J. Math. Anal. Appl. 356 (2009) 288–294

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Global positive periodic solutions of periodic n-species competition systems ☆

Jurang Yan

School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, PR China

ARTICLE INFO

Article history:
Received 8 December 2008
Available online 9 March 2009
Submitted by J.S.W. Wong

Keywords:
Positive periodic solution
Lotka–Volterra competition system
Gilpin–Ayala competition system
Fixed point theorem

ABSTRACT

In this paper, an easily verifiable necessary and sufficient condition for the existence of positive periodic solutions of generalized n-species Lotka–Volterra type and Gilpin–Ayala type competition systems is obtained. It improves a series of the well-known sufficiency theorems in the literature about the problems mentioned above. The method is based on a well-known fixed point theorem in a cone of Banach space. This approach can be applied to more general competition systems.

1. Introduction

For the last decades, the ecological competition systems of Lotka–Volterra type have been investigated extensively. Many interesting results concern with the global existence and attractivity of periodic solution, persistence and extinction of the population, etc., we refer to [1–13] and the references therein. However, the Lotka–Volterra type models have often been severely criticized. One of the criticisms is that in such a model, the per capita rate of change of the density of each species is a linear function of densities of the interacting species. In 1973, Gilpin, Ayala et al. [14,15] claimed that more complicated competition system are needed to study qualitative properties of the systems. To this aim, they proposed several competition models. One of the models is the following competition system,

$$\dot{N}_i(t) = r_i N_i \left(1 - \left(\frac{N_i}{K_i}\right)^{\theta_i} - \sum_{j=1,\, j\neq i}^{n} \alpha_{ij} \frac{N_j}{K_j}\right), \quad i = 1, 2, \dots, n, \tag{1.1}$$

where N_i is the population density of the ith species, r_i is the intrinsic exponential growth rate of the ith species, K_i is the environmental carrying capacity of species i in the absence of competition, θ_i provides a nonlinear measure of interspecific interference, and α_{ij} provides a measure of interspecific interference. Fan and Wang [16] further proposed delay Gilpin–Ayala type competition model,

$$\dot{y}_i(t) = y_i(t)\left[r_i(t) - \sum_{j=1}^{n} a_{ij}(t) y_j^{\theta_{ij}}\big(t - \tau_{ij}(t)\big)\right], \quad i = 1, 2, \dots, n. \tag{1.2}$$

By applying the coincidence degree theory, they obtained a set of easily verifiable sufficient conditions for the existence of at least one positive periodic solution of (1.2).

☆ This work is supported by the Fund of the Doctoral Program Research of the Education Ministry of China (20040108003).
E-mail address: jryan@sxu.edu.cn.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.03.013

Recently, Chen [17] and Xia, Han, Huang [18] investigated respectively the following n-species Gilpin–Ayala type competition systems

$$\dot{y}_i(t)=y_i(t)\left[r_i(t)-\sum_{j=1}^{n}a_{ij}(t)y_j^{\alpha_{ij}}(t)-\sum_{j=1}^{n}b_{ij}(t)y_j^{\beta_{ij}}\big(t-\tau_{ij}(t)\big)-\sum_{j=1}^{n}\int_{-\sigma_{ij}}^{0}c_{ij}(t,s)y_j^{\gamma_{ij}}(t+s)\,\mathrm{d}s\right],\quad i=1,2,\ldots,n,\tag{1.3}$$

and

$$\dot{y}_i(t)=y_i(t)\left[r_i(t)-\sum_{j=1}^{n}a_{ij}(t)y_j^{\alpha_{ij}}(t)-\sum_{j=1}^{n}b_{ij}(t)y_j^{\alpha_{ij}}\big(t-\tau_{ij}(t)\big)-\sum_{j=1}^{n}c_{ij}(t)y_i^{\alpha_{ii}}(t)y_j^{\alpha_{ij}}(t)\right],\quad i=1,2,\ldots,n.\tag{1.4}$$

They obtained respectively several interesting results on the permanence and extinction, the existence and global attractivity of almost periodic solution of (1.3) and (1.4).

Nevertheless, to the best of the author's knowledge, so far, no work has concerned the periodic systems (1.2)–(1.4) to establish sufficient and necessary conditions for the existence of positive periodic solutions of the systems.

In this paper, we investigate the following generalized periodic n-species Gilpin–Ayala type competition models in periodic environments with deviating arguments of the form

$$\dot{y}_i(t)=y_i(t)\left[r_i(t)-\sum_{j=1}^{n}a_{ij}(t)y_j^{\alpha_{ij}}(t)-\sum_{j=1}^{n}b_{ij}(t)y_j^{\beta_{ij}}\big(t-\tau_{ij}(t)\big)\right.$$
$$\left.-\sum_{j=1}^{n}c_{ij}(t)\int_{-\sigma_{ij}}^{0}K_{ij}(\xi)y_i^{\gamma_{ij}}(t+\xi)y_j^{\delta_{ij}}(t+\xi)\,\mathrm{d}\xi\right],\quad i=1,2,\ldots,n.\tag{S}$$

The purpose of this paper is to obtain a necessary and sufficient condition for the existence of positive periodic solutions (with strictly positive components) of the system (S). The method is based on the use of a fixed point theorem and the proof by contradiction. This approach in this paper may be used to more general Lotka–Volterra type competition systems and Gilpin–Ayala type competition systems.

Throughout this paper, we use $i,j=1,2,\ldots,n$, unless otherwise stated. For an ω-periodic ($\omega>0$) function $u(t)\in C(R,R)$, let $\bar{u}=\frac{1}{\omega}\int_0^{\omega}u(t)\,\mathrm{d}t$; a vector $u=(u_1,u_2,\ldots,u_n)^T$ is positive if $u_i>0$.

Let $R=(-\infty,\infty)$. We make the assumptions:

(H_1) $r_i,a_{ij},b_{ij},c_{ij},\tau_{ij}\in C(R,R)$ are ω-periodic functions and $\bar{r}_i>0$, $a_{ij},b_{ij},c_{ij}\geqslant 0$;
(H_2) $K_{ij}\in C([-\sigma_{ij},0],R)$, $K_{ij}\geqslant 0$, $\sigma_{ij}\geqslant 0$ is a constant and $\int_{-\sigma_{ij}}^{0}K_{ij}(t)\,\mathrm{d}t=1$;
(H_3) $\alpha_{ij},\beta_{ij},\delta_{ij}\geqslant 1$ are constants. γ_{ij} is a nonnegative constant.

Our main result is

Theorem 1.1. *Assume that* (H_1)–(H_3) *hold. Then condition*

$$m_0=\min_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}(\bar{a}_{ij}+\bar{b}_{ij}+\bar{c}_{ij})\right\}>0\tag{C}$$

is necessary and sufficient for system (S) *to have at least one positive ω-periodic solution.*

For some particular Lotka–Volterra type and Gilpin–Ayala type competition system, the existence results of positive periodic solutions have been established respectively in [4–6,8,9,11–13]. Clearly, Theorem 1.1 is an improvement and generalization of these results, which shall be stated in the last section of this paper.

The remainder of this paper is organized as follows. In Section 2, we introduce some notations and preliminaries. In Section 3 we prove the main theorem by using a well-known fixed point theorem in cones due to Krasnoselskii [19–21] and a simple proof by contradiction. As applications in Section 4, we study some particular cases of system (S) which have been investigated extensively in the references mentioned above.

2. Preliminaries

The following fixed point theorem is crucial in the arguments of our main results.

Lemma 2.1. *(See Krasnoselskii [19], Deimling [20], Guo and Lakshmikantham [21].) Let E be a Banach space and P be a cone in E. Assume that Ω_1,Ω_2 are open subsets of E with $0\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$. Let*

$$T:P\cap(\overline{\Omega}_2\backslash\Omega_1)\to P$$

be a continuous and completely continuous operator satisfying

(i) $\|Ty\| \leqslant \|y\|$ *for* $y \in P \cap \partial\Omega_1$;
(ii) *There exists* $\psi \in P\backslash\{0\}$ *such that* $y \neq Ty + \lambda\psi$ *for* $y \in P \cap \partial\Omega_2$ *and* $\lambda > 0$.

Then T *has a fixed point in* $P \cap (\overline{\Omega}_2\backslash\Omega_1)$. *The same conclusion remains valid if* (i) *holds for* $y \in P \cap \partial\Omega_2$ *and* (ii) *holds for* $y \in P \cap \partial\Omega_1$ *and* $\lambda > 0$.

We introduce a function space:

$$E = \{y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T \in C(R, R^n):\ y_i(t) = y_i(t+\omega)\}$$

and let $\|y\| = \sum_{i=1}^n |y_i|_0$, where $|y_i|_0 = \max_{t\in[0,\omega]} |y_i(t)|$. Then E is a Banach space endowed with the above norm.

Define an operator $T : E \to E$ by

$$(Ty)(t) = ((Ty)_1(t), (Ty)_2(t), \ldots, (Ty)_n(t)),$$

where

$$(Ty)_i(t) = \int_t^{t+\omega} G_i(t,s) y_i(s)\Bigg[\sum_{j=1}^n a_{ij}(s) y_j^{\alpha_{ij}}(s) + \sum_{j=1}^n b_{ij}(s) y_j^{\beta_{ij}}\big(s - \tau_{ij}(s)\big) + \sum_{j=1}^n c_{ij}(s) \int_{-\sigma_{ij}}^0 K(\xi) y_i^{\gamma_{ij}}(t+\xi) y_j^{\delta_{ij}}(t+\xi)\,d\xi\,ds\Bigg], \quad i = 1, 2, \ldots, n, \tag{2.1}$$

where

$$G_i(t,s) = \frac{\exp(-\int_t^s r_i(\tau)\,d\tau)}{1 - \exp(-\omega\bar{r}_i)}, \quad t \leqslant s \leqslant t+\omega.$$

Obviously, there exist two positive constants A and B such that

$$A \leqslant G_i(t,s) \leqslant B, \quad t \leqslant s \leqslant t+\omega. \tag{2.2}$$

Let $\sigma = \frac{A}{B}$. Now, we choose a set defined by

$$P = \{y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T \in E:\ y_i(t) \geqslant \sigma|y_i|_0\}.$$

Clearly, P is a cone in E. For convenience of expressions, we define an operator $F : P \to E$ by

$$(Fy)_i(t) = y_i(t)\Bigg[\sum_{j=1}^n a_{ij}(t) y_j^{\alpha_{ij}}(t) + \sum_{j=1}^n b_{ij}(t) y_j^{\beta_{ij}}\big(t - \tau_{ij}(t)\big) + \sum_{j=1}^n c_{ij}(t) \int_{-\sigma_{ij}}^0 K(\xi) y_i^{\gamma_{ij}}(t+\xi) y_j^{\delta_{ij}}(t+\xi)\,d\xi\Bigg].$$

Lemma 2.2. *The operator* T *maps* P *into* P, *that is,* $T(P) \subset P$.

Proof. In view of the definitions of P and F, for any $y \in P$,

$$(Ty)_i(t) = \int_t^{t+\omega} G_i(t,s)(Fy)_i(s)\,ds. \tag{2.3}$$

Thus,

$$(Ty)_i(t+\omega) = \int_{t+\omega}^{t+2\omega} G_i(t+\omega,s)(Fy)_i(s)\,ds = \int_t^{t+\omega} G_i(t+\omega,s+\omega)(Fy)_i(s+\omega)\,ds = \int_t^{t+\omega} G_i(t,s)(Fy)_i(s)\,ds = (Ty)_i(t).$$

Furthermore, for any $y \in P$, it follows from (2.1) and (2.2) that

$$|(Ty)_i|_0 \leqslant B\int_0^\omega (Fy)_i(s)\,ds \quad \text{and} \quad (Ty)_i(t) \geqslant A\int_0^\omega (Fy)_i(s)\,ds.$$

Hence

$$(Ty)_i(t) \geqslant \frac{A}{B}\big|(Ty)_i\big|_0 = \sigma\big|(Ty)_i\big|_0.$$

Thus $T(P) \subset P$ and the proof of Lemma 2.2 is completed. □

Lemma 2.3. *The operator $T : P \to P$ is continuous and completely continuous.*

Proof. By using a standard argument one can show that T is continuous on P. Now, we show that T is completely continuous. Let d be any positive constant and $S_d = \{y \in E\colon\ |y_i|_0 \leqslant d\}$ be a bounded set. For any $y \in S_d$, from (2.1) and (2.2) we have

$$\big|(Ty)_i\big|_0 \leqslant B\int_0^{\omega}(Fy)_i(s)\,\mathrm{d}s \leqslant \omega Bd\sum_{j=1}^{n}\big(d^{\alpha_{ij}}\bar{a}_{ij} + d^{\beta_{ij}}\bar{b}_{ij} + d^{\gamma_{ij}+\delta_{ij}}\bar{c}_{ij}\big) =: D_i.$$

Therefore, for any $y \in S_d$,

$$\|Ty\| = \sum_{i=1}^{n}\big|(Ty)_i\big|_0 \leqslant \sum_{i=1}^{n} D_i =: D,$$

which implies that $T(S_d)$ is a uniformly bounded set. On the other hand, in view of the definitions of T and F,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[(Ty)_i(t)\big] = r_i(t)(Ty)_i(t) - (Fy)_i(t).$$

Again, from (2.1) and (2.2) we obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\big[(Ty)_i(t)\big]\right| \leqslant r_i^u D + d\sum_{j=1}^{n}\big(\bar{a}_{ij}d^{\alpha_{ij}} + \bar{b}_{ij}d^{\beta_{ij}} + \bar{c}_{ij}d^{\gamma_{ij}+\delta_{ij}}\big) =: \tilde{D}_i \leqslant \tilde{D},$$

where $r_i^u = \max_{0\leqslant t\leqslant\omega}\{r_i(t)\}$ and $\tilde{D} = \max_{1\leqslant i\leqslant n}\{\tilde{D}_i\}$, which implies $\frac{\mathrm{d}}{\mathrm{d}t}[(Ty)(t)]$, for $y \in S_d$, is also uniformly bounded. Hence $T(S_d) \subset E$ is a family of uniformly bounded and equi-continuous functions. By the well-known Ascoli–Arzela theorem the operator T is completely continuous. The proof of Lemma 2.3 is completed. □

Lemma 2.4. *The system* (S) *has at least one positive ω-periodic solution provided T has a fixed point in P.*

Proof. Let $y \in P$ and $Ty = y$. Hence $(Ty)_i(t) = y_i(t)$. From (2.3),

$$\begin{aligned}
\dot{y}_i(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_t^{t+\omega} G_i(t,s)(Fy)_i(s)\,\mathrm{d}s\right)\\
&= r_i(t)(Ty)_i(t) - G_i(t,t+\omega)(Fy)_i(t+\omega) - G_i(t,t)(Fy)_i(t)\\
&= r_i(t)y_i(t) - (Fy)_i(t)\\
&= y_i(t)\left[r_i(t) - \sum_{j=1}^{n}a_{ij}(t)y_j^{\alpha_{ij}}(t) - \sum_{j=1}^{n}b_{ij}(t)y_j^{\beta_{ij}}\big(t-\tau_{ij}(t)\big) - \sum_{j=1}^{n}c_{ij}(t)\int_{-\sigma_{ij}}^{0}K_{ij}(\xi)y_i^{\gamma_{ij}}(t+\xi)y_j^{\delta_{ij}}(t+\xi)\,\mathrm{d}\xi\right],
\end{aligned}$$

which implies that $y(t)$ is a positive ω-periodic solution of (S). The proof is completed. □

3. The proof of main result

Proof of Theorem 1.1. (*Sufficiency*) Let

$$M_0 = \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}(\bar{a}_{ij} + \bar{b}_{ij} + \bar{c}_{ij})\right\}.$$

From condition (C) we conclude $M_0 \geqslant m_0 > 0$. Choose a constant $M \geqslant M_0$ so that $\frac{1}{\omega BM} < 1$. Let $r = \frac{1}{\omega BM}$ and

$$\Omega_1 = \big\{\big(y_1(t), y_2(t), \ldots, y_n(t)\big)^T \in E\colon\ |y_i|_0 < r,\ i = 1, 2, \ldots, n\big\}.$$

For any $y = y(t) \in P \cap \partial\Omega_1$, $\sigma|y_i|_0 \leqslant y_i(t) \leqslant |y_i|_0$. From (2.1) and (2.2), we have

$$\big|(Ty)_i\big|_0 < B\int_t^{t+\omega}(Fy)_i(s)\,\mathrm{d}s \leqslant \omega B|y_i|_0\sum_{j=1}^n\big(\bar a_{ij}r^{\alpha_{ij}}+\bar b_{ij}r^{\beta_{ij}}+\bar c_{ij}r^{\gamma_{ij}+\delta_{ij}}\big)$$
$$\leqslant \omega B|y_i|_0\sum_{j=1}^n(\bar a_{ij}+\bar b_{ij}+\bar c_{ij})r \leqslant \omega B|y_i|_0 M_0 r \leqslant |y_i|_0.$$

Hence for any $y(t)\in P\cap\partial\Omega_1$,

$$\|Ty\| = \sum_{i=1}^n\big|(Ty)_i\big|_0 \leqslant \sum_{i=1}^n|y_i|_0 = \|y\|,$$

which implies condition (i) in Lemma 2.1 is satisfied.

On the other hand, choose $0<m\leqslant m_0$ so that $\frac{1}{\omega\sigma Am}>1$. Let $R=\frac{1}{\omega\sigma Am}$ and

$$\Omega_2 = \big\{\big(y_1(t),y_2(t),\dots,y_n(t)\big)^T\in E:\ |y_i|_0<R,\ i=1,2,\dots,n\big\}.$$

Suppose $\psi=(\psi_1,\psi_2,\dots,\psi_n)^T\in P\backslash\{0\}$. We show that for any $y=y(t)\in P\cap\partial\Omega_2$ and $\lambda>0$, $y\neq Ty+\lambda\psi$. Otherwise, there exist $y_0\in P\cap\partial\Omega_2$ and $\lambda_0>0$ such that $y_0=Ty_0+\lambda_0\psi$. Let $\psi_{i_0}\neq 0$ $(1\leqslant i_0\leqslant n)$. Since $y_{i_0}(t)\geqslant\sigma|y_{i_0}|_0$, it follows that

$$y_{i_0}(t) = (Ty)_{i_0}(t)+\lambda_0\psi_{i_0} = \int_t^{t+\omega}G_{i_0}(t,s)(Fy)_{i_0}(s)\,\mathrm{d}s+\lambda_0\psi_{i_0}$$
$$\geqslant \omega\sigma A|y_{i_0}|_0\sum_{j=1}^n\big(\bar a_{ij}R^{\alpha_{ij}}+\bar b_{ij}R^{\beta_{ij}}+\bar c_{ij}R^{\gamma_{ij}+\delta_{ij}}\big)+\lambda_0\psi_{i_0}$$
$$\geqslant \omega\sigma A|y_{i_0}|_0\sum_{j=1}^n(\bar a_{ij}+\bar b_{ij}+\bar c_{ij})R+\lambda_0\psi_{i_0}$$
$$\geqslant \omega\sigma A|y_{i_0}|_0 m_0R+\lambda_0\psi_{i_0} \geqslant |y_{i_0}|_0+\lambda_0\psi_{i_0} > |y_{i_0}|_0,$$

which is a contradiction. This proves condition (ii) in Lemma 2.1 is also satisfied. By Lemmas 2.1 and 2.4, system (S) has at least one positive ω-periodic solution.

(*Necessity*) Suppose that (C) does not hold. Then there exists at least an i_0 $(1\leqslant i_0\leqslant n)$ such that

$$\bar a_{i_0j}=\bar b_{i_0j}=\bar c_{i_0j}=0,\quad j=1,2,\dots,n.$$

If (S) has a positive ω-periodic solution $y(t)=(y_1(t),y_2(t),\dots,y_n(t))^T$, then we have

$$\dot y_{i_0}(t)=r_{i_0}(t)y_{i_0}(t).$$

Therefore $0=\ln\frac{y_{i_0}(t+\omega)}{y_{i_0}(t)}=\int_t^{t+\omega}r_{i_0}(s)\,\mathrm{d}s=\omega\bar r_{i_0}>0$, which is a contradiction. The proof of Theorem 1.1 is completed. □

4. Applications

In this section, to illustrate the generality of our result, we apply Theorem 1.1 to some particular Lotka–Volterra type and Gilpin–Ayala type competition systems with (or without) deviating arguments which have been studied in the literature.

Consider the periodic n-species competition systems

$$\dot y_i(t)=y_i(t)\Bigg[r_i(t)-\sum_{j=1}^n a_{ij}(t)y_j\big(t-\tau_{ij}(t)\big)\Bigg],\quad i=1,2,\dots,n, \tag{4.1}$$

$$\dot y_i(t)=y_i(t)\Bigg[r_i(t)-a_{ii}(t)y_i(t)-\sum_{j=1,\,j\neq i}^n a_{ij}(t)\int_{-\sigma_{ij}}^0 K_{ij}(\xi)y_j(t+\xi)\,\mathrm{d}\xi\Bigg], \tag{4.2}$$

$$\dot y_i(t)=y_i(t)\Bigg[r_i(t)-\sum_{j=1}^n a_{ij}(t)y_j^{\alpha_{ij}}\big(t-\tau_{ij}(t)\big)\Bigg], \tag{4.3}$$

where r_i, a_{ij}, τ_{ij}, K_{ij}, σ_{ij} and α_{ij} are the same as in (H_1)–(H_3). Thus from Theorem 1.1 we have

Corollary 4.1. *Assume that $r_i, a_{ij}, \tau_{ij}, K_{ij}, \sigma_{ij}$ and α_{ij} are the same as in* (H_1)–(H_3). *Then condition*

$$\min_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}\bar{a}_{ij}\right\}>0$$

is a necessary and sufficient condition for (4.1) ((4.2) *or* (4.3)) *to have at least one positive ω-periodic solution.*

In [5] and [11], for (4.1) to have at least one positive ω-periodic solution, sufficient conditions are respectively

$$r_i(t)>0 \quad \text{and} \quad \sum_{j=1}^{n}\bar{a}_{ij}y_j=\bar{r}_i \quad \text{has a positive solution} \quad \big(\text{see Theorem 2.5 in [5]}\big)$$

and

$$\bar{a}_{ii}>0 \quad \text{and} \quad \bar{r}_i>\sum_{j=1,\, j\neq i}^{n}\frac{\bar{a}_{ij}\bar{r}_j}{\bar{a}_{ii}}\exp\big\{(\bar{r}_j+\bar{R}_j)\omega\big\} \quad \big(\text{see Theorem 2.1 in [11]}\big).$$

In [13], for (4.2) to have at least one positive ω-periodic solution, the sufficient conditions are $\bar{a}_{ii}>0$, $\bar{r}_i=\sum_{j=1}^{n}\bar{a}_{ij}e^{\bar{y}_j}$ has a unique solution and

$$\bar{r}_i>\sum_{j=1,\, j\neq i}^{n}\bar{a}_{ij}\left|\frac{r_j}{a_{jj}}\right|_0 \quad \big(\text{see Theorem 1 in [13]}\big).$$

In [16], for (4.3) to have at least one positive ω-periodic solution, the sufficient conditions are $\bar{a}_{ii}>0$,

$$\bar{r}_i-\sum_{j=1}^{n}\bar{a}_{ij}y_j^{\alpha_{ij}}=0 \quad \text{has positive solution}$$

and

$$\bar{r}_i>\sum_{j=1,\, j\neq i}^{n}\bar{a}_{ij}\left(\frac{\bar{r}_j}{\bar{a}_{jj}}\right)^{\alpha_{ij}/\alpha_{jj}}\exp\big(\alpha_{ij}(\bar{r}_j+\bar{R}_j)\omega\big) \quad \big(\text{see Theorem 2.1 in [16]}\big).$$

It is easy to see that Theorem 1.1 improves and generalizes the results mentioned above.

Consider the unsymmetrical May–Leonard model with periodic coefficients [4],

$$\begin{aligned}
\dot{y}_1(t)&=p_1(t)y_1(t)\big(1-y_1(t)-\alpha_1(t)y_2(t)-\beta_1(t)y_3(t)\big),\\
\dot{y}_2(t)&=p_2(t)y_2(t)\big(1-\beta_2(t)y_1(t)-y_2(t)-\alpha_2(t)y_3(t)\big),\\
\dot{y}_3(t)&=p_3(t)y_3(t)\big(1-\alpha_3(t)y_1(t)-\beta_3(t)y_2(t)-y_3(t)\big),
\end{aligned}\tag{4.4}$$

where $p_i, \alpha_i, \beta_i \in C(R,R)$ are ω-periodic functions and $p_i>0$, $\alpha_i, \beta_i \geqslant 0$. In [4], for (4.4) to have at least one positive ω-periodic solution, the sufficient conditions are

$$\alpha_1(t)\leqslant\frac{\bar{p}_1p_2(t)}{p_1(t)\bar{p}_2}, \qquad \alpha_2(t)\leqslant\frac{\bar{p}_2p_3(t)}{p_2(t)\bar{p}_3}, \qquad \alpha_3(t)\leqslant\frac{\bar{p}_3p_1(t)}{p_3(t)\bar{p}_1}\tag{4.5}$$

and

$$\frac{p_1(t)\bar{p}_2}{\bar{p}_1p_2(t)}\leqslant\beta_2(t), \qquad \frac{p_2(t)\bar{p}_3}{\bar{p}_2p_3(t)}\leqslant\beta_3(t), \qquad \frac{p_3(t)\bar{p}_1}{\bar{p}_3p_1(t)}\leqslant\beta_1(t),\tag{4.6}$$

where each of the inequalities for β_i is strict on some set $I_i\subset[0,\omega]$ of positive measure (see Corollary 1 in [4]).

But, since $p_i(t)>0$, $i=1,2,3$, from Theorem 1.1 we have the following result.

Corollary 4.2. *The system* (4.4) *has at least one positive ω-periodic solution.*

Corollary 4.2 implies that conditions (4.5) and (4.6) are not necessary for (4.4) to have at least one positive ω-periodic solution.

Consider the following two species periodic competition model [22],

$$\begin{aligned}
\dot{x}(t)&=x(t)\big(a(t)-b(t)x(t)-c(t)y(t)\big),\\
\dot{y}(t)&=y(t)\big(d(t)-e(t)x(t)-f(t)y(t)\big),
\end{aligned}\tag{4.7}$$

where $a,b,c,d,e,f\in C(R,R)$ are ω-periodic functions with $\bar{a},\bar{d}>0$, b,c,e,f are nonnegative. From Theorem 1.1 we have

Corollary 4.3. *For* (4.7) *to have at least one positive ω-periodic solution a necessary and sufficient condition is*

$$\bar{b} + \bar{e} > 0 \quad \textit{and} \quad \bar{c} + \bar{f} > 0.$$

In [22], for (4.7) to have at least one positive ω-periodic solution the following hypotheses are assumed

$$\bar{a} > 0, \quad \bar{d} > 0 \quad \text{and} \quad b, c, e, f > 0$$

and

$$\int_0^\omega \big(a(s) - c(s)y_0(s)\big)\,\mathrm{d}s > 0, \qquad \int_0^\omega \big(d(s) - e(s)x_0(s)\big)\,\mathrm{d}s > 0,$$

where $x_0(t)$ and $y_0(t)$ are respectively positive ω-periodic solution of $\dot{x} = x(a(t) - b(t)x)$ and $\dot{y} = y(d(t) - f(t)y)$ (see Theorem 2.1 in [22]).

For more general applications of the method in this paper, we may further consider some epidemic models in a periodic environment. For example, in [23] the authors studied the existence and stability of positive periodic solutions of the compartmental epidemic models in the periodic environments and established several interesting results by using the Poincaré map and the theory of the basic reproduction ratio. Under some appropriate conditions the method in this paper may be extended to study existence of positive periodic solutions of these models. For this subject, further investigations are still needed in the future.

Acknowledgments

The author would like to express his gratitude to the reviewers and editor for their valuable comments and careful reading of the manuscript.

References

[1] S. Ahmad, On the nonautonomous Lotka–Volterra competition equation, Proc. Amer. Math. Soc. 117 (1993) 199–204.
[2] S. Ahmad, A.C. Lazer, Average growth and extinction in a Lotka–Volterra system, Nonlinear Anal. 62 (2005) 545–557.
[3] S. Ahmad, A.C. Lazer, Average conditions for global asymptotic stability in a nonautonomous Lotka–Volterra system, Nonlinear Anal. 40 (2000) 37–49.
[4] A. Battauz, F. Zanolin, Coexistence states for periodic competition Kolmogorov systems, J. Math. Anal. Appl. 219 (1998) 178–199.
[5] X. Tang, X. Zou, On positive periodic solutions of Lotka–Volterra competition systems with deviating arguments, Proc. Amer. Math. Soc. 134 (2006) 2967–2974.
[6] X. Tang, D. Cao, X. Zou, Global attractivity of positive periodic solution to periodic Lotka–Volterra competition systems, J. Differential Equations 228 (2006) 580–610.
[7] K. Gopalsamy, Stability and Oscillation in Delay Differential Equations of Population Dynamics, Kluwer Academic, Dordrecht, 1992.
[8] K. Gopalsamy, Global asymptotical stability in a periodic Lotka–Volterra system, J. Austral. Math. Soc. Ser. B 29 (1985) 66–72.
[9] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, Boston, 1993.
[10] I. Györi, G. Ladas, Oscillation Theory of Delay Differential Equations, Oxford Science, Oxford, 1991.
[11] M. Fan, K. Wang, D.Q. Jiang, Existence and global attractivity of positive periodic solutions of periodic n-species Lotka–Volterra, Math. Biosci. 160 (1999) 47–61.
[12] Y.K. Li, On a periodic delay logistic type population model, Ann. Differential Equations 14 (1998) 29–36.
[13] Y.K. Li, Periodic solutions for delay Lotka–Volterra competitions, J. Math. Anal. Appl. 246 (2000) 230–244.
[14] M.E. Gilpin, F.J. Ayala, Global model of growth and competition, Proc. Natl. Acad. Sci. USA 70 (1973) 3590–3593.
[15] F.J. Ayala, M.E. Gilpin, J.G. Eherenfeld, Competition between species: Theoretical models and experimental tests, Theor. Popul. Biol. 4 (1973) 331–356.
[16] M. Fan, K. Wang, Global periodic solutions of a generalized n-species Gilpin–Ayala competition model, Comput. Math. Appl. 40 (2000) 1141–1151.
[17] F.D. Chen, Average conditions for permanence and extinction in nonautonomous Gilpin–Ayala competition model, Nonlinear Anal. Real World Appl. 4 (2006) 885–915.
[18] Y.H. Xia, M. Han, Z.K. Huang, Global attractivity of an almost periodic n-species nonlinear ecological competition model, J. Math. Anal. Appl. 337 (2008) 144–168.
[19] M.A. Krasnoselskii, Positive Solutions of Operator Equations, Noordhoff, Gorninggen, 1964.
[20] K. Deimling, Nonlinear Functional Analysis, Springer, New York, 1985.
[21] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, New York, 1988.
[22] P. Korman, Some new results on the periodic competition model, J. Math. Anal. Appl. 171 (1992) 131–138.
[23] W. Wang, X.Q. Zhao, Threshold dynamics for compartmental epidemic models in periodic environments, J. Dynam. Differential Equations 20 (2008) 699–717.