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Absolute Continuity of Positive Spectrum for Schrödinger Operators with Long-Range Potentials

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Absolute continuity in $(0, \infty)$ for Schrödinger operators $-\Delta + V(x)$, with long range potential $V = V_1 + V_2$ such that $\partial V_1/\partial r$, $V_2 = 0(r^{-1-\epsilon})$, $\epsilon > 0$, as $|x| \to \infty$, is shown by proving estimates on resolvents near the real axis. Completeness of the modified wave operators for a superposition of Coulomb potentials also follows. Singular local behavior of V is allowed.

1. INTRODUCTION AND STATEMENT OF RESULTS

The scattering and spectral theory of the Schrödinger operator $H = H_0 + V(x)$ (where H_0 is the usual self-adjoint Laplacian acting in $\mathscr{H} = \mathscr{L}^2(\mathbf{R}^n)$) has recently been brought into very satisfactory condition in the case where $V: \mathbf{R}^n \to \mathbf{R}$ has *short range*, i.e., $V(x) = O(r^{-\alpha})$, $\alpha > 1$, as $r \to \infty$. ($r = |x|$.) It was shown early in the history of the subject that the wave operators

$$\Omega_{\pm}(H, H_0) = \operatorname*{s-lim}_{t \to \pm\infty} e^{iHt} e^{-iH_0 t}$$

exist in such a case. Vectors in the range of $\Omega_+(\Omega_-)$ represent quantum mechanical states which are asymptotically free in the future (past). The following inclusions are automatically true:

$$\mathscr{R}(\Omega_{\pm}) \subset \mathscr{H}_{ac}(H) \subset \mathscr{H}_p(H)^{\perp} \subset \mathscr{R}(E_H((0, \infty))).$$

($\mathscr{R}(T)$ denotes the range of an operator T, $E_H(B)$ is the spectral projection for H corresponding to the set B; $\mathscr{H}_{ac}(H)$ is the subspace of absolute continuity for H, i.e., $\mathscr{H}_{ac}(H) = \{\varphi \in \mathscr{H} : |B| = 0 \Rightarrow$

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$E_H(B)\varphi = 0\}$; $\mathscr{H}_p(H)$ is the closed subspace spanned by the eigenvectors of H.)

It is hardly conceivable in physical terms how any of these inclusions could fail to be equality. The problem of *completeness of wave operators* is to prove equality at each stage.

$$\text{(A)} \quad \mathscr{H}_p(H)^{\perp} = \mathscr{R}(E_H((0, \infty)))$$

was shown under mild regularity conditions on V by Kato [1]. Progress on the problem

$$\text{(B)} \quad \mathscr{R}(\Omega_{\pm}) = \mathscr{H}_{ac}(H)$$

has been made by several authors; it was finally solved for general short range V by Kato and Kuroda [2, 3]. (Sometimes this part of the problem, or the weaker statement $\mathscr{R}(\Omega_+) = \mathscr{R}(\Omega_-)$ is called completeness of the wave operators.) Many have also made contributions to the problem of proving

$$\text{(C)} \quad \mathscr{H}_{ac}(H) = \mathscr{H}_p(H)^{\perp}.$$

It was recently solved by Agmon [4]. (Of course, (A), (B) and (C) are very much interrelated.)

Since Dollard [5] has shown that the wave operators do not exist in the case of the Coulomb potential $V(x) = c/|x|$, which just fails to be short-range, it might appear from a mathematical point of view that the last word has been spoken. But from the viewpoint of physics a scattering theory that leaves out the Coulomb potential can scarcely be considered satisfactory. Our goal is to extend this theory as far as possible to long range potentials. Conclusions (A) and (C) above do not refer to the wave operators, and in fact (A) has already been generalized to cover long-range potentials V that can be represented as the sum of a short-range potential and a long-range potential without too much oscillation [6, 7]. (A) can fail for potentials which barely violate this condition [7, 8]. For this and other reasons we feel that the potentials satisfying such a condition are the ones that can reasonably be studied.

The technical foundation of the results (B) and (C) is a pair of estimates controlling the resolvents $(H_0 - z)^{-1}$ and $(H - z)^{-1}$ as z approaches the positive real axis. An almost optimal result for $T = H_0$ is that for any interval $[a, b] \subset (0, \infty)$ there exists a constant c such that

$$\| h(T - z)^{-1} h \| \leqslant c, \qquad z \in \mathscr{N}(a, b), \tag{1.1}$$

where $\mathscr{N}(a, b) = \{z \in \mathbf{C}: 0 < |\operatorname{Im} z| < 1, a < \operatorname{Re} z < b\}$, and h is multiplication by a function which is $O(r^{-\alpha})$, $\alpha > \frac{1}{2}$, at ∞. The estimate (1.1) has been proved for $T = H_0 + V$, where V is short-range, for a set of intervals $[a, b]$ whose union is dense in $(0, \infty)$. We will extend this result in Lemma 1.1 to long-range potentials satisfying the type of condition previously imposed to prove (A).

We will consider short-range functions

$$\begin{gathered} F = (1 + r)^{-\gamma} (f_p + f_\infty), \qquad \gamma > 1, \\ f_p \in \mathscr{L}^p(\mathbf{R}^n), \quad p > \max(n/2, 1); \quad f_\infty \in \mathscr{L}^\infty(\mathbf{R}^n). \end{gathered} \tag{1.2}$$

A long-range function F without oscillation at infinity for us will be one which is continuously differentiable and

$$\begin{gathered} |\partial F/\partial r| \leqslant c(1 + r)^{-\gamma}, \qquad \gamma > 1, \\ \lim_{|x|\to\infty} F(x) = 0. \end{gathered} \tag{1.3}$$

We shall write $\langle\ ,\ \rangle$ for the inner product in $\mathscr{H}$, linear on the right, and $|\ |$ for the norm.

LEMMA 1.1. *Let V be a multiplication operator acting in $\mathscr{H} = \mathscr{L}^2(\mathbf{R}^n)$, where*

$$\begin{gathered} V = V_1 + V_2, \quad V_1 \text{ satisfies (1.3)}, \quad V_2 \text{ satisfies (1.2)}. \\ \text{with } \mathscr{D}(H) \subset \mathscr{D}(H_0^{1/2}) \end{gathered} \tag{1.4}$$

There is a unique self-adjoint operator H with $\mathscr{D}(H) \subset \mathscr{D}(H_0^{1/2})$ such that for all $\varphi, \psi \in \mathscr{D}(H)$

$$\langle \varphi, H_\varphi \rangle = \sum_{j=1}^{n} \left\langle \frac{\partial \varphi}{\partial x_j}, \frac{\partial \varphi}{\partial x_j} \right\rangle + \int_{\mathbf{R}^n} V(x)\, |\varphi(x)|^2\, dx.$$

The positive eigenvalues of H have finite multiplicity and can accumulate only at 0. *For any $|h(x)|^2 = F$ satisfying* (1.2) *and interval $[a, b] \subset (0, \infty)$ containing no eigenvalue of H there is a $C > 0$ such that $\| h(H - z)^{-1} h \| \leqslant C$ for all $z \in \mathscr{N}(a, b)$.*

Remark. We expect that the positive eigenvalues are absent, but we do not assume the local regularity on V necessary for the existing proofs of (A).

Because there are no wave operators, (B) cannot be deduced from this lemma as in the short-range case, but some results of interest in scattering theory, including (C), are consequences by means of standard methods. Next we shall give these results and indicate how they follow from Lemma 1.1. Finally we turn to the proof of the lemma itself.

THEOREM 1. *Let H be as in Lemma* 1.1. *Then*

(a) $\mathscr{H}_{ac}(H) = \mathscr{H}_p(H)^\perp$.

(b) *If* $|h(x)|^2$ *satisfies* (1.2) *and* $[a, b] \subset (0, \infty)$ *contains no eigenvalues of H, then there exists a constant C such that*

$$\int_{-\infty}^{\infty} | he^{-itH}E_H([a, b])\varphi |^2 \, dt \leqslant C | \varphi |^2.$$

(c) *If* $H' = H_0 + V'$ *also satisfies the conditions of Lemma* 1.1 *with* $V' = V_1' + V_2'$ *and* $V_1' = V_1$, *then*

$$\Omega_\pm(H', H) = \text{s-lim}\, e^{itH'}e^{-itH}[\mathscr{H}_{ac}(H)]$$

exists and has range $\mathscr{H}_{ac}(H')$. (*Denoting by* $[\mathscr{H}_{ac}(H)]$ *the projection on* $\mathscr{H}_{ac}(H)$).

Proof. For any self-adjoint T and z outside the spectrum of T,

$$\begin{aligned} \operatorname{Im} z \, |(T-z)^{-1}\varphi|^2 &= \tfrac{1}{2}\langle \varphi, [(T-z)^{-1} - (T-z^*)^{-1}]\varphi\rangle \\ &= \operatorname{Im}\langle \varphi, (T-z)^{-1}\varphi\rangle. \end{aligned} \tag{1.5}$$

By applying this to $T = H$, $\varphi = h\psi$, and using (1.1), we get

$$\begin{aligned} |\operatorname{Im} z| \, |(H-z)^{-1}h\psi|^2 &\leqslant \tfrac{1}{2}\{\| h(H-z)^{-1}h \| + \| h(H-z^*)^{-1}h\|\} \, |\psi|^2 \\ &\leqslant C|\psi|^2, \end{aligned}$$

for $z \in \mathscr{N}(a, b)$ if $[a, b] \subset (0, \infty)$ contains no eigenvalues. This means h is "H-smooth on (a, b)" [9] and the desired results are consequences of the theory of smooth operators. In particular, (a) follows from Theorem 2.2 [9], (b) from Lemma 2.1 and the equality of (2.3) and (2.4) [9], and (c) from Theorem 2.3 [9] (with $J = I$) since $|V_2 - V_2'| \leqslant ch^2$. ∎ (The theory of H-smooth operators was introduced and developed by Kato [10].)

The first conclusion of this theorem, which is just (C), has previously been proved in some special cases. Weidmann [8] proved it under a condition like (1.4) (but with less smoothness required) for radial potentials, and there are results for repulsive potentials [11, 12], "dilation analytic" potentials [13], potentials with no barriers

[9], and potentials with $|V| = O(r^{-\alpha})$, $|\nabla V| = O(r^{-1-\alpha})$, $\alpha > \frac{1}{2}$ (S. Agmon, private communication.)

Conclusion (b) implies that a quantum mechanical particle subject to a force $-\nabla V$ spends only a finite time in a bounded region B if its energy is in the interval (a, b). (Take $h = 1$ on B and 0 outside B.)

Conclusion (c) would give (B) if V were short-range (take $V_1 = 0$). For long-range potentials it only asserts the unitary equivalence of the absolutely continuous spectral parts of two Schrödinger operators whose difference is short-range; it does not relate H to H_0. For the Coulomb case, $H = H_0 + c/r$, Dollard defined *modified wave operators* [5] which do intertwine H and H_0;

$$\Omega_\pm{}^D = \operatorname*{s-lim}_{t \to \pm\infty} e^{-iHt} U_0(t),$$

where $U_0(t)$ is a one-parameter unitary family (not a group) which commutes with H_0 and more correctly approximates the asymptotic behavior of e^{-iHt}. The existence of $\Omega_\pm{}^D$ has now been shown for a large class of H [14–16], so the question of their completeness can be raised, i.e.,

$$(B^D)\ \mathscr{R}(\Omega_\pm{}^D) = \mathscr{H}_{ac}(H).$$

This has been proved only if V is radial [5, 15]. It could be extended to very short-range $(O(r^{-n-\epsilon}))$ perturbations H' of such potentials using the chain rule

$$\begin{aligned} \Omega_\pm{}^D(H', H_0) &= \lim_{t \to \pm\infty} e^{iH't} U_0(t) \qquad (1.6) \\ &= \{\lim_{t \to \pm\infty} e^{iH't} e^{-iHt} [\mathscr{H}_{ac}(H)]\} \{\lim_{t \to \pm\infty} e^{iHt} U_0(t)\} \\ &= \Omega_\pm(H', H)\, \Omega_\pm{}^D(H, H_0) \end{aligned}$$

and results using trace class conditions (e.g., Theorem X.4.9 [17]) to obtain the existence and completeness of $\Omega_\pm(H', H)$. But this is not sufficient to handle cases of physical interest like

EXAMPLE 1. $V(x) = \sum_{k=1}^N q_k |x - a_k|^{-1}$. (Unless $\sum_{k=1}^N q_k = 0$ where the short-range theory applies and the physics textbook argument about screening is valid, because cancellations yield $V(x) = O(r^{-2})$.) In general V differs from a Coulomb potential by $O(r^{-2})$.

However, we can use the same strategy, employing (c) of Theorem 1 for existence and completeness of $\Omega_\pm(H', H)$.

COROLLARY 1.1. *Let μ be a finite signed measure (i.e., a "charge") on $\mathbf{R}^3$ with support in a ball of radius R. Let*

$$V(x) = \int_{\mathbf{R}^n} | x - y |^{-1} \, d\mu(y)$$

and $H = H_0 + V$. The modified wave operators exist and are complete.

Proof. The Coulomb potential $V_c = 1/r$ satisfies the local hypotheses of Theorem 1, and V, being the convolution of μ with V_c satisfies them as well. We can show that $QV_c - V$ is short-range, where Q is the total charge $\mu(\mathbf{R}^n)$, and this will finish the proof, by (1.6) and Theorem 1(c).

$$| V(x) - QV_c(x)| = \left| \int_{|y| \leqslant R} (| x - y |^{-1} - | x |^{-1}) \, d\mu(y) \right|$$
$$\leqslant \int_{|y| \leqslant R} \frac{| y |}{| x - y | \, | x |} \, d \, | \mu | \, (y).$$

For $| x | > 2R$ and $| y | \leqslant R$ the integrand is dominated by $| x |^{-2}$, from which it follows that $V - QV_c = O(r^{-2})$ at infinity. ∎

Remark. If V is the sum of a long-range V_1 and short-range V_2, the theorems on the existence of $\Omega_\pm{}^D$ [15, 16] have made assumptions on V_1 slightly more stringent than (1.3). Thus Theorem 1(c) implies some new results on existence of $\Omega_\pm{}^D$, by the chain rule.

The modified wave operators are difficult to handle, and the information they provide is not as easily interpreted as for the ordinary wave operators. Since the operators $U_0(t)$ are more complicated than e^{-iH_0t}, theorems asserting the existence of $\Omega_\pm{}^D$ are at the same time less appealing to state and more difficult to prove than their short-range counterparts and, as we have noted, theorems asserting their completeness are scarce indeed. Several authors have proposed substitute asymptotic conditions which are more natural to state and imply the qualitative physical behavior expected in scattering [15, 18–20]. In most cases however, the only way to show that these conditions hold has been to prove existence and completeness of $\Omega_\pm{}^D$. In some special cases a direct proof was given of the condition put forth in [19], namely, the existence of

$$\omega_\pm(A)\varphi = \lim_{t \to \pm\infty} e^{iHt} A e^{-iHt} \varphi. \tag{1.7}$$

for $\varphi \in \mathscr{H}_{ac}$ where A is a continuous momentum observable ($=$ Fourier transform of multiplication by a bounded continuous function with a limit at infinity.) This proof can now be extended to a more general class of potentials.

COROLLARY 1.2. *Suppose* $H = H_0 + V$, $V = V_1 + V_2$, *as in Lemma* 1.1. *Suppose also that* $|\nabla V_1| \leqslant C(1 + r)^{-\gamma}$, $\gamma > 1$. *Then the limits* (1.7) *exist.*

Proof. First consider the operator $H_1 = H_0 + V_1$. The proof of Theorem 3.3 [9] can be applied to this case to show that the limits (1.7) exist for H_1 because $|\nabla V_1|^{1/2}$ is H_1-smooth on a dense collection of intervals. By a slight modification of the proof of Theorem 2.10 [19] (another version of the chain rule) this result extends to H by Theorem 1(c). ∎

Still lacking is a general proof of the completeness of $\Omega_\pm{}^D$, or of any isometries $\Omega_\pm$ such that (1.7) is given by $\Omega_\pm A \Omega_\pm{}^*$. Lemma 1.1 places in our hands analytic tools equal to those available in the short-range case; hopefully they will prove useful in solving this problem.

The remainder of the paper is devoted to the proof of Lemma 1.1. In Section 2 we define some basic concepts and prove a result necessary to handle the local singularities which will arise. In Sections 3 and 4 the main part of the argument is given, and in Section 5 we prove some rather standard facts needed in Sections 3, 4.

The program that has been carried out successfully for short-range potentials is to prove (1.1) for the unperturbed operator H_0 and then use a perturbation argument to extend it to H, a short-range perturbation of H_0. Proofs of (1.1) for H_0 have made essential use of the fact that this operator has constant coefficients, and the perturbation argument depends essentially on the fact that the perturbation is short-range. In Section 4 we show that the "unperturbed operator" need not have constant coefficients for the perturbation argument; its resolvent need only satisfy (1.1) and one other estimate. In Section 3 we establish these two estimates directly for $(H - z)^{-1}$ with $\operatorname{Re} z > \Lambda$, where $H = H_0 + V(x)$ if V belongs to a special class of long-range potentials, those which satisfy

$$(r/2)(\partial V/\partial r) + V \leqslant \Lambda, \tag{1.8}$$

as well as (1.3), by means of the "commutator methods" used in [9] to get an estimate weaker than (1.1). Lemma 1.2 below shows that every potential satisfying (1.4) is a short-range perturbation of one satisfying (1.3) and (1.8), so that the results of Sections 3, 4 apply.

EXAMPLE 2. A potential satisfying (1.8) for all $\Lambda > 0$ is

$$U(x) = -c(1 + (r^2/2))^{-\alpha}, \qquad c > 0, \quad 0 < \alpha < 1 \tag{1.9}$$

since

$$U(x) + (r/2)(\partial U/\partial r) = -c(1 + (1 - \alpha)(r^2/2))(1 + (r^2/2))^{-\alpha-1}. \tag{1.10}$$

LEMMA 1.2. *Suppose that V_1 is a continuously differentiable function satisfying* (1.3). *Then for any $\Lambda > 0$ there exist c and β such that if U_Λ is given by* (1.9), $|U_\Lambda(x)| \leqslant c(1 + r)^{-\beta}$, $\beta > 1$, *and*

$$(r/2)(\partial/\partial r)(V_1 + U_\Lambda) + (V_1 + U_\Lambda) < \Lambda.$$

Proof. By (1.3) the expression $\frac{1}{2}r\, \partial V_1/\partial r + V_1$ is less than Λ for r sufficiently large, and is bounded throughout $\mathbf{R}^n$. Take U_Λ as in (1.9) with $\alpha > \frac{1}{2}$. By (1.10), $-(\frac{1}{2}r\, \partial U_\Lambda/\partial r + U_\Lambda)$ can be made as large as necessary by proper choice of c on the set where $\frac{1}{2}r\, \partial V_1/\partial r \geqslant \Lambda$. ▮

Proof of Lemma 1.1. The operator H is defined by the Friedrichs extension (e.g., Theorem A [21] using estimates on the potential as a quadratic form [22, Proposition 3] to show lower boundedness.

Let $0 < a < b < \infty$, and choose $0 < \Lambda < a$. By Lemma 1.2 $V + U_\Lambda$ satisfies (1.8) so Theorem 3 (see Section 3) applies to $H_1 = H_0 + (U_\Lambda + V_1)$. Consequently, since $V_2 - U_\Lambda$ satisfies (1.2), Theorem 4 (see Section 4) says that $H = H_1 + (V_2 - U_\Lambda)$ has finitely many eigenvalues in $[a, b]$ (counting multiplicity) and that if $[a, b]$ has no eigenvalues then for $z \in \mathcal{N}(a, b)$, $\| h(H - z)^{-1}h \| \leqslant C$. ▮

2. SOME SPACES OF FUNCTIONS

Here we introduce some concepts and prove some facts that will be used later. The functions $\rho(r) = (1 + r^2)^{1/2}$ and $\rho_R(r) = \rho(r/R)$ will occur, raised to various powers to specify degrees of decay at infinity; they satisfy, for $k \leqslant 2$

$$|(d/dr)^k \rho_R{}^\alpha(r)| \leqslant R^{-k}B(\alpha)\rho_R{}^\alpha(r). \tag{2.1}$$

It will be important that this and other estimates improve as $R \to \infty$.

Let $\mathscr{H}_\alpha = \mathscr{L}^2(\mathbf{R}^n, \rho^{2\alpha}(r)\, dx)$ for $\alpha \in \mathbf{R}$. The norm in $\mathscr{H}_\alpha$ is

$$|\varphi|_\alpha = \left[\int_{\mathbf{R}^n} |\varphi(x)|^2 \rho^{2\alpha}(r)\, dx\right]^{1/2}.$$

We shall also need $\mathscr{H}_\alpha^1$, the space of functions in $\mathscr{L}^2$ whose distributional derivatives also belong to $\mathscr{L}^2$. The norm on this space is

$$| \varphi |_\alpha^{(1)} = \left(| \rho^\alpha \varphi |^2 + \sum_{j=1}^{n} |(\partial/\partial x_j)\, \rho^\alpha \varphi |^2 \right)^{1/2} = | \rho^\alpha \varphi |_0^{(1)}.$$

An equivalent norm is $(| \varphi |_\alpha^2 + \sum_{j=1}^N | \partial\varphi/\partial x_j |_\alpha^2)^{1/2}$ (because of (2.1)). Every function f in $\mathscr{H}_{-\alpha}$ defines a linear functional on $\mathscr{H}_\alpha$ by the formula

$$\langle f, \varphi\rangle = \int f(x)^* \,\varphi(x)\, dx.$$

If f and φ belong to $\mathscr{H}$ this is just the usual inner product (linear in the second variable). Let $\mathscr{H}_{-\alpha}^{-1}$ be the dual space of $\mathscr{H}_\alpha^1$. It can be identified with a space of distributions on $\mathbf{R}^n$ in such a way that the pairing agrees with $\langle\ ,\ \rangle$. Thus we have a family of Hilbert spaces with $\mathscr{H}_\alpha^1 \subset \mathscr{H}_\beta \subset \mathscr{H}_\gamma^{-1}$ if $\alpha \geqslant \beta \geqslant \gamma$. Each inclusion is dense.

We shall regard operators as objects independent of the spaces in which they act. Thus if T is an operator acting in $\mathscr{H}$ we shall also use the symbol T to denote the various (unique) extensions and restrictions of T to the spaces introduced above (adding a clarifying phrase where it seems needed). The norm of T as an operator from $\mathscr{H}_\beta^\tau$ to $\mathscr{H}_\alpha^\sigma$ will be denoted by ${}_\alpha^{(\sigma)}\| T \|_\beta^{(\tau)}$ ($\sigma, \tau = \pm 1$ or blank.)

H_0 has a unique extension to an operator mapping $\mathscr{H}^1$ into $\mathscr{H}^{-1}$. Because of (2.1) it maps $\mathscr{H}_\alpha^1$ into $\mathscr{H}_\alpha^{-1}$ for $\alpha > 0$, and hence it extends further to a map of $\mathscr{H}_{-\alpha}^1$ to $\mathscr{H}_{-\alpha}^{-1}$. The same is true for multiplication by V if

$$\int_{\mathbf{R}^n} | V(x)|\, | \varphi(x)|^2\, dx \leqslant d\, | H_0\varphi |^2 + m\, | \varphi |^2, \tag{2.2}$$

and therefore also for the self-adjoint operator H defined by the form $H_0 + V$ if $d < 1$. In this case $\langle\varphi, (H + b)^{\pm 1}\varphi\rangle^{1/2}$ is equivalent to $| \varphi |^{(\pm 1)}$ as a norm on $\mathscr{H}^{\pm 1}$ for large b and there exists (for any compact set of nonreal z) a constant C such that for all $\varphi \in \mathscr{H}$

$$\langle (H - z)^{-1}\, \varphi, (H + b)(H - z)^{-1}\, \varphi\rangle \leqslant C\langle\varphi, (H + b)^{-1}\, \varphi\rangle;$$

since $\mathscr{H}$ is dense in $\mathscr{H}^{-1}$ and $\mathscr{D}(H)$ is dense in $\mathscr{H}^1$ it follows that $(H - z)^{-1}$ extends to a bounded inverse to $H - z\colon \mathscr{H}^1 \to \mathscr{H}^{-1}$. This extension of the resolvent will be important in Section 4.

We shall need control over expressions like $| T(H - z)^{-1}\varphi |_{-\alpha}$ as $\operatorname{Im} z \to 0$, where H is Schrödinger operator and T is an unbounded operator in $\mathscr{H}$ with ${}_{-\alpha}\| T \|_{\alpha}^{(1)}$ finite. The following results reduce the problem to getting estimates on $|(H - z)^{-1}\varphi|_{-\alpha}$.

THEOREM 2. *Let $H = H_0 + V$, where V satisfies (2.2) with $d = 1 - c^{-1} < 1$ and let $R(z) = (H - z)^{-1}$. For any $\alpha \geqslant 0$, $0 < a < b < \infty$ write $D = (B(-\alpha)^2 + B + b)c$, where $B = 1 - d + m$. Then*

$$| \rho_R^{-\alpha} R(z)\varphi |^{(1)} \leqslant D \, | \rho_R^{-\alpha} R(z)\varphi | + c \, | \rho_R^{-\alpha} \varphi |^{(-1)} \tag{2.3}$$

for all $\varphi \in \mathscr{H}$, $R \geqslant 1$, and $z \in \mathscr{N}(a, b)$.

Proof. Let h be a smooth radial function on $\mathbf{R}^n$. Then for $\psi \in \mathscr{H}^1$ we get, by a straightforward calculation of commutators,

$$\begin{aligned} c^{-1}(| h\psi |^{(1)})^2 &= \langle h\psi, Hh\psi \rangle + B \, | h\psi |^2 \\ &= \operatorname{Re}\langle h^2\psi, H\psi \rangle + |(\partial h/\partial r)\psi|^2 + B \, | h\psi |^2. \end{aligned}$$

Taking $h = \rho_R^{-\alpha}$ and using (2.1) we obtain

$$\begin{aligned} c^{-1}(| \rho_R^{-\alpha} R(z)\varphi |^{(1)})^2 &\leqslant \operatorname{Re}\langle \rho_R^{-2\alpha} R(z)\varphi, (H - z) \, R(z)\varphi \rangle \\ &\quad + (\operatorname{Re}(z) + B + B(-\alpha)^2) \, | \rho_R^{-\alpha} R(z)\varphi |^2 \\ &\leqslant | \rho_R^{-\alpha} R(z)\varphi |^{(1)} \left\{ | \rho_R^{-\alpha}\varphi |^{(-1)} + \frac{D}{c} \, | \rho_R^{-\alpha} R(z)\varphi | \right\}. \end{aligned}$$

Dividing by $c^{-1} \, | \rho_R^{-\alpha} R(z)\varphi |^{(1)}$ gives (2.3). ∎

COROLLARY 2.1. *If $R(z)$ is bounded uniformly for $z \in \mathscr{N}(a, b)$ as an operator from $\mathscr{H}_\alpha$ to $\mathscr{H}_{-\alpha}$ then it is bounded in the same way from $\mathscr{H}_\alpha^{-1}$ to $\mathscr{H}_{-\alpha}^1$.*

Proof. By (2.3)

$$| R(z)\varphi |_{-\alpha}^{(1)} \leqslant D \, | R(z)\varphi |_{-\alpha} + c \, | \varphi |_{-\alpha}^{(-1)} \leqslant ({}_{-\alpha}\| R(z) \|_{\alpha}^{(-1)} D + c) \, | \varphi |_{\alpha}^{(-1)}. \tag{2.4}$$

But ${}_{-\alpha}\| R(z) \|_{\alpha}^{(-1)} = {}_{-\alpha}^{(1)}\| R(z)^* \|_{\alpha} = {}_{-\alpha}^{(1)}\| R(z^*) \|_{\alpha}$ and by (2.3) ${}_{-\alpha}^{(1)}\| R(z^*) \|_{\alpha} \leqslant c + {}_{-\alpha}\| R(z^*) \|_{\alpha} D$. Therefore if ${}_{-\alpha}\| R(z) \|_{\alpha} \leqslant C$ for all $z \in \mathscr{N}(a, b)$ we have (using (2.4))

$${}_{-\alpha}^{(1)}\| R(z) \|_{\alpha}^{(-1)} \leqslant c + (c + CD)D. \quad ∎$$

3. Resolvent Estimates for Potentials with No Barriers

In this section we derive the estimates which are necessary for the perturbation theory given in the next section. What we do is very similar to the work in [9], where we made use of the fact that estimates on $(H - z)^{-1}$ follow from essential positivity of $i[H, A]$ for a relatively H-bounded operator A. (The significance of positive commutators was discovered by Putnam (see [23]); their connection with resolvent estimates was pointed out by Kato [24] in a general setting. These ideas were first applied to Schrödinger operators in [11].) Here we get stronger estimates from the same commutator calculations.

In [9] we pointed out that the condition

$$V(x) + (r/2)(\partial V/\partial r)(x) < \Lambda, \quad x \in \mathbf{R}^n \tag{3.1}$$

says roughly that the force $-\nabla V$ sets up no effective potential barriers to a particle with energy greater than Λ. In the absence of such barriers a particle should move without hesitation along its path in from outer space and back out again. If h is a nonnegative continuous integrable function on $(0, \infty)$ a certain observable ($=$ self-adjoint operator) $A(h)$ can be said to reflect this steady progress in the sense that its commutator with H is almost positive for states φ of energy above Λ and thus $e^{iHt}A(h)\, e^{-iHt}$ tends to increase as the particle progresses. For $\varphi \in \mathscr{D}(H_0)$ we define

$$\begin{aligned} A(h)\,\varphi(x) &= -i\left\{g(r)\frac{x}{r}\cdot\nabla\varphi(x) + \nabla\cdot\left(\frac{x}{r}g(r)\,\varphi(x)\right)\right\} \\ &= -i\left\{2g(r)\frac{x}{r}\cdot\nabla\varphi(x) + \left(\frac{n-1}{r}g(r) + h(r)\right)\varphi(x)\right\}, \end{aligned} \tag{3.2}$$

where $g(r) = \int_0^r h(s)\, ds$.

The main things we need to know about $A(h)$ are expressions for its commutator with H, which will be given in Lemma 3.2, and the boundedness of $A(h)$ and related operators on various spaces, which is the content of Lemma 3.1.

Lemma 3.1. *Let F be a bounded vector-valued function on $\mathbf{R}^n$. Then, for any $\alpha \geqslant 0$ and $R > 1$*

$$|\,\rho_R^{-\alpha}F\cdot\nabla\varphi\,| \leqslant (1 + B(-\alpha))\,\|F\|_\infty\,|\,\rho_R^{-\alpha}\varphi\,|^{(1)} \tag{3.3}$$

In particular,

$${}_{-\alpha}\|\,A(h)\|_{-\alpha}^{(1)} \leqslant (1 + B(-\alpha))\,2\,\|h\|_1 + n\,\|h\|_\infty\,. \tag{3.4}$$

Proof. Using (2.1), we have

$$| \rho_R^{-\alpha} F \cdot \nabla \varphi | \leqslant | F \cdot \nabla \rho_R^{-\alpha} \varphi | + \| F \|_\infty B(-\alpha) | \rho_R^{-\alpha} \varphi |$$
$$\leqslant \| F \|_\infty | \rho_R^{-\alpha} \varphi |^{(1)} (1 + B(-\alpha)).$$

By (3.2)

$$A(h) = F \cdot \nabla + f \qquad \text{where} \quad \| F \|_\infty \leqslant 2 \| h \|_1 \quad \text{and} \quad \| f \|_\infty \leqslant n \| h \|_\infty .$$

Therefore by (3.3), we have

$$| A(h)\varphi |_{-\alpha} \leqslant (1 + B(-\alpha))\, 2 \| h \|_1 | \varphi |_{-\alpha}^{(1)} + n \| h \|_\infty | \varphi |_{-\alpha}$$
$$\leqslant \{(1 + B(-\alpha))\, 2 \| h \|_1 + n \| h \|_\infty\} | \varphi |_{-\alpha}^{(1)}$$

which is the desired result. ∎

Throughout this section we shall consider operators $H = H_0 + V$ where the radial distribution derivative of V satisfies (3.1) and for some $\gamma > 1$

$$(\partial V/\partial r)(x) \leqslant C(1 + r)^{-\gamma}. \tag{3.5}$$

This implies that V itself is bounded, so H is self-adjoint on $\mathscr{D}(H_0)$. One could allow more general V (as was done in [9]), but that will be unnecessary; all unbounded behavior will occur in the perturbation of a V satisfying (3.1) and (3.5). On the other hand, we do not assume (as we did in [9]) that $V(x) \to 0$ as $| x | \to \infty$ in this section. We would not need this extra generality either, but it may be instructive to note that one can get information about Schrödinger operators with such potentials. (Note that (3.5) implies that V tends to a limit in each radial direction, but these limits may depend on the direction. Our method will not give information about the part of the spectrum below the maximum of all such radial limits.)

LEMMA 3.2. *Suppose that V satisfies* (3.1) *and* (3.5) *and let $H = H_0 + V$. If $1 < \beta < \min(\gamma, 2)$, then for any $\epsilon > 0$ there exists a number $M(\epsilon)$ such that if $R \geqslant M(\epsilon)$ and $A(h)$ is the operator defined in* (3.2) *with $h(r) = \rho_R(r)^{-\beta}$ then for $\varphi \in \mathscr{D}(H)$,*

$$2 \operatorname{Re}\langle (H - \Lambda - \epsilon)\varphi, h\varphi \rangle \leqslant \operatorname{Im}\langle A(h)\varphi, H\varphi \rangle. \tag{3.6}$$

Proof. Note that the function g defined in (3.2) satisfies $g/r \geqslant h$

for our choice of h. Because of this, it is a straightforward calculation, starting with (2.13) of [9] that for $n \geqslant 2$, $\varphi \in \mathscr{D}(H)$,

$$2\,\mathrm{Re}\langle H\varphi, h\varphi\rangle \leqslant \mathrm{Im}\langle A(h)\varphi, H\varphi\rangle + \left\langle \varphi, 2h\left(V + \frac{r}{2}\frac{\partial V}{\partial r}\right)\varphi\right\rangle$$
$$+ \left\langle \varphi, \left\{\left(\frac{g}{r} - h\right)\left(r\frac{\partial V}{\partial r} + \frac{1}{2r^2}\right) - \frac{h''}{2}\right\}\varphi\right\rangle \qquad (3.7)$$

(Note that in [9] inner products are linear in the *first* variable.) The term $1/2r^2$ in (3.7) is actually necessary only in the case $n = 2$ where it arises from the term $(n - 3)(n - 1)/4r^2$ in the expression (2.9) for $-\Delta$ in [9]. For $n = 1$ one has (3.7) except that r is replaced by x, and no $1/2r^2$ term appears. The proof we give below for $n \geqslant 2$ thus extends to $n = 1$ but could actually be made simpler in this case.

By (3.1),

$$2h(V + (r/2)(\partial V/\partial r)) \leqslant 2\Lambda h.$$

Therefore it is enough to show that for R sufficiently large $| h'' | < \epsilon h$ and $(g/r - h)(r\,\partial V/\partial r + 1/2r^2) \leqslant \epsilon g'$. Let us write

$$g_1(r) = \int_0^r (1 + s^2)^{-\beta/2}\, ds.$$

Then for $1 \leqslant k \leqslant 3$

$$|g^{(k)}(r)| = R^{1-k}\left|g_1^{(k)}\left(\frac{r}{R}\right)\right| \leqslant R^{1-k}B(-\beta)\, g_1'\left(\frac{r}{R}\right) = B(-\beta)\, R^{1-k}h(r)$$

by (2.1). The necessary estimate on h'' is a consequence of this, with $k = 3$. Now consider

$$g(r)/r - h(r) = [g(r) - rg'(r)]/r = f(r/R),$$

where $f(r) = [g_1(r) - rg_1'(r)]/r$. Clearly $f(r) \leqslant c(1 + r)^{-1}$. (We shall use the letter C for a sequence of different constants below.) Since $g_1^{(2k)}(0) = 0$, we have $f(r) \leqslant Cr^2$. Combining these two estimates, we obtain $f(r) \leqslant Cr^2(1 + r)^{-3}$ so that

$$f(r/R) \leqslant C(r^2/R^2)\left(1 + \frac{r}{R}\right)^{-3} \leqslant C(r^2/R^2)\, g_1'(r/R),$$

since $\beta < 2$, which gives, for large R, $(g/r - h)\, 2r^{-2} \leqslant (\epsilon/2)h$. Finally, since

$$f\left(\frac{r}{R}\right) \frac{1}{g_1'\left(\frac{r}{R}\right)} \leqslant C \frac{r^2}{R^2}\left(1 + \frac{r}{R}\right)^{\beta-3} \leqslant C\left(\frac{r}{R}\right)^{\beta-1}$$

and $r\, \partial V/\partial r \leqslant Cr^{-\gamma+1}$, we have for x outside the unit ball in $\mathbf{R}^n$ and large R

$$(1/h)(g/r - g')\, r(\partial V/\partial r) \leqslant C(r^{\beta-\gamma}/R^{\beta-1}) \leqslant \epsilon/2.$$

But inside this ball, using the estimate $f(r) \leqslant Cr^2$ noted above, we have for large R

$$\frac{1}{h}\left(\frac{g}{r} - h\right) r \frac{\partial V}{\partial r} \leqslant C \frac{1}{R^2}\, r \frac{\partial V}{\partial r} \leqslant \frac{C}{R^2} \leqslant \frac{\epsilon}{2}. \quad \blacksquare$$

THEOREM 3. *Suppose* $H = H_0 + V$ *where* V *satisfies* (3.1) *and* (3.5) *and let* $R(z) = (H - z)^{-1}$. *Let* $a > \Lambda$ *and recall* $\mathcal{N}(a, b) = \{z \in \mathbf{C}\colon a < \operatorname{Re} z < b, \mid \operatorname{Im} z \mid > 0\}$. (1) *If* $\alpha > 1/2$ *there exists* $C(\alpha) > 0$ *such that for* $\varphi \in \mathscr{H}_\alpha$ *and* $z \in \mathcal{N}(a, b)$

$$\mid R(z)\varphi \mid_{-\alpha} \leqslant C(\alpha) \mid \varphi \mid_\alpha . \tag{3.8}$$

(2) *Given* $1/2 < \alpha < 1$ *and* $0 \leqslant \mu \leqslant \alpha$ *there exists* C *such that if* $\{z_k\} \subset \mathcal{N}(a, b)$, $\varphi \in \mathscr{H}_{\mu+\alpha}$, $\psi \in \mathscr{H}_{-\alpha}$ *satisfy*

(a) $\lim_{k\to\infty} z_k = \lambda \in [a, b]$

(b) $\lim_{k\to\infty} \mid R(z_k)\varphi - \psi \mid_{-\alpha} = 0$

(c) $\lim_{k\to\infty} \mid \operatorname{Im} z_k \mid \mid R(z_k)\varphi \mid^2 = 0$

then

$$\mid \psi \mid_{\mu-\alpha} \leqslant C \mid \varphi \mid_{\mu+\alpha} . \tag{3.9}$$

Remark. If we knew that $R(z)$ had boundary values $R(\lambda \pm i0)$ at each $\lambda \in [a, b]$, (3.8) would say $R(\lambda \pm i0)\colon \mathscr{H}_\alpha \to \mathscr{H}_{-\alpha}$ and (3.9) would assert that faster decrease of φ at ∞ gives faster decrease of $R(\lambda \pm i0)\varphi$ if (c) is satisfied, i.e., $\langle \varphi, [R(\lambda + i0) - R(\lambda - i0)]\varphi\rangle = 0$. (We do not show that boundary values exist; it can be done, but it is unnecessary and involves a considerable amount of work.)

Proof. Take h, as in Lemma 3.2, equal to $\rho_R^{-2\alpha}$ for some $R > M(\epsilon)$, where $\epsilon = \frac{1}{2}(a - \Lambda)$, and $1 < 2\alpha < \min(\gamma, 2)$. Then for $\varphi \in \mathscr{D}(H)$ and $\operatorname{Re} z \geqslant a$ we have

$$\begin{aligned} 2\epsilon\langle\varphi, h\varphi\rangle &\leqslant 2(\operatorname{Re} z - \Lambda - \epsilon)\langle\varphi, h\varphi\rangle \\ &\leqslant 2\operatorname{Re}\langle(H - \Lambda - \epsilon)\varphi, h\varphi\rangle + 2\operatorname{Re}\langle(z - H)\varphi, h\varphi\rangle \\ &\leqslant \operatorname{Im}\langle A(h)\varphi, H\varphi\rangle + 2\operatorname{Re}\langle(z - H)\varphi, h\varphi\rangle \\ &= \operatorname{Im}\langle A(h)\varphi, (H - z)\varphi\rangle + \operatorname{Im} z\langle\varphi, A(h)\varphi\rangle \\ &\quad + 2\operatorname{Re}\langle(z - H)\varphi, h\varphi\rangle. \end{aligned}$$

Now replace φ in this inequality by $R(z)\varphi$, $\varphi \in \mathscr{H}_\alpha$.

$$\begin{aligned} 2\epsilon \mid h^{1/2}R(z)\varphi \mid^2 \leqslant \operatorname{Im}\langle A(h)\, R(z)\varphi, \varphi\rangle + \operatorname{Im} z\langle R(z)\varphi, A(h)\, R(z)\varphi\rangle \\ - 2\operatorname{Re}\langle\varphi, hR(z)\varphi\rangle. \end{aligned} \tag{3.10}$$

(This is our basic inequality; in each term on the right side the singularity of $R(z)$ near the real axis is mitigated in some way.) Recall that by Lemma 3.1 $A(h)$ is bounded from $\mathscr{H}_\beta^1$ to $\mathscr{H}_\beta$ if $\beta \leqslant 0$. By Theorem 2 and (1.5),

$$\begin{aligned} \operatorname{Im} z\langle R(z)\varphi, A(h)\, R(z)\varphi\rangle &\leqslant \mid \operatorname{Im} z \mid \| A(h)\|^{(1)} \mid R(z)\varphi \mid^{(1)} \mid R(z)\varphi \mid \\ &\leqslant \mid \operatorname{Im} z \mid \| A(h)\|^{(1)} (D \mid R(z)\varphi \mid + c \mid \varphi \mid) \mid R(z)\varphi \mid \\ &= \| A(h)\|^{(1)} (D \mid \operatorname{Im}\langle\varphi, R(z)\varphi\rangle \mid \\ &\quad + c \mid \operatorname{Im} zR(z)\varphi \mid \mid \varphi \mid), \end{aligned} \tag{3.10a}$$

which together with (3.10) gives, using Theorem 2 again,

$$\begin{aligned} 2\epsilon \mid h^{1/2}R(z)\varphi \mid^2 &\leqslant {}_{-\alpha}\| A(h)\|_{-\alpha}^{(1)} (D \mid R(z)\varphi \mid_{-\alpha} + c \mid \varphi \mid_{-\alpha}) \mid \varphi \mid_\alpha \\ &\quad + (D \| A(h)\|^{(1)} + 2) \mid R(z)\varphi \mid_{-\alpha} \mid \varphi \mid_\alpha + c \| A(h)\|^{(1)} \mid \varphi \mid^2 \\ &\leqslant C(\mid R(z)\varphi \mid_{-\alpha} \mid \varphi \mid_\alpha + \mid \varphi \mid_{-\alpha} \mid \varphi \mid_\alpha). \end{aligned}$$

(We have used $\| h \|_\infty = 1$, $\| \operatorname{Im} zR(z)\| = 1$ and $\mid \varphi \mid^2 \leqslant \mid \varphi \mid_{-\alpha} \mid \varphi \mid_\alpha$.) Now since $\mid R(z)\varphi \mid_{-\alpha}$ is dominated by $\epsilon \mid h^{1/2}R(z)\varphi \mid$ we obtain the following after dividing by $\mid R(z)\varphi \mid_{-\alpha}$:

$$\begin{aligned} d \mid R(z)\varphi \mid_{-\alpha} &\leqslant C \mid \varphi \mid_a \left(1 + \frac{\mid \varphi \mid_{-\alpha}}{\mid R(z)\varphi \mid_{-\alpha}}\right) \\ &\leqslant C \mid \varphi \mid_\alpha \left(1 + \frac{\mid \varphi \mid_\alpha}{\mid R(z)\varphi \mid_{-\alpha}}\right) \end{aligned}$$

which implies that

$$d \mid R(z)\varphi \mid_{-\alpha} \leqslant (2C + d) \mid \varphi \mid_{\alpha},$$

proving (1).

For (2), note that we can estimate $\mid \psi \mid_{-\delta}$ using

$$\mid \psi \mid_{-\delta}^2 \leqslant C(\sup_{R>1} R^{-2\eta} \mid \rho_R^{-\alpha}\psi \mid^2 + \mid \psi \mid_{-\alpha}^2) \tag{3.11}$$

if $0 \leqslant \eta < \delta$ (Proposition 5.1) and for $\delta = 0$ we have

$$\mid \psi \mid_0 = \sup_{R>1} \mid \rho_R^{-\alpha} \mid \psi \mid.$$

Therefore an estimate on $\mid \psi \mid_{\mu-\alpha}$ can be obtained (i) with $0 \leqslant \eta < \alpha - \mu$ if $\mu < \alpha$, and (ii) with $\eta = 0$ if $\mu = \alpha$. Choose such an η which also satisfies

$$\alpha < 1 - \eta < \mu + \alpha.$$

This is consistent with (i) or (ii) since $\frac{1}{2} < \alpha < 1$. Then $\mid \varphi \mid_{1-\eta} \leqslant \mid \varphi \mid_{\mu+\alpha}$, so (3.9) will follow from

$$\mid \rho_R^{-\alpha}\psi \mid \leqslant CR^{\eta} \mid \varphi \mid_{1-\eta}. \tag{3.12}$$

But by (b), (3.10) and (3.10a)

$$\begin{aligned} 2\epsilon \mid \rho_R^{-\alpha}\psi \mid^2 &= 2\epsilon \lim_{k\to\infty} \mid \rho_R^{-\alpha} R(z_k)\varphi \mid^2 \\ &\leqslant \lim_{k\to\infty} \sup\{\operatorname{Im}\langle A(\rho_R^{-2\alpha})\, R(z_k)\varphi, \varphi\rangle \\ &\quad + \| A(\rho_R^{-2\alpha})\|^{(1)} \mid \operatorname{Im} z_k \mid \mid R(z_k)\varphi \mid (D \mid R(z_k)\varphi \mid + c \mid \varphi \mid) \\ &\quad + 2 \mid \rho_R^{-\alpha}\varphi \mid \mid \rho_R^{-\alpha} R(z_k)\varphi \mid\}. \end{aligned} \tag{3.13}$$

By assumption (c) the middle term on the right side converges to zero as $k \to \infty$. Now consider the first term;

$$\begin{aligned} |\langle A(\rho_R^{-2\alpha})\, R(z_k)\varphi, \varphi\rangle| &= |\langle(\rho^{\eta-1} A(\rho_R^{-2\alpha})\, \rho_R^{\alpha})\, \rho_R^{-\alpha} R(z_k)\varphi, \rho^{1-\eta}\varphi\rangle| \\ &\leqslant |(\rho^{\eta-1} A(\rho_R^{-2\alpha})\, \rho_R^{\alpha})(\rho_R^{-\alpha} R(z_k)\varphi)| \mid \varphi \mid_{1-\eta}. \end{aligned}$$

By (3.2)

$$\begin{aligned} \rho^{\eta-1} A(\rho_R^{-2\alpha})\, \rho_R^{\alpha} &= -i\rho^{\eta-1} \left\{2g(r)\frac{x}{r} \cdot \nabla + \frac{n-1}{r} g(r) + \rho_R^{-2\alpha}\right\} \rho_R^{\alpha} \\ &= -i\rho^{\eta-1} \left\{2g(r)\, \rho_R^{\alpha} \frac{x}{r} \cdot \nabla + \frac{n-1}{r} g\rho_R^{\alpha} + \rho_R^{-\alpha} + \frac{d\rho_R^{\alpha}}{dr}\right\}. \end{aligned}$$

The pure multiplication part of this operator is

$$\rho^{\eta-1}\left\{\rho_R{}^{\alpha}\,\frac{n-1}{r}\int_0^r \rho_R^{-2\alpha}(s)\,ds+\rho_R^{-\alpha}+\frac{d\rho_R{}^{\alpha}}{dr}\right\}$$

which is bounded independent of R since $\eta - 1 \leqslant -\alpha$ and $d\rho_R{}^{\alpha}/dr \leqslant B(\alpha)\,\rho_R{}^{\alpha}$ by (2.1). The other term is of the form $F \cdot \nabla$ with

$$R^{-\eta}\,|\,F\,| \leqslant 2(r/R)^{\eta-1}\,R^{-1}g(r)\,\rho_R{}^{\alpha} = 2f(r/R),$$

where $f(r) = r^{\eta-1}\int_0^r \rho^{-2\alpha}(s)\,ds\,\rho^{\alpha}(r)$ is a continuous function, bounded near zero since $\eta \geqslant 0$, and bounded near infinity since $1-\eta \geqslant \alpha$. Therefore $\|F\|_\infty \leqslant CR^{\eta}$, and

$$\begin{aligned}|\,\rho^{\eta-1}A(\rho_R^{-2\alpha})\,\rho_R^{\alpha})\,\rho_R^{-\alpha}R(z_k)\varphi\,| &\leqslant CR^{\eta}\,|\,\rho_R^{-\alpha}R(z_k)\varphi\,|^{(1)}\\ &\leqslant CR^{\eta}(D\,|\,\rho_R^{-\alpha}R(z_k)\varphi\,|+c\,|\,\rho_R^{-\alpha}\varphi\,|^{(-1)})\end{aligned}$$

(using Theorem 2.) Finally, having estimated the first two terms on the right in (3.13), we have

$$\begin{aligned}2\epsilon\,|\,\rho_R^{-\alpha}\psi\,|^2 &\leqslant \lim_{k\to\infty}|\,\rho_R^{-\alpha}R(z_k)\varphi\,|\,\{C'R^{\eta}\,|\,\varphi\,|_{1-\eta}+2\,|\,\rho_R^{-\alpha}\varphi\,|\}\\ &\quad + C'R^{\eta}\,|\,\varphi\,|_{1-\eta}\,|\,\rho_R^{-\alpha}\varphi\,|^{(1)}\\ &\leqslant |\,\rho_R^{-\alpha}\psi\,|\,\{C'R^{\eta}\,|\,\varphi\,|_{1-\eta}+2R^{\eta}\,|\,\varphi\,|_{1-\eta}\}+C'R^{\eta}\,|\,\varphi\,|_{1-\eta}^2\\ &\leqslant C''R^{\eta}\,|\,\varphi\,|_{1-\eta}\,(|\,\rho_R^{-\alpha}\psi\,|+|\,\varphi\,|_{1-\eta})\end{aligned}$$

which implies (3.12). (Divide by $|\,\rho_R^{-\alpha}\psi\,|$.) ∎

Remark. If we take the special case $H = H_0$, Theorem 3 gives a new method of obtaining estimates on $(H_0 - z)^{-1}$. Recently some other efficient methods have been introduced by Kuroda [25] and Agmon [4]. These apply to general elliptic operators with constant coefficients as well. The long-range problem in this context has not been studied at all, so far as we know.

4. Perturbation of Resolvent Estimates

Now we present the well-established theory of perturbation of the continuous spectrum in such a way that operators H_1 of the type considered in Section 3 can play the role of "unperturbed" operator. Our main point is that even though we do not know all about H_1 (as we do for H_0, the usual unperturbed operator, via

Fourier analysis) we do know enough to get information about a perturbed operator $H_1 + V$. Our method most closely resembles the one used by Agmon in unpublished work which was reported in some detail at the June 1971 Oberwolfach conference on scattering theory. This method leads to a more incisive result than is usually obtained because it is independent of theorems on absence of positive eigenvalues, which require more regularity than the rest of the argument. Agmon's method is not necessary in our case since such theorems are available. In fact an earlier version of our work (reported at the same conference) used such theorems in the traditional way.

As a rule, (1) One shows that if the perturbed resolvent is not bounded suitably near an interval of the real axis, then the perturbed operator has an eigenfunction ψ, which may not belong to $\mathscr{L}^2(\mathbf{R}^n)$, with eigenvalue in the interval and (2) One must rule out the possibility of too many such eigenvalues. This is traditionally done by using certain properties of ψ to show that it decays at infinity at a slightly quicker rate than was obvious at first, and this is enough to invoke theorems on nonexistence of eigenfunctions. But Agmon showed that the argument can be iterated to prove that ψ actually belongs to $\mathscr{L}^2(\mathbf{R}^n)$, and that only finitely many such eigenvalues can exist in any "noncritical" interval in the spectrum. We show here that this argument can be given using just the inequalities established in Section 3, (at least in the context of Schrödinger operators).

THEOREM 4. *Let $H_1 = H_0 + V_1$ where $H_0 = -\Delta$ and V_1 is bounded, and let $R_1(z) = (H_1 - z)^{-1}$. Suppose that conclusions* (1) *and* (2) *of Theorem* 3 *are satisfied by H_1 for any interval $[a, b] \subset (\Lambda, \infty)$. Then if V satisfies* (1.2), *and $[a, b] \subset (\Lambda, \infty)$, H_2, the self-adjoint operator defined by the form $H_0 + V_1 + V$, has only finitely many eigenvalues* (*counting multiplicity*) *in $[a, b]$ and if $[a, b]$ does not contain any eigenvalue of H_2 there exists, for any $\alpha > 1/2$, a $C > 0$ such that with $R_2(z) = (H_2 - z)^{-1}$ we have for all $z \in \mathscr{N}(a, b)$,*

$$| R_2(z)\varphi |_{-\alpha}^{(1)} \leqslant C | \varphi |_{\alpha}^{(-1)} \tag{4.1}$$

and

$$\| hR_2(z)h \| \leqslant C \tag{4.2}$$

if $| h |^2$ satisfies (1.2).

Proof. Our starting point is the resolvent equation. Regarding

H_1, V, and H_2 as bounded maps from $\mathscr{H}^1$ into $\mathscr{H}^{-1}$, and $R_1(z)$ and $R_2(z)$ from $\mathscr{H}^{-1}$ into $\mathscr{H}^1$, for $z \in \mathscr{N}(a, b)$ we have for $\varphi \in \mathscr{H}$

$$\varphi = (H_2 - z) R_2(z)\varphi = (H_1 - z) R_2(z)\varphi + VR_2(z)\varphi \tag{4.3}$$

and therefore

$$R_1(z)\varphi = (I + R_1(z)V) R_2(z)\varphi. \tag{4.4}$$

This equation holds as well for $\varphi \in \mathscr{H}^{-1}$ since $\mathscr{H}$ is dense in $\mathscr{H}^{-1}$, and the operators involved are all suitably bounded. Recall that $V = f(1 + r)^{-\gamma}$, $\gamma > 1$, as in (1.2). We may choose α and $\epsilon > 0$ so that $1 < 2\alpha = \gamma - \epsilon$; then $V: \mathscr{H}^1_{-\alpha} \to \mathscr{H}^{-1}_{\alpha+\epsilon}$. We may also assume $\alpha < 1$ without loss of generality.

LEMMA 4.1. *If $R_2(z)$ is not bounded as an operator from $\mathscr{H}^{-1}_\alpha$ to $\mathscr{H}^1_{-\alpha}$, uniformly for $z \in \mathscr{N}(a, b)$ then H_2 has an eigenvalue in $[a, b]$.*

Proof. If $R_2(z)$ is not so bounded, there exist sequences $\{z_k\} \subset \mathscr{N}(a, b)$ and $\{\varphi_k\} \subset \mathscr{H}^{-1}_\alpha$, $|\varphi_k|^{(-1)}_\alpha = 1$ with $|R_2(z_k)\varphi_k|^{(1)}_{-\alpha} \to \infty$ as $k \to \infty$. Since ${}^{(1)}_{-\alpha}\| R_2(z)\|^{(-1)}_\alpha \leqslant {}^{(1)}\| R_2(z)\|^{(-1)}$, which is uniformly bounded for z in any compact subset of $\mathscr{N}(a, b)$ (by Section 2) we must have $|\operatorname{Im} z_k| \to 0$, and we may assume (by passing to a subsequence) that $z_k \to \lambda \in [a, b]$. By (3.8) and Corollary 2.1

$$| R_1(z_k)\varphi_k |^{(1)}_{-\alpha} = |(1 + R_1(z_k)V) R_2(z_k)\varphi_k |^{(1)}_{-\alpha}$$

is bounded as $k \to \infty$. Now we can define a sequence of unit vectors $\psi_k = R_2(z_k)\varphi_k / | R_2(z_k)\varphi_k |^{(1)}_{-\alpha} \in \mathscr{H}^1_{-\alpha}$ so that $(1 + R_1(z_k)V)\psi_k \to 0$ in $\mathscr{H}^1_{-\alpha}$ as $k \to \infty$.

We have (as operators in $\mathscr{H}^1_{-\alpha}$)

$$R_1(z_k)V = R_1(i)(I + (i - z_k) R_1(z_k))V. \tag{4.5}$$

Let $1/2 < \alpha' < \alpha$. By (3.8) the factor $(I + (i - z_k) R_1(z_k))V$ has norm independent of k as an operator from $\mathscr{H}^1_{-\alpha}$ to $\mathscr{H}^{-1}_{-\alpha'}$ and $R_1(i)$ is compact from $\mathscr{H}^{-1}_{-\alpha'}$ to $\mathscr{H}_{-\alpha}$ (Proposition 5.2). Therefore $R_1(z_k)V$ maps the unit ball of $\mathscr{H}^1_{-\alpha}$ into a fixed compact set in $\mathscr{H}_{-\alpha}$ and since $|\psi_k|^{(1)}_{-\alpha} = 1$ we may assume (by passing to a subsequence again) that $R_1(z_k) V\psi_k$ is strongly convergent in $\mathscr{H}_{-\alpha}$. Then

$$|(1 + R_1(z_k)V)\psi_k|_{-\alpha} \to 0$$

implies the existence of $\lim_{k\to\infty} \psi_k = \psi \in \mathscr{H}_{-\alpha}$, with

$$| R_1(z_k) V\psi + \psi |_{-\alpha} \to 0 \qquad \text{as} \quad k \to \infty.$$

By Theorem 2, $|\psi_k|_{-\alpha}$ is bounded below as $k \to \infty$, so $\psi \neq 0$. Since $|\psi_k|_{-\alpha}^{(1)} = 1$, $\psi \in \mathscr{H}_{-\alpha}^1$.

The proof of Lemma 4.1 is completed by the following:

LEMMA 4.2. *Let $[a, b] \subset (\Lambda, \infty)$ and let $\psi \in \mathscr{H}_{-\alpha}^1$. A necessary and sufficient condition for $H_2\psi = \lambda\psi$, $\psi \in \mathscr{D}(H_2)$, $\lambda \in [a, b]$, is the existence of a sequence $\{z_k\} \subset \mathscr{N}(a, b)$ with $z_k \to \lambda$ and $|R_1(z_k) V\psi + \psi|_{-\alpha} \to 0$ as $k \to \infty$. There exists a compact operator K on $\mathscr{H}$ such that for any such ψ,*

$$|\psi| \leqslant |K\psi|. \tag{4.6}$$

This lemma will complete the proof of the theorem. For (A) it asserts that all eigenvectors with eigenvalues in $[a, b]$ must satisfy (4.6), but this relation can be satisfied by at most finitely many orthonormal vectors; and (B) on any interval not containing an eigenvalue (4.1) must be satisfied, by Lemma 4.1, and boundedness of $\| hR_2(z)h \|$ follows from (4.1).

Proof of Lemma 4.2. First suppose $\psi \in \mathscr{D}(H)$ and $H\psi = \lambda\psi$, $\lambda \in [a, b]$. Then

$$\begin{aligned} R_1(\lambda + i/k)\, V\psi &= R_1(\lambda + i/k)[(H_2 - \lambda - i/k) - (H_1 - \lambda - i/k)]\psi \\ &= -(i/k)\, R_1(\lambda + i/k)\psi - \psi \end{aligned}$$

(both sides regarded as the product of $R_1(\lambda + i/k)\colon \mathscr{H}^{-1} \to \mathscr{H}^1$ with an operator mapping $\mathscr{H}^1$ into $\mathscr{H}^{-1}$, acting on $\psi \in \mathscr{H}^1$.) The last expression approaches $-\psi$ in $\mathscr{H}_{-\alpha}$ as $k \to \infty$ since $k^{-1}R(\lambda + i/k)$ converges strongly to zero unless λ is an eigenvalue of H_1, which would contradict (3.8).

Now assume that $\{z_k\}$ is a sequence in $\mathscr{N}(a, b)$ convergent to $\lambda \in [a, b]$, $\psi \in \mathscr{H}_{-\alpha}^1$ and $|R_1(z_k) V\psi + \psi|_{-\alpha} \to 0$ as $k \to \infty$. We shall prove $\psi \in \mathscr{H}$. First we note that $V\psi \in \mathscr{H}_\alpha^{-1}$, $|R_1(z_k) V\psi|_{-\alpha}^{(1)} \leqslant C\, |V\psi|_\alpha^{(-1)}$ so that $R_1(z_k) V\psi + \psi$ converges weakly in $\mathscr{H}_{-\alpha}^1$, and

$$\begin{aligned} \lim_{k\to\infty} |\operatorname{Im} z_k|\, |R_1(z_k) V\psi|^2 &= \lim_{k\to\infty} |\operatorname{Im}\langle V\psi, R_1(z_k) V\psi\rangle| \\ &= |\operatorname{Im}\langle V\psi, -\psi\rangle| = 0. \end{aligned}$$

Thus (a)–(c) of Theorem 3 hold with $\varphi = V\psi$. Actually $V\psi$ has somewhat better decrease at infinity than its membership in $\mathscr{H}_\alpha^{-1}$ indicates, for $\psi \in \mathscr{H}_{-\alpha}^1$ and $V = O(r^{-2\alpha-\epsilon})$ at infinity, so $V\psi \in \mathscr{H}_{\alpha+\epsilon}$. (Therefore if $V\psi$ were locally square integrable, we could conclude from (3.9) that ψ decreases faster as well, giving even better decrease for $V\psi$. This argument could then be repeated until the conclusion $\psi \in \mathscr{H}$ was attained. This is Agmon's "bootstrap" procedure.) $V\psi \in \mathscr{H}^{-1}$ is locally singular, but the above argument can be applied to $\varphi = (i - \lambda) R_1(i) V\psi$. For

$$R_1(z_k) V\psi = [1 - (i - z_k) R_1(z_k)] R_1(i) V\psi$$

so that (b) holds for φ and ψ':

$$R_1(z_k)(i - \lambda) R_1(i) V\psi \to \psi + R_1(i) V\psi = \psi'$$

in $\mathscr{H}_{-\alpha}$ as $k \to \infty$. It will be enough to prove that $\psi' \in \mathscr{H}$, since we already have $R_1(i) V\psi \in \mathscr{H}$. We can apply (3.9) since (c) also holds:

$$\begin{aligned} |\operatorname{Im} z_k| \, | R_1(z_k)(i - \lambda) R_1(i) V\psi |^2 &= |\operatorname{Im} z_k| \, | i - \lambda | \, | R_1(i) R_1(z_k) V\psi | \\ &\leqslant C \, |\operatorname{Im} z_k| \, | R_1(z_k) V\psi |^2 \to 0 \end{aligned}$$

as $k \to \infty$. Now $V: \mathscr{H}^1_{-\alpha} \to \mathscr{H}^{-1}_{\alpha+\epsilon}$ and $R_2(i): \mathscr{H}_\delta \to \mathscr{H}_\delta$ for any δ, (Proposition 5.2) so that $R_2(i)V: \mathscr{H}^1_{-\alpha} \to \mathscr{H}^{\alpha}_{\alpha+\epsilon}$. Let

$$\beta = \sup\{\beta' < \alpha : \psi \in \mathscr{H}_{-\alpha+\beta'}\}.$$

Then ψ belongs to $\mathscr{H}_{-\alpha+\beta-\epsilon/2}$ so $\varphi = (i - \lambda) R_1(i) V\psi \in \mathscr{H}_{\alpha+\beta+\epsilon/2}$. If we assume $\beta + \epsilon/2 \leqslant \alpha$, (3.9) implies $\psi' \in \mathscr{H}_{-\alpha+\beta+\epsilon/2}$, and hence ψ also belongs to this space, which contradicts the definition of β. Therefore $\beta + \epsilon/2 > \alpha$, and $\varphi \in \mathscr{H}_{2\alpha}$. In this case (3.9) implies that ψ' and ψ belong to $\mathscr{H}$.

We also obtain the useful inequality

$$\begin{aligned} |\psi| = |\psi' - R_1(i) V\psi| &\leqslant |\psi'| + |R_1(i) V\psi| \\ &\leqslant C' \, |\rho^{2\alpha} R_1(i) V\psi|. \end{aligned} \tag{4.7}$$

Now we show that ψ is an eigenvector of H_2. We know that $\psi \in \mathscr{H}^1$. Let $\varphi \in \mathscr{H}^1_\alpha$. It is clear that H_1 and V map $\mathscr{H}^1_\alpha$ into $\mathscr{H}^{-1}_\alpha$ (using (2.1) in the case of H_1.) Then we have the following identity (in which each of the terms is a pairing of elements of dual spaces.)

$$\begin{aligned} \langle (H_2 - \lambda)\varphi, \psi\rangle &= \langle (H_1 - \lambda)\varphi, \psi\rangle + \langle V\varphi, \psi\rangle \\ &= \lim_{k\to\infty} \langle (H_1 - \lambda)\varphi, -R_1(z_k) V\psi\rangle + \langle V\varphi, \psi\rangle \\ &= \lim_{k\to\infty} \{-\langle \varphi, V\psi\rangle + (\lambda - z_k)\langle \varphi, R_1(z_k) V\psi\rangle + \langle V\varphi, \psi\rangle\} \\ &= -\langle \varphi, V\psi\rangle + \langle V\varphi, \psi\rangle = 0. \end{aligned}$$

Now $\mathscr{H}_\alpha^1$ is dense in $\mathscr{H}^1$, so this equation is true for all $\varphi \in \mathscr{H}^1$, and in particular for $\varphi \in \mathscr{D}(H_2)$, so $\psi \in \mathscr{D}(H_2)$ and $H_2\psi = \lambda\psi$.

Finally, we must prove (4.6). Having proved equivalence of our two conditions, we may assume $(H_2 - \lambda)\psi = 0$, which implies that $\psi = (\lambda - i)\, R_2(i)\psi$, and we may also assume that (4.7) holds. Then we have

$$\begin{aligned} |\,\psi\,| &\leqslant C' \,|\, R_1(i)\, VR_2(i)(\lambda - i)\psi\,|_{2\alpha} \\ &= C' \,|\, \lambda - i \,|\, |\, R_1(i)\, VR_2(i)\psi\,|_{2\alpha} \\ &\leqslant C' \sqrt{b^2+1}\, |\, R_1(i)\, VR_2(i)\psi\,|_{2\alpha}\,. \end{aligned}$$

Now we must show that $\rho^{2\alpha}R_1(i)\, VR_2(i)$ is compact. This is true because $VR_2(i)$ is bounded from $\mathscr{H}$ to $\mathscr{H}^{-1}_{2\alpha+\epsilon}$ and $R_1(i)$ is compact from $\mathscr{H}^{-1}_{2\alpha+\epsilon}$ to $\mathscr{H}_{2\alpha}$ (proved in Section 5, Proposition 5.2). ∎

5. Appendix

Here we prove some facts used above. They will probably come as no surprise to anyone. Proposition 5.1 is the proof of (3.11); Proposition 5.2 gives conditions for boundedness or compactness of certain operators which were invoked at several points in the proof of Theorem 4.

Proposition 5.1. *If $\psi \in \mathscr{H}_{-\alpha}$, $\alpha > 1/2$, $\delta > \eta > 0$*

$$|\,\psi\,|_{-\delta}^2 \leqslant C(\sup_{R>1} R^{-2\eta}\,|\,\rho_R^{-\alpha}\psi\,|^2 + |\,\psi\,|_{-\alpha}^2).$$

Proof. Suppose the expression on the right is finite. Then

$$\int_{|x|\geqslant 1} (1 + (R/r)^{-2})^{-\alpha}\,|\,\psi(x)|^2\, dx \leqslant C'R^{2\eta}.$$

Therefore if $\delta > \eta$

$$\begin{aligned} C' &\geqslant \int_1^\infty R^{-1-2\delta} \left\{ \int_{|x|\geqslant 1} (1 + (R/r)^{-2})^{-\alpha}\,|\,\psi(x)|^2\, dx \right\} dR \\ &= \int_{|x|\geqslant 1} \left\{ \int_{1/r}^\infty (rR)^{-1-2\delta}(1 + R^{-2})^{-\alpha}\, r\, dR \right\} |\,\psi(x)|^2\, dx \\ &\geqslant \int_1^\infty R^{-1-2\delta}(1 + R^{-2})^{-\alpha}\, dR \int_{|x|\geqslant 1} r^{-2\delta}\,|\,\psi(x)|^2\, dx \\ &= C'' \int_{|x|\geqslant 1} r^{-2\delta}\,|\,\psi(x)|^2\, dx. \quad ∎ \end{aligned}$$

Proposition 5.2. *Let* $H = H_0 + V$ *acting in* $\mathscr{L}^2(\mathbf{R}^n) = \mathscr{H}$, *where* V *is a bounded function in* $\mathscr{L}^p(\mathbf{R}^n)$ *for some* $p < \infty$. *For* z *outside the spectrum of* H, $(H - z)^{-1}$ *is compact as an operator from* $\mathscr{H}_{-\alpha}$ *to* $\mathscr{H}^1_{-\beta}$ *and from* $\mathscr{H}^{-1}_{\beta}$ *to* $\mathscr{H}_{\alpha}$ *if* $\beta > \alpha \geqslant 0$, *and bounded from* $\mathscr{H}^{-1}_{\alpha}$ *to* $\mathscr{H}^1_{\alpha}$ *for all* α.

Proof. Let $\mathscr{F}$ denote the Fourier transform. If $f, g \in \mathscr{L}^p(\mathbf{R}^n)$ for $2 < p < \infty$ then $\| \mathscr{M}_g \mathscr{F}^* \mathscr{M}_f \mathscr{F} \| \leqslant c \| g \|_p \| f \|_p$ by the Hölder and Hausdorff–Young inequalities, and $\mathscr{M}_g \mathscr{F}^* \mathscr{M}_f \mathscr{F}$ is Hilbert–Schmidt if $f, g \in \mathscr{L}^2(\mathbf{R}^n)$. Since any function in $\mathscr{L}^p(\mathbf{R}^n)$ can be approximated in $\mathscr{L}^p$ norm by functions in $\mathscr{L}^p \cap \mathscr{L}^2(\mathbf{R}^n)$ it follows that $\mathscr{M}_g \mathscr{F}^* \mathscr{M}_f \mathscr{F}$ is the limit in operator norm of a sequence of Hilbert–Schmidt operators and is therefore compact. (This argument was suggested by a remark of W. Faris).

It follows that if $g \in \mathscr{L}^p(\mathbf{R}^n)$ for $p > n$, $\mathscr{M}_g(H_0 - z)^{-1} (H_0 + 1)^{1/2}$ has compact closure; in other words $\mathscr{M}_g(H_0 - z)^{-1}$ is compact from $\mathscr{H}^{-1}$ to $\mathscr{H}$.

The same is true for $\mathscr{M}_g(H - z)^{-1}$, since $\mathscr{M}_g(H - z)^{-1} = \mathscr{M}_g(H_0 - z)^{-1} [I - V(H - z)^{-1}]$ and $I - V(H - z)^{-1}$ is bounded from $\mathscr{H}^{-1}$ to $\mathscr{H}^{-1}$.

Now note that if $\psi \in \mathscr{S}(\mathbf{R}^n)$,

$$(H\rho^\alpha - \rho^\alpha H)\psi = -\rho^{\alpha-2}(2\alpha x \cdot \nabla + h)\psi,$$

where h is a bounded function. Suppose $\beta \geqslant \alpha \geqslant 0$ and let $\varphi = (H - z)\psi$, $\psi \in \mathscr{S}(\mathbf{R}^n)$; φ is rapidly decreasing and

$$\begin{aligned} &\rho^{-\beta}(H - z)^{-1} \rho^\alpha \varphi \\ &\quad = \rho^{-\beta}(H - z)^{-1} \rho^\alpha (H - z)\psi \qquad (5.1)\\ &\quad = \rho^{-\beta}(H - z)^{-1}\{(H - z)\rho^\alpha - \rho^{\alpha-2}(2\alpha x \cdot \nabla + h)\}\psi \\ &\quad = \rho^{\alpha-\beta}(H - z)^{-1} \varphi - \rho^{-\beta}(H - z)^{-1} \rho^{\alpha-2}(2\alpha x \cdot \nabla + h)(H - z)^{-1} \varphi. \end{aligned}$$

Suppose $\alpha < 1$. The operator $\rho^{\alpha-\beta}(H - z)^{-1}$ was shown to be compact from $\mathscr{H}^{-1}$ to $\mathscr{H}$ if $\beta > \alpha$ and bounded if $\beta = \alpha$, $\rho^{-\beta}(H - z)^{-1} \rho^{\alpha-1}$ is compact on $\mathscr{H}$, and $[2\alpha\rho^{-1}x \cdot \nabla - \rho^{-1}h](H - z)^{-1}$ is bounded from $\mathscr{H}^{-1}$ to $\mathscr{H}$. Since $\mathscr{S}(\mathbf{R}^n)$ is dense in $\mathscr{H}^1$, $(H - z)\mathscr{S}$ is dense in $\mathscr{H}^{-1}$ and the relation (5.1), proved for $\varphi \in (H - z)\,\mathscr{S}(\mathbf{R}^n)$ implies that the closure of $\rho^{-\beta}(H - z)\,\rho^\alpha$ applied to $\varphi \in \mathscr{H}^{-1}$ is given by (5.1) and therefore this operator is compact from $\mathscr{H}^{-1}$ to $\mathscr{H}$ if $\beta > \alpha$, and bounded if $\beta = \alpha$.

Now suppose that compactness if $\beta > \alpha$, and boundedness if $\beta = \alpha$, has been shown for all $\alpha < k$. The above argument shows that compactness if $\beta > \alpha$ and boundedness if $\beta = \alpha$ follows for $\alpha < k + 1$.

By Theorem 2 this implies that $(H - z)^{-1}\colon \mathscr{H}_{-\alpha}^{-1} \to \mathscr{H}_{-\alpha}^{1}$ is bounded for $\alpha > 0$, and, by taking adjoints, for $\alpha \leqslant 0$.

We also have $(H - z)^{-1}\colon \mathscr{H}_{-\beta} \to \mathscr{H}_{-\alpha}$ compact if $\beta > \alpha \geqslant 0$. Let $\{\varphi_k\}$ be a sequence of vectors in $\mathscr{H}_{-\alpha}$ which converges weakly to zero. By Theorem 2

$$|(H - z)^{-1} \varphi_k|_{-\beta}^{(1)} \leqslant D\,|(H - z)^{-1} \varphi_k|_{-\beta} + |\varphi_k|_{-\beta}^{(-1)}. \tag{5.2}$$

The first term on the right hand side converges to zero since $(H - z)^{-1}\colon \mathscr{H}_{-\alpha} \to \mathscr{H}_{-\beta}$ is compact. We have

$$|\varphi_k|_{-\beta}^{-1} = |(H_0 + 1)^{-1/2} \rho^{-\beta+\alpha} \rho^{-\alpha} \varphi_k|.$$

$(H_0 + 1)^{-1/2} \rho^{-\beta+\alpha}$ is the adjoint of $\rho^{-\beta+\alpha}(H_0 + 1)^{-1/2}$ which is compact since $\rho^{-\beta+\alpha}$ and ρ^{-1} are both in $\mathscr{L}^p(\mathbf{R}^n)$ for $p > n$, $n/(\beta - \alpha)$. Since $\rho^{-\alpha}\varphi_k$ converges to zero the second term in (5.2) converges to zero, so $(H - z)^{-1}\colon \mathscr{H}_{-\alpha} \to \mathscr{H}_{-\beta}^{1}$ is compact if $\beta > \alpha \geqslant 0$. It follows by taking adjoints that for z outside the spectrum of H, $(H - z)^{-1}$ is compact from $\mathscr{H}_{\beta}^{-1}$ to $\mathscr{H}_{\alpha}$. ∎

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