\) \((l = 0, 1, \ldots, 2k - 2)\). Numerically, this means that

\[
(-1)^l \tilde{\chi}(\text{Skel}_l(\Delta)) \geq \left( \frac{2k}{l+1} \right) \quad \text{for } l = 0, 1, \ldots, 2k - 2.
\]

6. Concluding Remarks

Several questions in this area are still open:

1. In Section 5 we showed that many of the coefficients of the \( cd \)-index of any odd-dimensional Eulerian Buchsbaum (over \( k \), \( \text{char } k = 0 \)) simplicial complex are positive. It would be interesting to find out what other coefficients of the \( cd \)-index of such complexes are nonnegative. For example, is it true that for any odd-dimensional simplicial manifold \( \Delta \) and any monomial \( w \), \([wc] \Phi_{cd} \geq 0? \text{ Do similar results hold in the even-dimensional case?}\)

2. In Section 4 we obtained simple recursion formulas for \( \Phi^n_{\text{cd}} \)-polynomials (see Lemmas 4.1 and 4.2). These polynomials appear from the \( cd \)-indexes of semisuspended shelling components of an \( n \)-dimensional simplex. Similar polynomials can be defined in the case of an \( n \)-dimensional cube as follows (see [6] for more details): let \( C^n \) be a boundary complex of an \( n \)-cube and let \( V^n = \Phi_{C^n} \). We say that a collection of facets \( \mathcal{F} = \{F_1, \ldots, F_{r+1}\} \) of \( C^n \) has type \((r, s)\) if there are exactly \( s \) pairs of parallel facets in \( \mathcal{F} \). For \( r > 0 \) and \( s \geq 0 \) such that \( r + s \leq n \) choose any collection of facets \( \mathcal{F} \subset C^n \) of type \((r, s)\). Define the polynomial \( V^n_{r,s} \) as the \( cd \)-index of the semisuspension of the cubical complex generated by \( \mathcal{F} \). In particular, \( V^n = V^n_{1,n-1} \).

The same arguments we used in Section 4 provide the following recursion formulas for \( V^n_{r,s} \)-polynomials:

\[
[w c](V^n_{r+1,s} - V^n_{r,s}) = [w](V^{n-1}_{r,s} - V^{n-1}_{r+1,s}) \quad (11)
\]

\[
[w d](V^n_{r+1,s} - V^n_{r,s}) = [w c](V^{n-1}_{r,s} + V^{n-1}_{r,n-1-r,s} - V^{n-1}). \quad (12)
\]

Since (as observed in [6, Eqn. (3.4)])

\[
V^n_{r+1,s} - V^n_{r,s} = 2(V^n_{r,s+1} - V^n_{r,s}).
\]

Eqns (11) and (12) together with the boundary condition \( V^n_{1,0} = V^{n-1}c \) give a complete recipe for computing \( V^n_{r,s} \)-polynomials.

Using recurrences of Lemmas 4.1 and 4.2 it can be shown by a simple induction argument that \( \Phi^n_{\text{cd}} \)-polynomials also satisfy the following recursion formula (which is due to Ehrenborg and Readdy [7]):

\[
\tilde{\Phi}^{n+1} = G(\tilde{\Phi}^n),
\]

where \( G \) is a derivation on the ring \( \mathbb{Z}(c, d) \) (see [7] for appropriate definitions), thus, giving a different proof of their result on the interpretation of \( \Phi^n_{\text{cd}} \)-polynomials in terms of simsun permutations (see [7, Corollary 8.2]).

Do formulas (11) and (12) lead to a similar combinatorial interpretation for \( V^n_{r,s} \)-polynomials?
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