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Cycle decompositions IV: complete directed graphs and fixed length directed cycles

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Abstract

We establish necessary and sufficient conditions for decomposing the complete symmetric digraph of order n into directed cycles of length m, where $2 \leqslant m \leqslant n$.

Keywords: Directed graph; Decomposition; Directed cycle

1. Introduction

Throughout this paper, K_n will denote the complete graph of order n, $K_n - I$ will denote the complete graph of even order n with a 1-factor removed, and K_n^* will denote the complete symmetric digraph of order n, the digraph with all possible arcs.

It is natural to ask when K_n admits a decomposition into cycles of some fixed length. Since the existence of such a decomposition requires the degrees of the vertices to be even, it follows that n must be odd. However, this question can be extended to complete graphs with an even number of vertices by removing a 1-factor. The two necessary conditions are that $3 \leqslant m \leqslant n$ and that m must divide the number

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of edges in either K_n or $K_n - I$. In [1], it is shown that such a decomposition exists when m and n have the same parity and in [7], a decomposition is given when m and n have opposite parity thereby completing the solution for K_n and $K_n - I$.

A natural extension of this question is to ask when K_n^* admits a decomposition into directed cycles of length m. The necessary conditions then become $2 \leqslant m \leqslant n$ and $m \mid n(n-1)$. Bermond and Faber [4] have conjectured that the necessary conditions are sufficient as long as $(n,m) \neq (6,3), (4,4), (6,6)$. Bermond [2] showed that the necessary conditions are sufficient if $m \in \{10, 12, 14\}$. Bermond and Faber [4] further showed that the necessary conditions are sufficient if $m \in \{4, 6, 8, 16\}$ and $(n,m) \neq (4,4), (6,6)$. They further resolved the problem for m even and a divisor of $n-1$. Tillson [9] has shown that K_n^* can be decomposed into directed Hamilton cycles if n is even and $n \neq 4, 6$. Bermond [3] has also completely settled the case when $m = 3$ and has given some results for other odd lengths.

The goal of this paper is to completely settle the directed cycle decomposition problem for K_n^*, that is, to prove the following theorem.

1.1. Theorem. *For positive integers m and n, with $2 \leqslant m \leqslant n$, the digraph K_n^* can be decomposed into directed cycles of length m if and only if m divides the number of arcs in K_n^* and $(n,m) \neq (4,4), (6,3), (6,6)$.*

An immediate consequence of Theorem 1.1 is the following result. The multigraph λK_n has λ edges between every pair of distinct vertices.

1.2. Corollary. *For positive integers m and n with $3 \leqslant m \leqslant n$, the multigraph $2K_n$ can be decomposed into cycles of length m if and only if m divides the number of edges in $2K_n$.*

Proof. Let m and n be positive integers with $3 \leqslant m \leqslant n$. Clearly, if $2K_n$ decomposes into cycles of length m, then $m \mid n(n-1)$. On the other hand, suppose $m \mid n(n-1)$. If, $(n,m) \neq (4,4), (6,3), (6,6)$, then K_n^* can be decomposed into directed cycles of length m by Theorem 1.1. Ignoring direction provides a decomposition of $2K_n$ into cycles of length m. For the remaining three cases, let $V(2K_n) = \{v_1, v_2, \ldots, v_n\}$. If $(n,m) = (4,4)$, then $\{(v_1, v_2, v_3, v_4), (v_1, v_2, v_4, v_3), (v_1, v_4, v_2, v_3)\}$ is a decomposition of $2K_4$ into 4-cycles. If $(n,m) = (6,6)$, then $\{(v_1, v_2, v_3, v_4, v_5, v_6), (v_1, v_2, v_3, v_5, v_6, v_4), (v_1, v_3, v_5, v_2, v_6, v_4), (v_1, v_3, v_6, v_2, v_4, v_5), (v_1, v_5, v_2, v_4, v_3, v_6)\}$ is a decomposition of $2K_6$ into 6-cycles. Finally, if $(n,m) = (6,3)$, then $\{(v_1, v_2, v_4), (v_2, v_3, v_5), (v_3, v_4, v_6), (v_4, v_5, v_1), (v_5, v_6, v_2), (v_6, v_1, v_3), (v_1, v_2, v_3), (v_3, v_4, v_5), (v_5, v_6, v_1), (v_2, v_4, v_6)\}$ is a decomposition of $2K_6$ into 3-cycles. □

The following lemma, sometimes called the "doubling" lemma, provides a useful tool for directed cycle decompositions.

1.3. Lemma. *If K_n can be decomposed into cycles of length m, then K_n^* can be decomposed into directed cycles of length m.*

Proof. For each m-cycle C in a decomposition of K_n, obtain two directed cycles of length m by giving C the two possible orientations in each direction. The resulting collection of directed cycles of length m clearly forms a decomposition of K_n^*. □

The proof of Theorem 1.1 for m and n both odd is an immediate corollary of [1]. To see this, note that if m and n are both odd with m dividing $n(n-1)$, the number of arcs in K_n^*, then clearly m divides $n(n-1)/2$, the number of edges in K_n. Hence, we apply the doubling lemma to a decomposition of K_n into cycles of length m to obtain a decomposition of K_n^* into directed cycles of length m.

Thus, we complete the proof of Theorem 1.1 in the rest of this paper by showing the necessary conditions are sufficient when n and m have opposite parity and $(n,m) \neq (6,3)$, or n and m are both even with $(n,m) \neq (4,4), (6,6)$.

2. Definitions and preliminaries

Let us begin this section with a few definitions.

2.1. Definition. The directed cycle on m vertices is denoted by $\overrightarrow{C_m}$.

2.2. Definition. For vertices x and y in a digraph D, we will use the notation xy to denote the arc from x to y.

2.3. Definition. For a graph G, the notation G^* denotes the digraph obtained from G by replacing each edge xy of G with arcs xy and yx.

2.4. Definition. For a graph G, we write $G = H_1 \oplus H_2$ if G is the edge-disjoint union of the subgraphs H_1 and H_2. If $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, where $H_1 \cong H_2 \cong \cdots \cong H_k \cong H$, then the graph G can be *decomposed* into subgraphs isomorphic to H and we say that G is *H-decomposable*. We shall also write $H \mid G$.

Similarly, a digraph D can be decomposed into copies of a digraph H if the arc set of D can be partitioned into sets, each inducing a copy of H. We will also say that D is H-decomposable and write $H \mid D$.

We shall use Cayley graphs and circulant graphs for several proofs. Accordingly, we define them now.

2.5. Definition. Let S be a subset of a finite group Γ satisfying

1. $1 \notin S$ where 1 denotes the identity of Γ, and
2. $S = S^{-1}$, that is, $s \in S$ implies that $s^{-1} \in S$.

A subset S satisfying the above conditions is called a *Cayley subset*. The *Cayley graph* $X(\Gamma; S)$ is defined to be the graph whose vertices are the elements of Γ and

there is an edge between vertices g and h if and only if $h = gs$ for some $s \in S$. We call S the *connection set* and say that $X(\Gamma; S)$ is a *Cayley graph on the group* Γ.

2.6. Definition. A Cayley graph $X(\Gamma; S)$ is called a *circulant graph* when Γ is a cyclic group. For a cyclic group Γ of order n, we will write $X(n; S)$ for $X(\Gamma; S)$.

For a circulant graph $X(n; S)$ whose underlying group is Z_n, we have S is a subset of $\{1, 2, \ldots, n-1\}$ satisfying $s \in S$ if and only if $n - s \in S$. We will often write $-s$ for $n - s$ when n is understood. Denoting the vertices of $X(n; S)$ by $u_0, u_1, \ldots, u_{n-1}$, then there is an edge between u_i and u_j if and only if $j - i \in S$.

The *circulant digraph* is defined similarly except that the connection set S need not be a Cayley subset.

2.7. Definition. Let n be a positive integer and $S \subseteq \{1, 2, \ldots, n-1\}$. The *circulant digraph* $\overrightarrow{X}(n; S)$ with *connection set* S is the digraph whose vertices are $u_0, u_1, \ldots, u_{n-1}$ with an arc from u_i to u_j if and only if $j - i \in S$. Again, we will often write $-s$ for $n - s$ when n is understood.

Many of our decompositions arise from the action of a permutation on a fixed subdigraph. The next definition makes this precise.

2.8. Definition. Let ρ be a permutation of the vertex set V of a digraph D. For any subset U of V, ρ acts as a function from U to V by considering the restriction of ρ to U. If H is a subdigraph of D with vertex set U and if for each $xy \in E(H)$, we have $\rho(x)\rho(y) \in E(D)$, then $\rho(H)$ is the subdigraph of D with vertex set $\rho(U)$ and arc set $\{\rho(x)\rho(y) : xy \in E(H)\}$.

Note that $\rho(H)$ may not be defined for all subdigraphs H of D since ρ is not necessarily an automorphism.

2.9. Definition. If G_1 and G_2 are vertex-disjoint graphs, then the *join* of G_1 and G_2, denoted $G_1 \bowtie G_2$, is the graph obtained by taking the union of G_1 and G_2 together with all possible edges between G_1 and G_2. If D_1 and D_2 are vertex-disjoint digraphs, then $D_1 \bowtie D_2$ is the digraph obtained by taking the union of D_1 and D_2 together with all possible arcs from D_1 to D_2 and from D_2 to D_1.

2.10. Definition. For a subset A of $\mathbb{Z}_n$, the notation $\pm A$ will denote the set $\{\pm a \in \mathbb{Z}_n \mid a \in A\}$, and for an integer x, the notation $A + x$ will denote the set $\{a + x \in \mathbb{Z}_n \mid a \in A\}$.

3. The case when m is even and n is odd

In this section, we prove the following theorem.

3.1. Theorem. *For positive integers m and n with m even, n odd, and $2 \leqslant m \leqslant n$, the digraph K_n^* can be decomposed into directed cycles of length m if and only if $m \mid n(n-1)$.*

Since every complete directed graph of order at least 2 may be decomposed trivially into directed cycles of length 2, we may assume $m \geqslant 4$ for the remainder of this section. Our first goal towards proving Theorem 3.1 is to determine bounds on the value of n in terms of m.

3.2. Lemma. *Let $m \geqslant 4$ be an even positive integer. If K_n^* is $\overrightarrow{C_m}$-decomposable for all odd n satisfying $m < n < 2m$ with $m \mid n(n-1)$, then K_n^* is $\overrightarrow{C_m}$-decomposable for all odd $n > m$ satisfying $m \mid n(n-1)$.*

Proof. We begin by showing that K_{m+1}^* can be decomposed into directed cycles of length m. Let the vertices of K_{m+1}^* be labelled with $u, u_0, u_1, \ldots, u_{m-1}$ and let $K_{m+1}^* = \overrightarrow{X}(m; S) \bowtie \{u\}$, where $S = \pm\{1, 2, \ldots, (m-2)/2\} \cup \{m/2\}$. Let ρ denote the permutation $(u_0\ u_1 \ldots u_{m-1})(u)$. Observe that if L is any subdigraph of K_{m+1}^*, then $\rho(L)$ is well-defined since $\rho \in \mathrm{Aut}(K_{m+1}^*)$. Let C be the directed m-cycle

$$C : u, u_0, u_{-1}, u_1, u_{-2}, u_2, \ldots, u_{m/2+1}, u_{m/2-1}, u,$$

where all subscripts are taken modulo m. Note that C uses an arc of each length in S except length 1. Thus it follows that $\{C, \rho(C), \rho^2(C), \ldots, \rho^{m-1}(C)\}$ together with the directed cycle $u_0, u_1, \ldots, u_{m-1}, u_0$ is a partition of the arc set of K_{m+1}^* into directed m-cycles.

Suppose that K_n^* can be decomposed into directed m-cycles whenever m is even, n is odd, $m \mid n(n-1)$, and $m < n < 2m$. Let m and n be positive integers with m even and n odd such that $4 \leqslant m < n$ and $m \mid n(n-1)$. Write $n = qm + r + 1$ for integers q and r with $0 \leqslant r < m-1$. Observe that $m \mid n(n-1)$ implies $m \mid r(r+1)$ as well. Label one vertex of K_n^* with x and partition the remaining vertices of K_n^* into $q-1$ sets of m vertices and one set of $m+r$ vertices. Each set of m vertices together with vertex x induces a subdigraph isomorphic to K_{m+1}^*, and the arcs between any two of these sets of m vertices induce a subdigraph isomorphic to $K_{m,m}^*$. The remaining set of $m+r$ vertices together with vertex x induces a subdigraph isomorphic to K_{m+r+1}^*, and the arcs between the set of $m+r$ vertices and any one of the sets of m vertices induce a subdigraph isomorphic to $K_{m,m+r}^*$. By a result of Sotteau [8], we have $\overrightarrow{C_m} \mid K_{m,m}^*$ and $\overrightarrow{C_m} \mid K_{m,m+r}^*$. Since $m \mid (m+r+1)(m+r)$, we have $\overrightarrow{C_m} \mid K_{m+r}^*$ by hypothesis. Above it was shown that $\overrightarrow{C_m} \mid K_{m+1}^*$ and thus $\overrightarrow{C_m} \mid K_n^*$. □

Observe that if m divides $n(n-1)$ evenly, then m divides $n(n-1)/2$. Hence a decomposition of the complete graph K_n into m-cycles exists by [7]. The doubling

lemma then gives a decomposition of K_n^* into directed m-cycles. Therefore, we may assume that $n(n-1)$ is an odd multiple of m. This leads to the following lemma.

3.3. Lemma. *If m, n, and r are positive integers such that $n = m + r + 1$, with m even, n odd, and $n(n-1) \equiv m \pmod{2m}$, then $r \equiv 0 \pmod 4$.*

Proof. First, $n(n-1) \equiv m \pmod{2m}$ implies that $n(n-1) = mk$ for some positive odd integer k. Observe that $n = m + r + 1$, with n odd, m even and $n(n-1)$ an odd multiple of m, implies that r is even and that $r(r+1) = mt$ with t even. Suppose, contrary to the conclusion of the lemma, that $r \equiv 2 \pmod 4$. Then $r = 4\ell + 2$ for some positive integer ℓ. So, $r(r+1) = (4\ell+2)(4\ell+3) = 2(8\ell^2 + 10\ell + 3)$. Therefore, $r(r+1)/2 = mt/2 = 8\ell^2 + 10\ell + 3$, and since t is even, we have a contradiction. Hence, $r \equiv 0 \pmod 4$. □

The next lemma will be very useful in proving Theorem 3.1.

3.4. Lemma. *Let m and n be positive integers satisfying m even, n odd, $4 \leqslant m < n < 2m$, and $n(n-1) \equiv m \pmod{2m}$. If $A = \{a_1, a_2, \ldots, a_{m/2}\}$, where $a_1, a_2, \ldots, a_{m/2}$ are positive integers satisfying $a_1 < a_2 < \cdots < a_{m/2} < n/2$, then $\overrightarrow{C_m} \mid \overrightarrow{X}(n; \pm A)$.*

Proof. Label the vertices of the circulant digraph $\overrightarrow{X}(n, \pm A)$ with $u_0, u_1, \ldots, u_{n-1}$. We have $u_i u_j \in E(\overrightarrow{X}(n; \pm A))$ if and only if $j - i \in \pm A$. Let ρ denote the permutation $(u_0\ u_1\ \cdots\ u_{n-1})$. If L is any subdigraph of $\overrightarrow{X}(n; \pm A)$, then $\rho(L)$ is well-defined since $\rho \in \mathrm{Aut}(\overrightarrow{X}(n; \pm A))$.

Suppose first that $m = 2k$ with k odd. To describe a directed walk in $\overrightarrow{X}(n; \pm A)$, we will specify the starting vertex and the lengths of the arcs and the order in which they are encountered. Let P be the directed trail of length $m-1$ starting at u_0, where the lengths of the $m-1$ arcs of P are $a_1, -a_2, a_3, \ldots, -a_{k-1}, -a_k, a_{k-1}, -a_{k-2}, \ldots, -a_1$ and these arcs are encountered in this order. Note that P uses precisely one arc of each length in $\pm A$, except for an arc of length a_k. Also note that alternate vertices starting with u_{a_1} on P will have strictly increasing subscripts, while alternating vertices starting with u_0 on P will have strictly decreasing subscripts. Thus, the vertices of P are distinct so that P is a path. Also, we have that

$$a_1 - a_2 + a_3 - \cdots - a_{k-1} - a_k + a_{k-1} - a_{k-2} + \cdots - a_1 \equiv -a_k \pmod n$$

and thus P followed by the arc of length a_k gives a directed m-cycle C. Hence, it follows that $\{C, \rho(C), \rho^2(C), \ldots, \rho^{(n-1)}(C)\}$ is a decomposition of $\overrightarrow{X}(n; \pm A)$ into directed m-cycles.

When $m = 2k$ with k even, we let P be the directed trail of length $m-1$ starting at u_0 where the lengths of the arcs of P are

$$a_1, -a_2, a_3, \ldots, a_{k-1}, a_k, -a_{k-1}, a_{k-2}, \ldots, -a_1$$

and these arcs are encountered in this order. As before, clearly P is a directed path and

$$a_1 - a_2 + a_3 - \cdots + a_{k-1} + a_k - a_{k-1} + a_{k-2} + \cdots - a_1 \equiv a_k \pmod{n}.$$

Thus P followed by the arc of length $-a_k$ gives a directed m-cycle C. Then

$$\{C, \rho(C), \rho^2(C), \ldots, \rho^{(n-1)}(C)\}$$

is a decomposition of $\overrightarrow{X}(n; \pm A)$ into directed m-cycles. □

We now present the proof of Theorem 3.1.

Proof of Theorem 3.1. Let m and n be integers with n odd, m even, $3 < m < n$, and $n(n-1) \equiv 0 \pmod{m}$. As mentioned earlier, we may assume that $n(n-1)$ is an odd multiple of m. In addition, by Lemma 3.2, we also may assume that $n < 2m$, say $n = m + r + 1$ for some even integer r with $0 \leqslant r < m - 1$. Now if $r = 0$, we are done by a construction given in the proof of Lemma 3.2. Thus, we assume that $0 < r < m - 1$. We have seen that when $n(n-1)$ is an odd multiple of m, we have that $r(r+1)$ is an even multiple of m.

Let $A = \{1, 2, \ldots, (n-1)/2\}$. We think of the digraph K_n^* as the circulant $\overrightarrow{X}(n; \pm A)$. Label the vertices of the circulant digraph $\overrightarrow{X}(n; \pm A)$ with $u_0, u_1, \ldots, u_{n-1}$. The length of the arc from u_i to u_j is $j - i$. The proof of Theorem 3.1 proceeds as follows. Suppose that B is a subset of A such that $|B| = r/2$, and that we can decompose the circulant digraph $\overrightarrow{X}(n; \pm B)$ into directed m-cycles. Then since $\overrightarrow{X}(n; \pm A) = \overrightarrow{X}(n; \pm(A \backslash B)) \oplus \overrightarrow{X}(n; \pm B)$ and the circulant digraph $\overrightarrow{X}(n; \pm(A \backslash B))$ can be decomposed into directed m-cycles by Lemma 3.4, it follows that we have a decomposition of $\overrightarrow{X}(n; \pm A)$ into directed m-cycles. Thus, the rest of the proof consists of determining a convenient set B of $r/2$ lengths such that the circulant digraph $H = \overrightarrow{X}(n; \pm B)$ can be decomposed into directed m-cycles.

By Lemma 3.3, we know that $r \equiv 0 \pmod{4}$. Let $r = 2^e a$ where a is odd and $e \geqslant 2$. Thus $r(r+1) = 2^e a(2^e a + 1)$. Since $r(r+1)$ is an even multiple of m, we have that $m = 2^d a' b'$, where $a' \mid a$, $b' \mid (2^e a + 1)$, and $1 \leqslant d < e$. Then

$$n = m + r + 1 = 2^d a' b' + 2^e a + 1 = b'\left(2^d a' + \frac{2^e a + 1}{b'}\right).$$

Partition the vertices of the circulant H into b' segments, each with $\ell = 2^d a' + (2^e a + 1)/b'$ vertices. Each segment will contribute $2^d a'$ arcs to a directed m-cycle. We proceed by cases, depending on the value of d.

Case 1. Suppose first that $d \geqslant 2$: Define the directed path $P_{0,0}$ by

$$P_{0,0} : u_0, u_2, u_{-1}, u_3, u_{-2}, \ldots, u_{2^{d-2}a'}, u_{-2^{d-2}a'+1}, u_{2^{d-2}a'+2}, u_{-2^{d-2}a'},$$

$$u_{2^{d-2}a'+3}, u_{-2^{d-2}a'-1}, \ldots, u_{2^{d-1}a'+1}, u_{-2^{d-1}a'+1}, u_{\ell+1}, u_\ell.$$

The definition of $P_{0,0}$ does not make sense when $2^{d-2}a' = 1$, that is, $d = 2$ and $a' = 1$. In this case let

$$P_{0,0} : u_0, u_3, u_{-1}, u_{\ell+1}, u_\ell.$$

Let $P_{0,1} = \rho^\ell(P_{0,0})$, where ρ is the permutation from the proof of Lemma 3.4. Since $\ell > 2^d a'$ implies that $2^{d-1}a' + 1 < \ell - 2^{d-1}a' + 1$, the vertices of $P_{0,1}$ are distinct from the vertices of $P_{0,0}$ except for u_ℓ, which is the last vertex of $P_{0,0}$ and the first vertex of $P_{0,1}$. Similarly, the paths

$$P_{0,0}, P_{0,1} = \rho^\ell(P_{0,0}), P_{0,2} = \rho^{2\ell}(P_{0,0}), \ldots, P_{0,b'-1} = \rho^{(b'-1)\ell}(P_{0,0})$$

are vertex-disjoint except that the path $P_{0,i}$ begins at the last vertex of $P_{0,i-1}$ for $1 \leqslant i \leqslant b' - 1$ and $P_{0,b'-1}$ ends at u_0. Thus $C_0 = P_{0,0} \cup P_{0,1} \cup \cdots \cup P_{0,b'-1}$ is a directed m-cycle and the lengths of the arcs of C_0 are $-1, 2, -3, \ldots, -(2^{d-1}a' - 1), 2^{d-1}a' + \ell, 2^{d-1}a' + 1, -(2^{d-1}a' + 2) \ldots, -2^d a'$. Let C_0' denote the directed m-cycle obtained by reversing the orientation of all the arcs of C_0. The family $\mathscr{C}_0$ of directed m-cycles defined by

$$\mathscr{C}_0 = \{\rho^i(C_0), \rho^i(C_0') \mid 0 \leqslant i \leqslant \ell - 1\}$$

is a decomposition of $\overrightarrow{X}(n; B_0)$ into directed m-cycles where

$$B_0 = \pm\{1, 2, 3, \ldots, 2^{d-1}a' - 1, 2^{d-1}a' + \ell, 2^{d-1}a' + 1, 2^{d-1}a' + 2, \ldots, 2^d a'\}.$$

Let $b = r/(2^{d+1}a') = 2^{e-d-1}a/a'$. If $b > 1$, then obtain the path $P_{i,0}$ by adding $i\ell$ to the subscripts of every other vertex of $P_{0,0}$ starting with u_2, that is,

$$\begin{aligned} P_{i,0} : {} & u_0, u_{2+i\ell}, u_{-1}, u_{3+i\ell}, u_{-2}, \ldots, u_{2^{d-2}a'+i\ell}, u_{-2^{d-2}a'+1}, \\ & u_{2^{d-2}a'+2+i\ell}, u_{-2^{d-2}a'}, u_{2^{d-2}a'+3+i\ell}, u_{-2^{d-2}a'-1}, \ldots, u_{2^{d-1}a'+1+i\ell}, \\ & u_{-2^{d-1}a'+1}, u_{(i+1)\ell+1}, u_\ell, \end{aligned}$$

for $1 \leqslant i \leqslant b - 1$. Next, obtain the paths $P_{i,1}, P_{i,2}, \ldots, P_{i,b'-1}$ by letting powers of ρ^ℓ act on $P_{i,0}$ in the same way they acted on $P_{0,0}$. Furthermore, the path $P_{i,j}$ begins at the last vertex of $P_{i,j-1}$ for $1 \leqslant j \leqslant b' - 1$ and the last vertex of $P_{i,b'-1}$ is u_0. Thus, for each i with $1 \leqslant i \leqslant b - 1$, we have that $C_i = P_{i,0} \cup P_{i,1} \cup P_{i,2} \cup \cdots \cup P_{i,b'-1}$ is an m-cycle, where the lengths of the arcs of C_i are $-(1 + i\ell), 2 + i\ell, -(3 + i\ell), \ldots, -(2^{d-1}a' - 1 + i\ell), 2^{d-1}a' + (i + 1)\ell, 2^{d-1}a' + 1 + i\ell, -(2^{d-1}a' + 2 + i\ell), \ldots, -(2^d a' + i\ell)$. As before, let C_i' denote the directed m-cycle obtained by reversing the orientation of all the arcs of C_i.

Let $B = \pm\{1 + i\ell, 2 + i\ell, 3 + i\ell, \ldots, 2^{d-1}a' - 1 + i\ell, 2^{d-1}a' + (i + 1)\ell, 2^{d-1}a' + 1 + i\ell, \ldots, 2^d a' + i\ell \mid 0 \leqslant i \leqslant b - 1\}$. Now the longest arc in B has length $b\ell + 2^{d-1}a'$ in absolute value. Since $2^e a < 2^d a' b'$ implies that $2^{e-d}a/a' + 1 \leqslant b'$, it follows that

$$\begin{aligned} b\ell + 2^{d-1}a' &= \frac{2^{e-d-1}a\ell}{a'} + 2^{d-1}a' = \frac{2^{e-d-1}a\ell}{a'} + \frac{\ell}{2} - \frac{2^e a + 1}{2b'} \\ &\leqslant \frac{b'\ell}{2} - \frac{2^e a + 1}{2b'} < \frac{b'\ell}{2} = \frac{n}{2}. \end{aligned}$$

Therefore, the lengths of B are distinct and hence $|B| = r/2$. Let $\mathscr{C}_j = \{\rho^i(C_j), \rho^i(C'_j) \mid 0 \leqslant i \leqslant \ell - 1\}$ for $j = 0, 1, \ldots, b-1$. Thus, the collection

$$\{\mathscr{C}_0, \mathscr{C}_1, \ldots, \mathscr{C}_{b-1}\}$$

is a partition of the arc set of the circulant digraph $H = \overrightarrow{X}(n; B)$ into directed m-cycles.

Case 2. Suppose that $d = 1$: In this case, note that $\ell = 2a' + (2^e a + 1)/b'$ and each segment will contribute $2a'$ arcs toward a directed m-cycle. Define the directed path $P_{0,0}$ by

$$P_{0,0} : u_0, u_1, u_{-1}, u_2, u_{-2}, \ldots, u_{-(a'-1)/2}, u_{(a'+1)/2}, u_{-(a'+3)/2}, u_{(a'+3)/2}, \ldots,$$
$$u_{a'-1}, u_{-a'}, u_{a'}, u_{-\ell}.$$

Let $P_{0,1} = \rho^\ell(P_{0,0})$, $P_{0,2} = \rho^{2\ell}(P_{0,0}), \ldots$, $P_{0,b'-1} = \rho^{(b'-1)\ell}(P_{0,0})$, where ρ is the permutation from the proof of Lemma 3.4. Since $\ell > 2a'$, the internal vertices of these directed paths are distinct. Therefore, $C_0 = P_{0,0} \cup P_{0,1} \cup \cdots \cup P_{0,b'-1}$ is a directed cycle of length m. Let C'_0 denote the directed m-cycle obtained by reversing the orientation of all the arcs of C_0. The family $\mathscr{C}_0$ of directed m-cycles defined by

$$\mathscr{C}_0 = \{\rho^i(C_0), \rho^i(C'_0) \mid 0 \leqslant i \leqslant \ell - 1\}$$

is a decomposition of the circulant $\overrightarrow{X}(n; B_0)$ into directed m-cycles, where

$$B_0 = \pm\{1, 2, \ldots, a', a' + 2, a' + 3, \ldots, 2a', \ell + a'\}.$$

The family $\mathscr{C}_0$ uses $4a'$ lengths. If $r > 4a'$, then we must obtain another family of directed m-cycles as in Case 1. However, we cannot add ℓ to the subscripts of every other vertex of $P_{0,0}$ starting with u_1 as was done in Case 1 since both lengths a' and $\ell + a'$ are used in $P_{0,0}$. However, we can add 2ℓ to the subscripts of every other vertex of $P_{0,0}$ without duplicating lengths. We can then add 4ℓ, 6ℓ and so on as long as we do not exceed $n/2$. Unfortunately, this may not use r lengths. This suggests trying to find a second family of directed cycles arising from another initial path and then modifying them by adding even multiples of ℓ to the subscripts of every other vertex. This is precisely what we do.

Define the directed path $Q_{0,0}$ by

$$Q_{0,0} : u_0, u_{a'+1}, u_{a'-\ell}, u_{a'+2}, u_{a'-1-\ell}, u_{a'+3}, \ldots, u_{(a'+3)/2-\ell}, u_{2a'-(a'-1)/2},$$
$$u_{(a'-1)/2-\ell}, u_{2a'-(a'-3)/2}, \ldots, u_{2a'-1}, u_{1-\ell}, u_{2a'}, u_{-\ell}.$$

Let $Q_{0,1} = \rho^\ell(Q_{0,0})$, $Q_{0,2} = \rho^{2\ell}(Q_{0,0}), \ldots$, $Q_{0,b'-1} = \rho^{(b'-1)\ell}(Q_{0,0})$. The internal vertices of these directed paths are distinct because $\ell > 2a'$. Therefore, $C_1 = Q_{0,0} \cup Q_{0,1} \cup \cdots \cup Q_{0,b'-1}$ is a directed cycle of length m. Let C'_1 denote the directed m-cycle obtained by reversing the orientation of all the arcs of C_1. The family $\mathscr{C}_1$ of directed m-cycles defined by

$$\mathscr{C}_1 = \{\rho^i(C_1), \rho^i(C'_1) \mid 0 \leqslant i \leqslant \ell - 1\}$$

is a decomposition of $\overrightarrow{X}(n; B_1)$ into directed m-cycles where

$$B_1 = \pm\{a'+1, \ell+1, \ell+2, \ldots, \ell+a'-1, \ell+a'+1, \\ \ell+a'+2, \ldots, \ell+2a'\}.$$

Note that length $\pm(\ell+a')$ appears in $\mathscr{C}_0$ but not in $\mathscr{C}_1$. Similarly, length $\pm(a'+1)$ appears in $\mathscr{C}_1$ but not in $\mathscr{C}_0$.

Now we must show that we can modify the preceding two families of directed m-cycles often enough to use r arc lengths. Recall that $r \equiv 0 \pmod 4$, that is, $e \geqslant 2$.

As previously noted, the family $\mathscr{C}_0$ uses $4a'$ lengths. If $r = 4a'$, then we may stop. If $r > 4a'$, then the families $\mathscr{C}_0$ and $\mathscr{C}_1$ use a total of $8a'$ lengths. If $r = 8a'$, then we may stop. If $r > 8a'$, then let $P_{2,0}$ be the directed path obtained from $P_{0,0}$ by adding 2ℓ to the subscripts of every other vertex of $P_{0,0}$ starting with u_1. We obtain a family of directed m-cycles, denoted $\mathscr{C}_2$, from $P_{2,0}$ in the same way $\mathscr{C}_0$ is obtained from $P_{0,0}$. If necessary, we obtain a family $\mathscr{C}_4$ from $P_{0,0}$ by adding 4ℓ and so on. We call these $\mathscr{C}_0$-based families. We can perform similar operations on $P_{1,0}$ to get $\mathscr{C}_1$-based families.

To show that r arc lengths can be used, we must calculate the maximum number of times the preceding extensions can be carried out. Since $a' < \ell/2$ and the longest arc length used in $\mathscr{C}_0$ is $\ell + a'$ (in absolute value), if $b' \equiv 1 \pmod 4$, we can use $\mathscr{C}_0$- based families $(b'-1)/4$ times. If $b' \equiv 3 \pmod 4$, we can use $\mathscr{C}_0$-based families $(b'+1)/4$ times. The longest arc length used in $\mathscr{C}_1$ is $\ell + 2a'$ so that if $b' \equiv 1 \pmod 4$, we can use $\mathscr{C}_1$-based families $(b'-1)/4$ times, and if $b' \equiv 3 \pmod 4$, we can use $\mathscr{C}_1$-based families $(b'-3)/4$ times.

Thus, if $b' \equiv 1 \pmod 4$, we have $(b'-1)/2$ families of directed cycles. Similarly, if $b' \equiv 3 \pmod 4$, we also have $(b'-1)/2$ families of directed cycles. Therefore, we always can use as many as $4a'(b'-1)/2 = 2a'b' - 2a' = m - 2a'$ arc lengths. Since both m and r even multiples of a', and since $m > r$, it follows that $m - 2a' \geqslant r$.

The preceding argument means we can find a set of arc lengths of the form $\pm A$, where $|\pm A| = r$, such that $\overrightarrow{X}(n, \pm A)$ decomposes into directed m-cycles. We use Lemma 3.4 to complete the proof.

When $a' = 1$, the directed paths $P_{0,0}$ and $Q_{0,0}$ collapse to $P_{0,0} = u_0, u_1, u_{-\ell}$ and $Q_{0,0} = u_0, u_2, u_{-\ell}$. Everything else is done the same. □

4. The case when m is even and n is even

When the order of the complete digraph K_n^* is even, we no longer are able to use corresponding results from the undirected case because there the graph involved is $K_n - I$ instead of K_n. Nevertheless, the methods are similar. In this section, we will prove the following.

4.1. Theorem. *For positive even integers m and n, with $2 \leqslant m \leqslant n$, the digraph K_n^* can be decomposed into directed cycles of length m if and only if $m \mid n(n-1)$ and $(n,m) \neq (4,4)$ or $(6,6)$.*

The result is trivially true for directed cycles of length 2. The result for $m = 4$ and $m = 6$ is proved in [4]. Hence, for the rest of the proof, we assume $m \geqslant 8$. Our first goal towards proving Theorem 4.1 is to determine bounds on the value of n in terms of m.

4.2. Lemma. *Let $m \geqslant 8$ be an even integer. If K_n^* is $\overrightarrow{C_m}$-decomposable for all even n satisfying $m \leqslant n < 2m$ with $m \mid n(n-1)$, then K_n^* is $\overrightarrow{C_m}$-decomposable for all even $n \geqslant m$ satisfying $m \mid n(n-1)$.*

Proof. Suppose that K_n^* can be decomposed into directed m-cycles whenever m and n are even, $8 \leqslant m \leqslant n < 2m$, and $m \mid n(n-1)$. Let m and n be even positive integers such that $8 \leqslant m \leqslant n$ and $m \mid n(n-1)$. Recall that $\overrightarrow{C_m} \mid K_m^*$ for $m \geqslant 8$ from [9]. Write $n = qm + r$, where $0 \leqslant r < m$. Partition the vertex set of K_n^* into $q-1$ sets of m vertices and one set of $m + r$ vertices. Each subdigraph induced by a set of m vertices is isomorphic to K_m^* and can be decomposed into directed m-cycles by Tillson [9]. The subdigraph induced by the set of $m + r$ vertices can be partitioned into directed m-cycles by assumption since $m \mid n(n-1)$ implies $m \mid (m+r)(m+r-1)$. The subdigraph induced by the arcs between any two of the parts is isomorphic to $K_{m,m}^*$ or $K_{m+r,m}^*$, both of which are decomposable into directed m-cycles by Sotteau [8]. This completes the proof. □

In proving Theorem 4.1, we will always use a central vertex, that is, we will think of K_n^* as the digraph $X(n-1;S) \bowtie \{u\}$ where $S = \pm\{1, 2, \ldots, (n-2)/2\}$. Throughout the rest of this section, the vertices of K_n^* will be labelled with $u_0, u_1, \ldots, u_{n-2}, u$ where the vertices of the subcirculant are $u_0, u_1, \ldots, u_{n-2}$ and u is the central vertex.

The following lemma is used extensively throughout this section but before stating and proving it, we would like to introduce some useful notation.

4.3. Definition. Let $u_0, u_1, \ldots, u_{n-1}$ be the vertices of a circulant digraph $\overrightarrow{X}(n;A)$ and picture them cyclically ordered according to increasing subscript. We then use $[u_i, u_j]$ to denote the *closed interval* of vertices $\{u_i, u_{i+1}, \ldots, u_j\}$. Similarly, $[u_i, u_j)$ and $(u_i, u_j]$ denote intervals containing only one of the endpoints, while (u_i, u_j) denotes the *open interval*. For example, $(u_{n-2}, u_2]$ contains the vertices u_{n-1}, u_0, u_1, u_2.

4.4. Lemma. *Let n be a positive even integer and let $a_1, a_2, \ldots, a_t$ be positive integers with $1 \leqslant a_1 < a_2 < \cdots < a_t < (n-2)/2$. Let $A = \pm\{a_1, a_2, \ldots, a_t\}$. Then the circulant digraph $\overrightarrow{X}(n-1; A \cup \{n/2\})$ has a directed path of length $2t+1$ from u_0 to $u_{n/2}$ using*

one arc of each of the lengths in $A \cup \{n/2\}$ such that none of the vertices in the intervals (u_0, u_{a_1}) or $(u_{n/2}, u_{n/2+a_1})$ are used.

Proof. Let P be the directed trail of length $2t+1$ starting at u_0 where the lengths of the $2t+1$ arcs of P are $a_1, -a_2, a_3, \ldots, a_t, n/2, -a_t, a_{t-1}, \ldots, a_2, -a_1$ if t is odd, or $a_1, -a_2, a_3, \ldots, -a_t, n/2, a_t, -a_{t-1}, \ldots, a_2, -a_1$ if t is even, and these arcs are encountered in precisely this order. Since alternating vertices on P starting with u_{a_1} will have strictly increasing subscripts while alternating vertices starting on P with u_0 will have strictly decreasing subscripts, it follows that P is a path. Also, since the sum of the arc lengths is $n/2$, it must be that the terminal vertex of P is $u_{n/2}$. Finally, since the first arc of P has length a_1 while the last arc has length $-a_1$, it is clear that none vertices in the intervals $(u_0, u_{a_1}) \cup (u_{n/2}, u_{n/2+a_1})$ are used on P. □

Using the same techniques as in Lemma 4.4 and replacing the length $n/2$ with $(n-2)/2$, another directed path of length $2t+1$ is created from u_0 to $u_{(n-2)/2}$ in $\overrightarrow{X}(n-1; A \cup \{(n-2)/2\})$ such that none of vertices in the intervals $(u_0, u_{a_1}) \cup (u_{(n-2)/2}, u_{(n-2)/2+a_1})$ are used. Similarly, by using the negatives of the arc lengths encountered on the path P in Lemma 4.4 in the same order and starting at u_0, another directed path is created from u_0 to $u_{(n-2)/2}$ in $\overrightarrow{X}(n-1; A \cup \{(n-2)/2\})$ of length $2t+1$ such that none of the vertices in the intervals $(u_{-a_1}, u_0) \cup (u_{(n-2)/2-a_1}, u_{(n-2)/2})$ are used.

We now present the proof of Theorem 4.1.

Proof of Theorem 4.1. Let m and n be positive even integers with $8 \leqslant m \leqslant n < 2m$ and $n(n-1) \equiv 0 \pmod m$. Let $n = 2^e a$, a odd, so that $n-1 = 2^e a - 1$. Let $m = 2^d a' b'$, where $d \leqslant e$, $a' \mid a$, and $b' \mid (2^e a - 1)$. Note that $\gcd(a', b') = 1$ since $\gcd(n, n-1) = 1$, $a' \mid n$, and $b' \mid (n-1)$. Next, if $b' = 1$, then $m \mid n$. Thus either $m = n$ or $m \leqslant n/2$, and our assumption that $n < 2m$ implies $m = n$. The case $m = n$ is handled by Tillson [9], and hence we assume throughout this section that $b' \geqslant 3$.

As was mentioned earlier, we will use a central vertex, that is, let $K_n^* = \overrightarrow{X}(n-1; S) \bowtie \{u\}$ where $S = \{\pm 1, \pm 2, \ldots, \pm(n-2)/2\}$ and let the vertices of K_n^* be denoted by $u_0, u_1, \ldots u_{n-2}, u$. The proof proceeds as follows. We create one directed m-cycle by taking a directed $(m-2)$-path, formed from $m-2$ arcs of distinct length, together with arcs through the central vertex u. This directed m-cycle is rotated through all $n-1$ positions, using the permutation $\rho = (u_0 u_1 \cdots u_{n-2})(u)$, thereby using all arcs incident with u and all arcs of the $m-2$ distinct arc lengths. The number of unused arc lengths is then

$$(n-2) - (m-2) = n - m = 2^d(2^{e-d}a - a'b').$$

Next, we partition the $n-1$ vertices $\{u_0, u_1, \ldots, u_{n-2}\}$ into b' segments, each of which has $\ell = (2^e a - 1)/b'$ vertices. Each segment will contribute $2^d a'$ arcs toward a directed m-cycle. Therefore, the number of distinct families of directed m-cycles we

need to construct is given by

$$F = \frac{n-m}{2^d a'} = \frac{2^d(2^{e-d}a - a'b')}{2^d a'} = \frac{2^{e-d}a - a'b'}{a'},$$

and observe that $F < b'$ since $n - m < m$.

The overall strategy is to choose the latter families first and then show the $m-2$ lengths remaining can be ordered to form a directed path. Using this directed $(m-2)$-path and the central vertex completes the decomposition.

Since $2^d a'b' = m < n-1 = 2^e a - 1$, dividing both sides by b' yields $2^d a' < (2^e a - 1)/b' = \ell$. Thus, the number of arcs a segment must contribute is strictly less than the number of vertices in the segment. *Case* 1. Suppose that $F = 1$: First we want to show that $2^d a' = 2$ is impossible. If $2^d a' = 2$, then $d = 1$ and $a' = 1$. From the assumption $F = 1$, we have

$$1 = \frac{2^{e-d}a - a'b'}{a'} = 2^{e-1}a - b'.$$

Thus, $b' = 2^{e-1}a - 1$. Since $n - 1 = 2^e a - 1$ and $\gcd(2^e a - 1, 2^{e-1}a - 1) = 1$, we have that $\gcd(n-1, b') = 1$. However, since $b' \mid (n-1)$, this implies that $b' = 1$. We have already seen that $b' = 1$ gives $m = n$ and that we are done in this case. Hence, we assume that $2^d a' \geqslant 4$ when $F = 1$.

Define the directed path P_0 as follows:

$$P_0 : u_0, u_1, u_{-1}, u_2, u_{-2}, \ldots, u_{-(2^{d-1}a'-1)}, u_{2^{d-1}a'}, u_{(b'+1)\ell/2}.$$

Then

$$C_0 = P_0 \cup \rho^{\ell}(P_0) \cup \cdots \cup \rho^{(b'-1)\ell}(P_0)$$

is a directed m-cycle since $\gcd(b', (b'+1)/2) = 1$. This is the only directed m-cycle required as $F = 1$.

The family $\{C_0, \rho(C_0), \ldots, \rho^{\ell-1}(C_0)\}$ is a decomposition of the circulant $\overrightarrow{X}(n-1; B)$ into directed m-cycles where

$$B = \left\{1, -2, 3, -4, 5, \ldots, -(2^d a' - 2), 2^d a' - 1, \frac{(b'+1)\ell}{2} - 2^{d-1}a'\right\}.$$

Thus what remains is to find a directed m-cycle decomposition of the digraph $\overrightarrow{X}(n-1; S\backslash B) \bowtie \{u\}$ where

$$S\backslash B = \left\{-1, 2, -3, 4, -5, \ldots, 2^d a' - 2, -(2^d a' - 1), \frac{(b'-1)\ell}{2} + 2^{d-1}a'\right\}$$
$$\cup \pm\left(\left\{2^d a', 2^d a' + 1, \ldots, \frac{n-2}{2}\right\} \backslash \left\{\frac{(b'-1)\ell}{2} + 2^{d-1}a'\right\}\right).$$

Note that $|S\backslash B| = m - 2$ and that the sum of the lengths in $S\backslash B$ is $(b'-1)\ell/2$.

From Lemma 4.4, there is a directed path P from u_0 to $u_{(n-2)/2}$ such that P has exactly one arc of each length

$$\pm 2^d a', \pm(2^d a'+1), \ldots, \pm\left(\frac{(b'-1)\ell}{2}+2^{d-1}a'-1\right),$$
$$\pm\left(\frac{(b'-1)\ell}{2}+2^{d-1}a'+1\right), \pm\left(\frac{(b'-1)\ell}{2}+2^{d-1}a'+2\right), \ldots,$$
$$\pm(n-4)/2, (n-2)/2$$

and uses none of the vertices in $(u_{-2^d a'}, u_0) \cup (u_{(n-2)/2-2^d a'}, u_{(n-2)/2})$. We now wish to augment P so that each of the remaining lengths is used. Consider the trail P' where

$$P' : P, u_{(n-2)/2-2^d a'+1}, u_{(n-4)/2}, u_{-1}, u_{-(2^d a'-2)}, u_{-2}, u_{-(2^d a'-3)},$$
$$u_{-3}, \ldots, u_{-(2^{d-1}a'-1)}, u_{-2^{d-1}a'}, u_{(b'-1)\ell/2},$$

where the last $2^d a'+1$ lengths encountered on P' are

$$-(2^d a'-1), 2^d a'-2, -(n-2)/2, -(2^d a'-3), 2^d a'-4, \ldots,$$
$$-1, 2^{d-1}a' + \frac{(b'-1)\ell}{2}.$$

To show that P' is indeed a path, we need only show that the vertex $u_{(b'-1)\ell/2}$ lies in the interval $(u_{(n-2)/2-2^d a'+1}, u_{(n-4)/2})$. Thus, it suffices to show that

$$\frac{n-2}{2} - 2^d a' + 1 < \frac{(b'-1)\ell}{2} < \frac{n-4}{2}.$$

Since $4 \leqslant 2^d a' < \ell$, we have that $\ell \geqslant 5$. Hence $(n-1)-\ell = b'\ell - \ell = (b'-1)\ell < n-4$ or $(b'-1)\ell/2 < (n-4)/2$.

Next we show that

$$\frac{n-2}{2} - 2^d a' + 1 < \frac{(b'-1)\ell}{2}.$$

Now

$$\frac{n-2}{2} - \frac{(b'-1)\ell}{2} = \frac{\ell-1}{2},$$

and thus the proof of Case 1 is finished provided we show that $(\ell-1)/2 < 2^d a' - 1$. We know that $n-1 < 2m-1$, or $\ell b' < 2^{d+1}a'b' - 1$. Thus $\ell < 2^{d+1}a' - 1/b'$, or $\ell - 1 < 2^{d+1}a' - (b'+1)/b'$. Hence

$$\frac{\ell-1}{2} < 2^d a' - \frac{b'+1}{2b'},$$

and since $(\ell-1)/2$ is an integer, we have

$$\frac{\ell-1}{2} \leqslant 2^d a' - 1.$$

From the last inequality, we need only show $(\ell-1)/2 \neq 2^d a' - 1$. Suppose, to the contrary, that $(\ell-1)/2 = 2^d a' - 1$. Since $F = 1$, we have that $n - m = 2^d a'$. Using the fact that $m = 2^d a' b'$, we obtain $n = 2^d a' + 2^d a' b'$. Since $(\ell-1)/2 = 2^d a' - 1$ and $n = b'\ell + 1$, we have that $n = b'(2^{d+1}a' - 1) + 1$. Thus

$$2^d a' + 2^d a' b' = 2^{d+1} a' b' - b' + 1,$$

or

$$2^d a' b' - b' = 2^d a' - 1,$$

or

$$b'(2^d a' - 1) = 2^d a' - 1.$$

Since $2^d a' \geqslant 4$, it must be that $b' = 1$, producing a contradiction.

Therefore, P' is indeed a path of length $m-2$ and can be completed to a directed m-cycle $C : u, P', u$. Thus $\{C, \rho(C), \rho^2(C), \ldots, \rho^{n-2}(C)\}$ is a decomposition of $\overrightarrow{X}(n-1; S\backslash B) \bowtie \{u\}$ into directed m-cycles.

Case 2. Suppose F is odd and $F \geqslant 3$: First, since F is odd, we have that $n \equiv 0 \pmod 4$ and $e > d$. Suppose first that $d > 1$. Let P_0 be defined as follows:

$$\begin{aligned} P_0 : {} & u_0, u_{(n-2^d a')/2}, u_{-1}, u_{(n-2^d a'+2)/2}, \ldots, u_{-(2^{d-2}a'-1)}, \\ & u_{(n-2^{d-1}a'-2)/2}, u_{-(2^{d-2}a'+1)}, u_{(n-2^{d-1}a')/2}, u_{-(2^{d-2}a'+2)}, \\ & u_{(n-2^{d-1}a'+2)/2}, \ldots, u_{-2^{d-1}a'}, u_{(n-2)/2}, u_{(b'+1)\ell/2}. \end{aligned}$$

In the case that $d = 1$ and $a' > 1$, let

$$\begin{aligned} P_0 : {} & u_0, u_{(n-2a')/2}, u_{-1}, u_{(n-2a'+2)/2}, u_{-2}, \ldots, u_{(n-a'-3)/2}, u_{-(a'-1)/2}, \\ & u_{(n-a'-1)/2}, u_{-(a'+3)/2}, u_{(n-a'+1)/2}, u_{-(a'+5)/2}, \\ & u_{(n-a'+3)/2}, \ldots, u_{-a'}, u_{(n-2)/2}, u_{(b'+1)\ell/2}, \end{aligned}$$

and in the case that $d = 1$ and $a' = 1$, let $P_0 = u_0, u_{(n-2)/2}, u_{(b'+1)\ell/2}$.

The last arc in P_0 has length $(\ell+1)/2$ and the preceding arc has length $-(n - 2^d a')/2$. As long as $(\ell+1)/2 < (n - 2^d a')/2$, the vertices are distinct. To verify this inequality holds, recall that $2^d a' < \ell$. Thus,

$$n - 2^d a' > n - \ell = (b'-1)\ell + 1 > \ell + 1,$$

since $b' \geqslant 3$ and $n = b'\ell + 1$. So P_0 is indeed a directed path.

Since P_0 starts at u_0 and terminates at $u_{(b'+1)\ell/2}$, we have that

$$C_0 = P_0 \cup \rho^{\ell}(P_0) \cup \cdots \cup \rho^{(b'-1)\ell}(P_0)$$

is a directed m-cycle. Note here that the internal vertices of the paths $\rho^i(P_0)$ are pairwise disjoint since $(b'+1)\ell/2 - 2^{d-1}a' > (n-2)/2$ and $(n - 2^d a')/2 >$

$(b'-1)\ell/2$. Thus $\{C_0, \rho(C_0), \ldots, \rho^{\ell-1}(C_0)\}$ is a decomposition of $\overrightarrow{X}(n-1; A)$ into directed m-cycles where

$$A = \{\pm(n-2^d a')/2, \pm(n-2^d a'+2)/2, \ldots, \pm(n-4)/2, (n-2)/2, (\ell+1)/2\}.$$

For $i = 1, 2, \ldots, (F-1)/2$, let P_{2i-1} be the directed path of length $2^d a'$ defined as follows:

$$P_{2i-1} : u_0, u_{i\ell+1}, u_{-1}, u_{i\ell+2}, u_{-2}, \ldots, u_{i\ell+2^{d-1}a'-1},$$

$$u_{-(2^{d-1}a'-1)}, u_{i\ell+2^{d-1}a'}, u_{(b'+1)\ell/2}.$$

Then $C_{2i-1} = P_{2i-1} \cup \rho^{\ell}(P_{2i-1}) \cup \cdots \cup \rho^{(b'-1)\ell}(P_{2i-1})$ is a directed m-cycle. Let τ be the permutation fixing u_0 and mapping u_i to u_{-i} for $1 \leqslant i \leqslant (n-2)/2$. Let $C_{2i} = \tau(C_{2i-1})$ for $i = 1, 2, \ldots, (F-1)/2$, and let B denote the set of arc lengths used in the directed m-cycles $C_1, C_2, \ldots, C_{F-1}$; thus

$$B = \pm\left\{ i\ell+1, i\ell+2, \ldots, i\ell+2^d a'-1, \right.$$
$$\left. \frac{(b'-2i+1)\ell}{2} - 2^{d-1}a' \mid 1 \leqslant i \leqslant (F-1)/2 \right\}.$$

We now show that all of the arc lengths used in the directed m-cycles $C_1, C_2, \ldots, C_{F-1}$ are distinct. It suffices to show that

$$\frac{(b'-2(\frac{F-1}{2})+1)\ell}{2} - 2^{d-1}a' \neq j\ell + \alpha$$

for any j and α with $1 \leqslant j \leqslant (F-1)/2$ and $1 \leqslant \alpha \leqslant 2^d a'-1$. Thus, suppose to the contrary, that

$$\frac{(b'-F)\ell}{2} + \ell - 2^{d-1}a' = j\ell + \alpha$$

for some j $(1 \leqslant j \leqslant (F-1)/2)$ and α $(1 \leqslant \alpha \leqslant 2^d a'-1)$. Then, clearly, $\ell - 2^{d-1}a' \equiv \alpha \pmod{\ell}$, and since $\alpha \leqslant 2^d a' - 1 < \ell$, it follows that $\ell - 2^{d-1}a' = \alpha$. Therefore, $(b'-F)/2 = j$. This implies that $\ell - 2^{d-1}a' \leqslant 2^d a' - 1$ and $(b'-F)/2 \leqslant (F-1)/2$. Now the second inequality implies that $(b'+1)/2 \leqslant F$, and the first gives $\ell \leqslant 3 \cdot 2^{d-1}a' - 1$, or $b'\ell \leqslant 3(2^{d-1}a'b') - b'$, or $n \leqslant 3(2^{d-1}a'b') - b' + 1$, or $2n \leqslant 3 \cdot 2^d a'b' - 4$ since $b' \geqslant 3$. Dividing both sides by $2^d a'$ yields

$$2\left(\frac{2^{e-d}a}{a'}\right) \leqslant 3b' - 1$$

since $2(2^{e-d}a/a')$ is an integer. Thus $2(F+b') \leqslant 3b'-1$ which implies $F \leqslant (b'-1)/2$, contradicting the fact that $(b'+1)/2 \leqslant F$. Hence, all lengths used in $C_1, C_2, \ldots, C_{F-1}$ are distinct.

We still must show that $A \cap B = \emptyset$, that is, we must show that none of the lengths used in C_0 are used in $C_1, C_2, \ldots, C_{F-1}$. The undirected lengths of C_0 start with $(n-2)/2$ and decrease successively by one until reaching $n/2 - 2^{d-1}a'$ and also include the isolated length $(\ell+1)/2$. The longest undirected length in

$C_1, C_2, \ldots, C_{F-1}$ is either $(F-1)\ell/2 + 2^d a' - 1$ or $(b'-1)\ell/2 - 2^{d-1}a'$. The latter quantity certainly is less than $n/2 - 2^{d-1}a'$. The former quantity is less than $n/2 - 2^{d-1}a'$ as well since $2^d a' < \ell$ and $n - m < m = 2^d a' b'$ implies $b' > F$, that is, since F is odd, $F \leqslant b' - 2$. As far as the isolated length $(\ell+1)/2$ is concerned, the shortest length in $C_1, C_2, \ldots, C_{F-1}$ is $\ell + 1$ since $(b' - F + 2)\ell/2 - 2^{d-1}a' \geqslant 3\ell/2$, which certainly is greater than $(\ell+1)/2$.

Then, the collection $\{C_i, \rho(C_i), \ldots, \rho^{\ell-1}(C_i) \mid 0 \leqslant i \leqslant F-1\}$ is a decomposition of $\overrightarrow{X}(n-1; A \cup B)$ into ℓF directed m-cycles. To complete the proof of this case, we must show the remaining $m-2$ arc lengths can be used to form a directed path P' of length $m-2$. Then, letting $C : u, P', u$ gives the decomposition $\{C, \rho(C), \ldots, \rho^{n-2}(C)\}$ of $\overrightarrow{X}(n-1; S\backslash(A \cup B)) \bowtie \{u\}$ into directed m-cycles. There are two subcases to consider.

Case 2.1. Assume $\ell \equiv 3 \pmod 4$: Start a directed path Q_1 as follows:

$$Q_1 : u_{\ell+1}, u_{(\ell+1)/2}, u_{(\ell+3)/2}, u_{(\ell-1)/2}, u_{(\ell+5)/2}, \ldots, u_{(3\ell-1)/4},$$
$$u_{(\ell+5)/4}, u_{(3\ell+3)/4}, u_{(\ell-3)/4}, u_{(3\ell+7)/4}, \ldots, u_1, u_\ell, u_0.$$

Note that Q_1 uses all vertices of the interval $[u_0, u_{\ell+1}]$, except $u_{(\ell+1)/4}$, and uses arcs of lengths

$$1, -2, 3, -4, \ldots, -\frac{\ell-3}{2}, \frac{\ell-1}{2}, -\frac{\ell+1}{2}, -\frac{\ell+3}{2}, \frac{\ell+5}{2}, -\frac{\ell+7}{2}, \ldots, \ell-1, -\ell.$$

In the special case that $\ell = 3$, let $Q_1 : u_4, u_2, u_3, u_0$.

The sum of the lengths not used in $C_0, C_1, \ldots, C_{F-1}$ is $(b'-1)\ell/2$. Let

$$T = \pm\left\{1, 2, \ldots, \frac{\ell-1}{2}, \frac{\ell+3}{2}, \frac{\ell+5}{2}, \ldots, \ell\right\} \cup \left\{-\frac{(\ell+1)}{2}\right\},$$

and let $\mathscr{L} = S\backslash(A \cup B \cup T)$. The sum of the lengths in $\mathscr{L}$ is $-(n-2)/2 = n/2$ and the shortest length in $\mathscr{L}$ is at least $\ell + 2^d a' > \ell + 1$ since $F \geqslant 3$. Using Lemma 4.4, we can find a directed path P from u_0 to $u_{n/2}$ such that P has exactly one arc of each length in $\mathscr{L}$ and uses none of the vertices in the intervals $(u_0, u_{\ell+2^d a'})$ and $(u_{n/2}, u_{n/2+\ell+2^d a'})$.

Now, we must use the small lengths of T not used in Q_1. The completion is the directed path

$$Q_2 : u_{n/2}, u_{n/2+\ell}, u_{n/2+1}, u_{n/2+\ell-1}, \ldots, u_{(2n+\ell-3)/4}, u_{(2n+3\ell+3)/4},$$
$$u_{(2n+\ell+5)/4}, u_{(2n+3\ell-1)/4}, \ldots, u_{(b'+1)\ell/2+2}, u_{(b'+1)\ell/2+1}.$$

Letting $P' : Q_1, P, Q_2$ gives the desired directed path of length $m-2$.

Case 2.2. Assume $\ell \equiv 1 \pmod 4$: Since $\ell = 1$ is impossible, we may assume $\ell \geqslant 5$. In this case we define Q_1 to be the following directed path:

$$Q_1 : u_{\ell+1}, u_{(\ell+1)/2}, u_{(\ell-3)/2}, u_{(\ell+3)/2}, u_{(\ell-5)/2}, \ldots, u_{(3\ell-3)/4},$$
$$u_{(\ell-1)/4}, u_{(3\ell+5)/4}, u_{(\ell-5)/4}, \ldots, u_1, u_\ell, u_0.$$

Note that Q_1 uses all vertices of the interval $[u_0, u_{\ell+1}]$, except $u_{(\ell-1)/2}$ and $u_{(3\ell+1)/4}$, and uses arcs of lengths

$$-2, 3, -4, \ldots, -\frac{\ell-1}{2}, -\frac{\ell+1}{2}, \frac{\ell+3}{2}, -\frac{\ell+5}{2}, \ldots, \ell-1, -\ell.$$

Let $\mathscr{L}$ and T denote the same sets of lengths as in Case 2.1 and again let P be a directed path from u_0 to $u_{n/2}$ such that P has exactly one arc of each length in $\mathscr{L}$ and uses none of the vertices in the intervals $(u_0, u_{\ell+2^d a'})$ and $(u_{n/2}, u_{n/2+\ell+2^d a'})$. To use the remaining lengths in T, consider the directed path

$$Q_2 : u_{n/2}, u_{n/2+\ell}, u_{n/2+1}, u_{n/2+\ell-1}, \ldots, u_{(n/2)+(3\ell+5)/4}, u_{(n/2)+(\ell-1)/4},$$
$$u_{(n/2)+(\ell+3)/4}, u_{(n/2)+(3\ell+1)/4}, \ldots, u_{2+(b'+1)\ell/2}, u_{1+(b'+1)\ell/2}.$$

Letting $P' : Q_1, P, Q_2$ gives a directed path of length $m-2$.

Case 3. Assume F is even: We have to make certain the sum of the arc lengths not used in the construction of the directed paths for the F families of directed m-cycles is not zero modulo $n-1$. It was easy to do for odd F by using a single directed path and taking the rest of the directed paths in pairs which used all the plus–minus lengths in some set of lengths. We cannot do this for even F. Instead, we define two directed m-cycles such that the union of the arc lengths in these two directed cycles does not sum to zero modulo $n-1$. When $F>2$, we obtain the remaining directed cycles in pairs as before.

Another unfortunate aspect of the even F case is the existence of more cases to consider. The first division into two main cases arises from the relationship between ℓ and $3\cdot 2^{d-1}a'-1$.

Case 3.1. Assume $\ell \leqslant 3\cdot 2^{d-1}a'-1$: Earlier we saw that this assumption implies $F \leqslant (b'-1)/2$. Since $F \geqslant 2$, we know $b' \geqslant 5$ and $F \leqslant b'-3$. We now consider several subcases depending on F.

Subcase 3.1.1. Assume that $F=2$: If $d=1$, let P_0 and P_1 be the directed walks of length $2a'$ defined as follows:

$$P_0 : u_0, u_1, u_{-1}, u_2, u_{-2}, \ldots, u_{(\ell-a'-2)/2}, u_{-(\ell-a'-2)/2},$$
$$u_{(\ell-a'+2)/2}, u_{-(\ell-a')/2}, u_{(\ell-a'+4)/2}, \ldots, u_{-(a'-1)}, u_{a'+1}, u_\ell,$$

and

$$P_1 : u_0, u_{-1}, u_1, u_{-2}, u_2, \ldots, u_{-(\ell-a'-2)/2}, u_{(\ell-a'-2)/2}, u_{-(\ell-a')/2},$$
$$u_{(\ell-a'+2)/2}, u_{-(\ell-a'+2)/2}, \ldots, u_{a'}, u_{-a'}, u_{(b'-1)\ell/2};$$

while if $d \geqslant 2$, let P_0 and P_1 be directed walks of length $2^d a'$ defined as follows:

$$P_0 : u_0, u_1, u_{-1}, u_2, u_{-2}, \ldots, u_{(\ell-2^{d-1}a'-1)/2}, u_{-(\ell-2^{d-1}a'-1)/2},$$
$$u_{(\ell-2^{d-1}a'+3)/2}, u_{-(\ell-2^{d-1}a'+1)/2}, u_{(\ell-2^{d-1}a'+5)/2}, \ldots, u_{-(2^{d-1}a'-1)},$$
$$u_{2^{d-1}a'+1}, u_\ell,$$

and

$$P_1 : u_0, u_{-1}, u_1, u_{-2}, u_2, \ldots, u_{(\ell-2^{d-1}a'-3)/2}, u_{-(\ell-2^{d-1}a'-1)/2},$$
$$u_{(\ell-2^{d-1}a'+1)/2}, u_{-(\ell-2^{d-1}a'+1)/2}, \ldots, u_{2^{d-1}a'}, u_{-2^{d-1}a'}, u_{(b'-1)\ell/2}.$$

Both P_0 and P_1 are directed paths because the condition $\ell \leqslant 3(2^{d-1}a') - 1$ implies $\ell - 2^{d-1}a' < 2^d a'$. Let $C_0 = P_0 \cup \rho^{\ell}(P_0) \cup \cdots \cup \rho^{(b'-1)\ell}(P_0)$ and $C_1 = P_1 \cup \rho^{\ell}(P_1) \cup \cdots \cup \rho^{(b'-1)\ell}(P_1)$. Let A denote the set of arc lengths used in C_0 and C_1, that is,

$$A = \pm\{1, 2, \ldots, \ell - (2^{d-1}a' + 1), \ell - (2^{d-1}a' - 1), \ldots, 2^d a'\}$$
$$\cup \left\{\ell - 2^{d-1}a', \frac{(b'-1)\ell}{2} + 2^{d-1}a'\right\}.$$

Thus $\{C_i, \rho(C_i), \ldots, \rho^{\ell-1}(C_i) \mid i = 0, 1\}$ is a decomposition of $\overrightarrow{X}(n-1; A)$ into directed m-cycles.

The sum of the unused arc lengths is $(b'-1)\ell/2$ so that finding a directed path P' of length $m-2$ using the remaining arc lengths is feasible. Then letting $C : u, P', u$ gives the decomposition $\{C, \rho(C), \ldots, \rho^{n-2}(C)\}$ of $\overrightarrow{X}(n-1; S\backslash A) \bowtie \{u\}$ into directed m-cycles. Before continuing, note that $\ell \geqslant 2^d a' + 1$, or $2\ell - 2^d a' > \ell + 1$, and hence $(\ell+1)/2 \leqslant \ell - 2^{d-1}a'$.

Suppose first that $(\ell+1)/2 < \ell - 2^{d-1}a'$. Let

$$\mathscr{L} = S\backslash\left(A \cup \left\{-\frac{(b'-1)\ell}{2} - 2^{d-1}a', \frac{n-2}{2}, -(\ell - 2^{d-1}a')\right\}\right).$$

By Lemma 4.4, there is a path P from u_0 to $u_{n/2}$ such that none of the vertices in the intervals $(u_0, u_{2^d a'+1})$ and $(u_{n/2}, u_{n/2+2^d a'+1})$ appear on P and P has exactly one arc of each length in $\mathscr{L}$. Consider the directed path $Q : u_{(\ell+1)/2}, u_{((b'+2)\ell-2^d a'+1)/2}, u_{\ell-2^{d-1}a'}, u_0$, and note that Q is a path since $(\ell+1)/2 < \ell - 2^{d-1}a'$. Let P' denote the directed trail obtained by appending Q to the beginning of P. Now P' will be a directed path as long as $u_{((b'+2)\ell-2^d a'+1)/2} \in (u_{n/2}, u_{n/2+2^d a'+1})$, and $u_{(\ell+1)/2}, u_{\ell-2^{d-1}a'} \in (u_0, u_{2^d a'+1})$.

Since $(\ell+1)/2 < \ell - 2^{d-1}a' < 2^d a'$, we have $u_{(\ell+1)/2}, u_{\ell-2^{d-1}a'} \in (u_0, u_{2^d a'+1})$. Next

$$\frac{(b'+2)\ell - 2^d a' + 1}{2} = \frac{(b'+1)\ell}{2} + \frac{\ell+1}{2} - 2^{d-1}a',$$

and since

$$\frac{n}{2} = \frac{(b'-1)\ell}{2} + \frac{\ell+1}{2} < \frac{(b'+1)\ell}{2}$$

and $2^d a' < \ell + 1 < 3 \cdot 2^{d-1} a'$, we have

$$\begin{aligned}\frac{n}{2} &< \frac{(b'+1)\ell}{2} + \frac{\ell+1}{2} - 2^{d-1}a' \\ &= \left(\frac{(b'-1)\ell}{2} + \frac{\ell+1}{2}\right) + \ell - 2^{d-1}a' \\ &< \frac{n}{2} + 2^d a'.\end{aligned}$$

Thus, $u_{((b'+2)\ell - 2^d a' + 1)/2} \in (u_{n/2}, u_{n/2+2^d a'+1})$.

When $\ell = 2^d a' + 1$, we have $(b'-1)\ell/2 + 2^{d-1}a' = (n-2)/2$ and $(\ell+1)/2 = \ell - 2^{d-1}a'$. Let

$$\begin{aligned}\mathscr{L} &= S \setminus \left(A \cup \left\{-\frac{\ell+1}{2}\right\}\right) \\ &= \pm\left\{(2^d a' + 1), 2^d a' + 2, \ldots, \frac{n-4}{2}\right\} \cup \left\{-\frac{n-2}{2}\right\}.\end{aligned}$$

By Lemma 4.4, we can find a directed path P from u_0 to $u_{n/2}$ such that none of the vertices in the intervals (u_0, u_ℓ) and $(u_{n/2}, u_{n/2+\ell})$ appear on P and P has exactly one arc of each length in $\mathscr{L}$. Letting P' be the directed path constructed by placing the arc from $u_{(\ell+1)/2}$ to u_0 at the beginning of P gives the required directed path of length $m-2$.

Subcase 3.1.2. Assume that F is even and $F \geqslant 4$: For $i = 1, 2, \ldots, (F-2)/2$, define P_{2i} by

$$P_{2i} : u_0, u_{i\ell+1}, u_{-1}, u_{i\ell+2}, u_{-2}, \ldots, u_{-(2^{d-1}a'-1)}, u_{2^{d-1}a'+i\ell}, u_{(b'-1)\ell/2}.$$

As in Case 2, let $P_{2i+1} = \tau(P_{2i})$ for $i = 1, 2, \ldots, (F-2)/2$. We again have two isolated paths whose definitions are given by

$$P_0 : u_0, u_{(b'-3)\ell/2+1}, u_{-1}, u_{(b'-3)\ell/2+2}, u_{-2}, \ldots, u_{-(2^{d-1}a'-1)}, u_{(b'-3)\ell/2+2^{d-1}a'}, u_{-\ell}$$

and

$$\begin{aligned}P_1 : &u_0, u_{-(b'-3)\ell/2-1}, u_1, u_{-(b'-3)\ell/2-2}, u_2, \ldots, \\ &u_{2^{d-1}a'-1}, u_{-(b'-3)\ell/2-2^{d-1}a'}, u_{(b'+1)\ell/2}.\end{aligned}$$

Recall that $b' \geqslant 5$ so that all the preceding constructions produce directed paths. In the usual way, define $C_i : P_i \cup \rho^\ell(P_i) \cup \cdots \cup \rho^{(b'-1)\ell}(P_i)$ for $i = 0, 1, \ldots, F-1$, and observe that each C_i is a directed m-cycle. The sum of the arc lengths used in $C_0, C_1, \ldots, C_{F-1}$ is $(b'-1)\ell/2$ so that the sum of the unused arc lengths is $(b'+1)\ell/2$. We now show that arc lengths used in constructing $C_0, C_1, \ldots, C_{F-1}$ are distinct. Let A denote the set of arc lengths used in C_0 and C_1 and B denote the set of

arc lengths used in $C_2, C_3, \ldots, C_{F-1}$, that is,

$$A = \pm\left\{\frac{(b'-3)\ell}{2}+1, \frac{(b'-3)\ell}{2}+2, \ldots, \frac{(b'-3)\ell}{2}+2^d a'-1\right\}$$
$$\cup\left\{-(\ell-2^{d-1}a'), -\left(\frac{(b'-1)\ell}{2}+2^{d-1}a'\right)\right\}$$

and

$$B = \pm\left\{i\ell+1, i\ell+2, \ldots, i\ell+2^d a'-1, \frac{(b'-2i-1)\ell}{2}-2^{d-1}a' \mid 1\leqslant i\leqslant\frac{F-2}{2}\right\}.$$

Now $F\leqslant(b'-1)/2$ implying that $F-2<(b'-1)/2$. First, consider the set of arc lengths in B. If two lengths coincide, then, for some i and j with $1\leqslant i,j\leqslant(F-2)/2$, we have

$$\frac{(b'-2i-1)\ell-2^d a'}{2}\in\{j\ell+1, j\ell+2, \ldots, j\ell+2^d a'-1\}.$$

This implies $1\leqslant(b'-2i-3)/2\leqslant(F-2)/2$ which gives $b'-(F-2)-3\leqslant F-2$, or $F\geqslant(b'+1)/2$, producing a contradiction. Thus, all lengths in B are distinct.

Clearly, the arc lengths in A are distinct, and thus it remains to show that $A\cap B=\emptyset$. The shortest undirected length in B is either $\ell+1$ or $(b'-F+1)\ell/2-2^{d-1}a'$, but since $F\leqslant b'-3$, the second value is at least $2\ell-2^{d-1}a'>\ell+1$. So $\ell+1$ is the shortest undirected length. The shortest undirected length in A is $\ell-2^{d-1}a'$, which is strictly smaller than $\ell+1$.

The longest undirected length in B is either $(F-2)\ell/2+2^d a'-1$ or $(b'-3)\ell/2-2^{d-1}a'$. The arcs of A, other than $\ell-2^{d-1}a'$, have undirected length at least $(b'-3)\ell/2+1$, which certainly exceeds $(b'-3)\ell/2-2^{d-1}a'$. Since $F\leqslant b'-3$ and $2^d a'<\ell$, we have

$$\frac{(F-2)\ell}{2}+2^d a'-1\leqslant\frac{(b'-5)\ell}{2}+2^d a'-1<\frac{(b'-3)\ell}{2}+1.$$

Finally, the longest undirected length used is $(b'-1)\ell/2+2^{d-1}a'$ which is at most $(n-2)/2$. Therefore, all the arc lengths in $A\cup B$ are distinct and $\{C_i, \rho(C_i), \ldots, \rho^{\ell-1}(C_i) \mid 0\leqslant i\leqslant F-1\}$ is a decomposition of $\overrightarrow{X}(n-1; A\cup B)$ into directed m-cycles.

The rest of Subcase 3.1.2 consists of showing that the remaining arc lengths may be used to form a directed path P' of length $m-2$. Then, letting $C: u, P', u$, gives the decomposition $\{C, \rho(C), \ldots, \rho^{n-2}(C)\}$ of $\overrightarrow{X}(n-1; S\backslash(A\cup B))\bowtie\{u\}$ into directed m-cycles. There are several subcases. First, we consider the case when $2^d a'<\ell-1$. Let

$$T = \pm\{1, 2, \ldots, \ell-2^{d-1}a'-1\}\cup\{\ell-2^{d-1}a'\}\cup$$
$$\pm\{\ell-2^{d-1}a'+1, \ell-2^{d-1}a'+2, \ldots, \ell\}\cup$$
$$\left\{\frac{(b'-1)\ell}{2}+2^{d-1}a', \frac{n-2}{2}\right\},$$

and let $\mathscr{L} = S\backslash(A \cup B \cup T)$. The sum of the arc lengths in $\mathscr{L}$ is $n/2$. By Lemma 4.4, there is a directed path P from u_0 to $u_{n/2}$ which uses all the lengths of $\mathscr{L}$ and avoids vertices in the intervals $(u_0, u_{\ell+2^d a'}) \cup (u_{n/2}, u_{n/2+\ell+2^d a'})$. The object now is to use the lengths in T to complete P to the directed path P'.

Suppose first that $d = 1$. Let Q_1 be the directed path given by

$$Q_1 : u_{n/2}, u_{n/2+\ell-a'}, u_{n/2+2\ell-a'}, u_{n/2+\ell-a'+1}, u_{n/2+2\ell-a'-1},$$

$$u_{n/2+\ell-a'+2}, \ldots, u_{(n+2\ell-a'-1)/2}, u_{(n+4\ell-3a'+1)/2}, u_{(n+2\ell-a'+3)/2},$$

$$u_{(n+4\ell-3a'-1)/2}, \ldots, u_{(n+3\ell-2a'+3)/2}, u_{(n+3\ell-2a'+1)/2},$$

and note that the lengths of the arcs on Q_1 are $\ell - a', \ell, -(\ell-1), \ldots, \ell - a' + 1, -(\ell - a' - 1), \ell - a' - 2, \ldots, -1$.

Also since

$$\frac{n}{2} + 2\ell - a' = \frac{n}{2} + \ell + (\ell - a') < \frac{n}{2} + \ell + 2a'$$

(because $\ell \leqslant 3a' - 1$), all of the vertices of Q_1 lie on the interval $[u_{n/2}, u_{n/2+\ell+2a'})$. Next let Q_2 be the directed path given by

$$Q_2 : u_{(n+3\ell-2a'+1)/2}, u_{\ell+1}, u_1, u_\ell, u_2, \ldots, u_{\ell-(a'-3)/2}, u_{(a'+1)/2},$$

$$u_{\ell-(a'+1)2}, u_{(a'+3)/2}, \ldots, u_{(\ell+3)/2}, u_{(\ell-1)/2}, u_{(\ell+1)/2}, u_{(b'+1)\ell/2},$$

and observe that the lengths of the arcs on Q_2 are $(b'-1)\ell/2 + a', -\ell, \ell - 1, -(\ell-2), \ldots, -(\ell - a' + 1), \ell - a' - 1, -(\ell - a' - 2), \ldots, 1, (n-2)/2$. Next,

$$\frac{n}{2} < \frac{(b'+1)\ell}{2} = \frac{b'\ell}{2} + \frac{\ell}{2} < \frac{n}{2} + \ell - a',$$

since $a' < \ell/2$. Thus $u_{(b'+1)\ell/2}$ does not appear on Q_1. Hence the directed path $P' : P, Q_1, Q_2$ has length $m-2$ and contains an arc of each length in $S\backslash(A \cup B)$.

We now move to the case when $d \geqslant 2$ and recall we are assuming $2^d a' < \ell - 1$. In this case, much of the directed path P' is the same as the one constructed when $d = 1$. As before, by Lemma 4.4, there is a directed path P from u_0 to $u_{n/2}$ using all lengths in $\mathscr{L}$ and avoiding vertices of $(u_0, u_{\ell+2^d a'})$ and $(u_{n/2,n/2+\ell+2^d a'})$. We now wish to augment P so that each length in T is used. Let Q_1 be the directed path given by

$$Q_1 : u_{n/2}, u_{n/2+\ell-2^{d-1}a'}, u_{n/2+2\ell-2^{d-1}a'}, u_{n/2+\ell-2^{d-1}a'+1}, \ldots,$$

$$u_{n/2+2\ell-3\cdot 2^{d-2}a'+1}, u_{n/2+\ell-2^{d-2}a'}, u_{n/2+2\ell-3\cdot 2^{d-2}a'-1},$$

$$u_{n/2+\ell-2^{d-2}a'+1}, \ldots, u_{(n+3\ell-2^d a'-3)/2}, u_{(n+3\ell-2^d a'+1)/2},$$

and observe that the lengths encountered on Q_1 are $\ell - 2^{d-1}a', \ell, -(\ell-1), \ell-2, \ldots, -(\ell - 2^{d-1}a' + 1), \ell - 2^{d-1}a' - 1, -(\ell - 2^{d-1}a' - 2), \ldots, -3, 2$. As before, since $\ell \leqslant 3 \cdot 2^{d-1}a' - 1$ implies $n/2 + 2\ell - 2^{d-1}a' < n/2 + \ell + 2^d a'$, all the vertices on Q_1 lie on the interval $[u_{n/2}, u_{n/2+\ell+2^d a'})$. Next let Q_2 be the directed path given by

$$Q_2 : u_{(n+3\ell-2^d a'+1)/2}, u_{\ell+1}, u_1, u_\ell, u_2, \ldots, u_{2^{d-2}a'}, u_{\ell-2^{d-2}a'+1}, u_{\ell-2^{d-2}a'},$$

$$u_{2^{d-2}a'+1}, u_{\ell-2^{d-2}a'-1}, \ldots, u_{(\ell+3)/2}, u_{(\ell-1)/2}, u_{(\ell+1)/2}, u_{(b'+1)\ell/2}.$$

The lengths of the arcs on Q_2 are $(b'-1)\ell/2+2^{d-1}a', -\ell, \ell-1, -(\ell-2), \ldots, \ell-2^{d-1}a'+1, -1, -(\ell-2^{d-1}a'-1), \ell-2^{d-1}a'-2, \ldots, 1$. Using the same techniques as in the case when $d=1$, it is easy to verify that $P': P, Q_1, Q_2$ is a directed path of length $m-2$ containing an arc of each length in $S\backslash(A\cup B)$.

We now move to the special case when $2^d a' = \ell - 1$. We have seen previously that this condition implies $\ell - 2^{d-1}a' = (\ell+1)/2$ and $((b'-1)\ell + 2^d a')/2 = (n-2)/2$. Let

$$T = \pm\left\{1, 2, \ldots, \frac{\ell-1}{2}\right\} \cup \left\{\frac{\ell+1}{2}\right\} \cup \pm\left\{\frac{\ell+3}{2}, \frac{\ell+5}{2}\ldots, \ell\right\},$$

and let $\mathscr{L} = S\backslash(A\cup B\cup T)$. The sum of the arc lengths in $\mathscr{L}$ is $(n-2)/2$, and note that the shortest length in $\mathscr{L}$ is $2\ell-1$. The proof of Lemma 4.4 allows us to interchange the roles of $n/2$ and $(n-2)/2$, and thus there is a directed path P from u_0 to $u_{(n-2)/2}$ using all lengths in $\mathscr{L}$ and avoiding vertices of the intervals $(u_0, u_{2\ell-1})$ and $(u_{(n-2)/2}, u_{(n+4\ell-4)/2})$. To complete the proof of Case 3.1.2, we need to use the lengths in T to complete the directed path P to a directed path P' using all $m-2$ lengths.

Suppose first that $d=1$, which implies $\ell \equiv 3 \pmod 4$. Let Q_1 be the directed path

$$Q_1 : u_{(\ell+1)/2}, u_{(\ell+3)/2}, u_{(\ell-1)/2}, \ldots, u_{(\ell+5)/4}, u_{(3\ell+3)/4},$$
$$u_{(\ell-3)/4}, u_{(3\ell+7)/4}, u_{(\ell-7)/4}, \ldots, u_\ell, u_0.$$

The lengths of the arcs on Q_1 are $1, -2, \ldots, (\ell-1)/2, -(\ell+3)/2, (\ell+5)/2, \ldots, -\ell$, and the vertices of Q_1 lie in the interval $[u_0, u_{2\ell-1})$. Next, let Q_2 be the directed path

$$Q_2 : u_{(n-2)/2}, u_{\ell+(n-2)/2}, u_{n/2}, u_{\ell+(n-4)/2}, \ldots, u_{(2n+\ell-7)/4}, u_{(2n+3\ell-1)/4},$$
$$u_{(2n+\ell+1)/4}, u_{(2n+3\ell-5)/4}, \ldots, u_{(n+\ell-3)/2}, u_{(n+\ell+1)/2}, u_{(n+\ell-1)/2}, u_{\ell+n/2}.$$

The arcs lengths encountered on Q_2 are $\ell, -(\ell-1), \ell-2, \ldots, (\ell+3)/2, -(\ell-1)/2, (\ell-3)/2, -(\ell-5)/2, \ldots, -1, (\ell+1)/2$, and the vertices of Q_2 lie entirely in the interval $[u_{(n-2)/2}, u_{(n-2)/2+2\ell-1})$. Thus, $P': Q_1, P, Q_2$ is the required directed path.

Now suppose $d \geqslant 2$, which gives $\ell \equiv 1 \pmod 4$. Let Q_1 be the following directed path

$$Q_1 : u_{(\ell+1)/2}, u_{(\ell-3)/2}, u_{(\ell+3)/2}, \ldots, u_{(\ell+3)/4}, u_{(3\ell-3)/4},$$
$$u_{(\ell-1)/4}, u_{(3\ell+5)/4}, u_{(\ell-5)/4}, \ldots, u_\ell, u_0$$

and note that the lengths encountered on Q_1 are $-2, 3, \ldots, (\ell-3)/2, -(\ell-1)/2, (\ell+3)/2, \ldots, -\ell$, and the vertices of Q_1 lie in the interval $[u_0, u_{2\ell-1})$. Next, let Q_2 be the directed path

$$Q_2 : u_{(n-2)/2}, u_{\ell+(n-2)/2}, u_{n/2}, u_{\ell+(n-4)/2}, \ldots, u_{(2n+\ell-9)/4},$$
$$u_{(2n+3\ell+1)/4}, u_{(2n+\ell-5)/4}, u_{(2n+\ell-1)/4}, u_{(2n+3\ell-3)/4}, u_{(2n+\ell+3)/4},$$
$$u_{(2n+3\ell-7)/4}, \ldots, u_{(n+\ell-3)/2}, u_{(n+\ell+1)/2}, u_{(n+\ell-1)/2}, u_{\ell+n/2}.$$

Now the arcs lengths encountered on Q_2 are $\ell, -(\ell-1), \ell-2, \ldots, (\ell+5)/2, -(\ell+3)/2, 1, (\ell-1)/2, -(\ell-3)/2, \ldots, -1, (\ell+1)/2$, and the vertices of Q_2 lie entirely in the interval $[u_{(n-2)/2}, u_{(n-2)/2+2\ell-1})$. Thus, $P' : Q_1, P, Q_2$ is the required directed path thereby completing the proof of Subcase 3.1.2.

Case 3.2. Assume $\ell \geqslant 3 \cdot 2^{d-1}a'$: This inequality and the fact that $F + b' = 2^{e-d}a/a'$ imply $F > b'/2$. Thus, when $F = 2$ it follows that $b' = 3$. This, in turn, implies $2^d a' \geqslant 4$ since we are assuming $m \geqslant 8$. The arc lengths used in the construction of the F directed paths of length $2^d a'$ appear complicated, but have been chosen so the absolute values of the lengths are sequential when $d = 1$ and nearly sequential when $d \geqslant 2$, and yet still sum to $-\ell$.

Subcase 3.2.1. Assume that $F = 2$: As noted earlier, this implies that $b' = 3$ and $2^d a' \geqslant 4$. Suppose first that $d = 1$. Then $2a' \geqslant 4$, that is, $a' \geqslant 3$. Let P_0 be the directed path of length $2a'$ defined by

$$P_0 : u_0, u_{(\ell-3a'+2)/2}, u_{-1}, u_{(\ell-3a'+4)/2}, \ldots, u_{-(a'-1)}, u_{(\ell-a')/2}, u_{-\ell}.$$

Recall that τ is the permutation fixing u_0 and mapping u_i to u_{-i} for all $1 \leqslant i \leqslant (n-2)/2$. Let P_1 be the directed path of length $2a'$ obtained from $\tau(P_0)$ by replacing the last arc from $u_{-(\ell-a')/2}$ to u_ℓ with the arc from $u_{-(\ell-a')/2}$ to $u_{-\ell}$. Let $C_0 : P_0 \cup \rho^{\ell}(P_0) \cup \rho^{2\ell}(P_0)$ and $C_1 : P_1 \cup \rho^{\ell}(P_1) \cup \rho^{2\ell}(P_1)$. The directed cycles C_0 and C_1 have length $6a' = m$ and arcs with lengths in the set A where

$$A = \pm\left\{\frac{\ell-3a'+2}{2}, \frac{\ell-3a'+4}{2}, \ldots, \frac{\ell+a'-2}{2}\right\} \cup \left\{-\frac{3\ell-a'}{2}, -\frac{\ell+a'}{2}\right\}.$$

Clearly, the lengths used in constructing C_0 and C_1 are distinct, and thus $\{C_i, \rho(C_i), \ldots, \rho^{\ell-1}(C_i) \mid i = 0, 1\}$ is a decomposition of $\overrightarrow{X}(n-1; A)$ into directed m-cycles.

The rest of the case when $d = 1$ consists of showing that the remaining arc lengths may be used to form a directed path P' of length $m-2$. Then, letting $C : u, P', u$ gives the decomposition $\{C, \rho(C), \ldots, \rho^{n-2}(C)\}$ of $\overrightarrow{X}(n-1; S\backslash A) \bowtie \{u\}$ into directed m-cycles. Now, the sum of the arc lengths in A is ℓ so that the sum of the remaining arc lengths is 2ℓ since $b' = 3$. Let

$$T = \pm\left\{1, 2, \ldots, \frac{\ell-3a'}{2}\right\} \cup \left\{\frac{\ell+a'}{2}, \frac{3\ell-a'}{2}, \frac{n-2}{2}\right\}$$

and let $\mathscr{L} = S\backslash(A \cup T)$. Note that $(n-2)/2 = (3\ell-1)/2$ so that $(n-2)/2 \neq (3\ell-a')/2$ because $a' > 1$. The sum of the arc lengths in $\mathscr{L}$ is $n/2$, and the shortest length in absolute value is $(\ell+a'+2)/2$. By Lemma 4.4, there is a directed path P from u_0 to $u_{n/2}$ using arcs of all the lengths in $\mathscr{L}$ and avoiding vertices in the intervals $(u_0, u_{(\ell+a')/2}]$ and $(u_{n/2}, u_{(n+\ell+a')/2}]$. We now show that the remaining arc lengths, the elements of T, can be used to complete P to a directed path P' of length $m-2$.

First extend the directed path P by appending the directed 2-path

$$u_{n/2}, u_{(n+\ell+a')/2}, u_{(\ell+a')/2}$$

using arc lengths $(\ell+a')/2$ and $(n-2)/2$. We complete the directed path in two different ways depending on the parity of $(\ell-3a')/2$.

When $(\ell-3a')/2$ is even, complete the directed path to P' by appending the following directed path:

$$u_{(\ell+a')/2}, u_{2a'}, u_{(\ell+a'-2)/2}, u_{2a'+1}, \ldots, u_{(\ell+5a'+4)/4}, u_{(\ell+5a'-4)/4},$$

$$u_{(\ell+5a')/4}, u_{n/2+(\ell+3a'-2)/4}, u_{n/2+(3\ell-3a'-2)/4}, u_{n/2+(\ell+3a'+2)/4},$$

$$u_{n/2+(3\ell-3a'-6)/4}, \ldots, u_{(n+\ell-3)/2}, u_{(n+\ell+1)/2}, u_{(n+\ell-1)/2}.$$

When $(\ell-3a')/2$ is odd, complete the directed path to P' by appending the following directed path:

$$u_{(\ell+a')/2}, u_{2a'}, u_{(\ell+a'-2)/2}, u_{2a'+1}, \ldots, u_{(\ell+5a'-6)/4}, u_{(\ell+5a'+2)/4},$$

$$u_{(\ell+5a'-2)/4}, u_{n/2+(\ell+3a'-4)/4}, u_{n/2+(3\ell-3a'-4)/4}, u_{n/2+(\ell+3a')/4},$$

$$u_{n/2+(3\ell-3a'-8)/4}, \ldots, u_{(n+\ell+1)/2}, u_{(n+\ell-3)/2}, u_{(n+\ell-1)/2}.$$

It is easy to verify that P' is indeed a directed path of length $m-2$ containing an arc of each length in $S\backslash A$.

Now suppose that $d \geqslant 2$. Let P_0 be the directed path of length $2^d a'$ defined by

$$P_0 : u_0, u_{(\ell-3\cdot 2^{d-1}a'+1)/2}, u_{-1}, u_{(\ell-3\cdot 2^{d-1}a'+3)/2},$$
$$u_{-2}, \ldots, u_{-(2^{d-1}a'-1)}, u_{(\ell-2^{d-1}a'-1)/2}, u_{-\ell}.$$

Let P_1 be the directed path of length $2^d a'$ obtained from $\tau(P_0)$ by replacing the last arc from $u_{-(\ell-2^{d-1}a'-1)/2}$ to u_ℓ with the arc from $u_{-(\ell-2^{d-1}a'-1)/2}$ to $u_{-\ell}$. Let $C_0 = P_0 \cup \rho^\ell(P_0) \cup \rho^{2\ell}(P_0)$ and $C_1 = P_1 \cup \rho^\ell(P_1) \cup \rho^{2\ell}(P_1)$. Then $\{C_i, \rho(C_i), \ldots, \rho^{\ell-1}(C_i) \mid i = 0, 1\}$ is a decomposition of $\overrightarrow{X}(n-1; A)$ into directed m-cycles where

$$A = \pm\left\{\frac{\ell-3\cdot 2^{d-1}a'+1}{2}, \frac{\ell-3\cdot 2^{d-1}a'+3}{2}, \ldots, \frac{\ell+2^{d-1}a'-3}{2}\right\}$$
$$\cup\left\{-\frac{3\ell-2^{d-1}a'-1}{2}, -\frac{\ell+2^{d-1}a'+1}{2}\right\}.$$

The rest of the case when $d \geqslant 2$ consists of showing that the remaining arc lengths may be used to form a directed path P' of length $m-2$. Then, letting $C : u, P', u$ gives the decomposition $\{C, \rho(C), \ldots, \rho^{n-2}(C)\}$ of $\overrightarrow{X}(n-1; S\backslash A) \bowtie \{u\}$ into directed m-cycles.

The sum of the arc lengths in A is ℓ so that the sum of the unused arc lengths is 2ℓ since $b'=3$. Let

$$T=\pm\left\{1,2,\ldots,\frac{\ell-3\cdot 2^{d-1}a'-1}{2},\frac{\ell+2^{d-1}a'-1}{2}\right\}$$
$$\cup\left\{\frac{3\ell-2^{d-1}a'-1}{2},\frac{\ell+2^{d-1}a'+1}{2},\frac{n-2}{2}\right\},$$

and let $\mathscr{L}=S\backslash(A\cup T)$. The sum of the arc lengths in $\mathscr{L}$ is $n/2$, and $(\ell+2^{d-1}a'+3)/2$ is the shortest length in absolute value. From Lemma 4.4, there is a directed path P from u_0 to $u_{n/2}$ containing an arc of each length in $\mathscr{L}$ and using none of the vertices in the intervals $(u_0,u_{(\ell+2^{d-1}a'+1)/2}]$ and $(u_{n/2},u_{(n+\ell+2^{d-1}a'+1)/2}]$. We now show that the remaining arc lengths in T can be used to complete P to a directed path P' of length $m-2$.

First extend the directed path P by appending the directed 4-path

$$u_{n/2},u_{(n+\ell+2^{d-1}a'+1)/2},u_{(n+2)/2},u_1,u_{(\ell+2^{d-1}a'+1)/2}$$

using arc-lengths

$$\frac{\ell+2^{d-1}a'+1}{2},\frac{-(\ell+2^{d-1}a'-1)}{2},\frac{n-2}{2},\frac{\ell+2^{d-1}a'-1}{2}.$$

The continuation now depends on the parity of $(\ell-3\cdot 2^{d-1}a'-1)/2$. When the latter value is even, continue with the directed path

$$u_{(\ell+2^{d-1}a'+1)/2},u_{2^da'+1},u_{(\ell+2^{d-1}a'-1)/2},u_{2^da'+2},\ldots,$$

$$u_{(\ell+5\cdot 2^{d-1}a'-1)/4},u_{(\ell+5\cdot 2^{d-1}a'+3)/4}$$

which uses lengths $-(\ell-3\cdot 2^{d-1}a'-1)/2$, $(\ell-3\cdot 2^{d-1}a'-3)/2$, $-(\ell-3\cdot 2^{d-1}a'-5)/2,\ldots,1$.

Next, let P' be the directed path established thus far together with the continuation:

$$u_{(\ell+5\cdot 2^{d-1}a'+3)/4},\ u_{n/2+(\ell+3\cdot 2^{d-1}a'-1)/4},\ u_{n/2+(3\ell-3\cdot 2^{d-1}a'-3)/4},$$

$$u_{n/2+(\ell+3\cdot 2^{d-1}a'+3)/4},u_{n/2+(3\ell-3\cdot 2^{d-1}a'-7)/4},\ldots,u_{2\ell+1},u_{2\ell}$$

using arc lengths $(3\ell-2^{d-1}a'-1)/2,(\ell-3\cdot 2^{d-1}a'-1)/2,-(\ell-3\cdot 2^{d-1}a'-3)/2,(\ell-3\cdot 2^{d-1}a'-5)/2,\ldots,-1$. Since $n-1<2m$, it follows that $\ell<2^{d+1}a'$. Thus $3\ell-3\cdot 2^{d-1}a'-3<2\ell+2^da'+2$, and $d\geqslant 2$ and $\ell>3\cdot 2^{d-1}a'$ gives $3\ell-3\cdot 2^{d-1}a'-3>4$ so that $u_{n/2+(3\ell-3\cdot 2^{d-1}a'-3)/4}\in(u_{(n+2)/2},u_{(n+\ell+2^{d-1}a'+1)/2})$. Next $(\ell+3\cdot 2^{d-1}-1)/4>1$ since $3\cdot 2^{d-1}a'\geqslant 6$. Thus, $u_{n/2+(\ell+3\cdot 2^{d-1}a'-1)/4}\in(u_{(n+2)/2},\ u_{(n+\ell+2^{d-1}a'+1)/2})$. Hence P' is a directed path of length $m-2$ containing an arc of each length in $S\backslash A$.

When $(\ell-3\cdot 2^{d-1}a'-1)/2$ is odd, the continuation of P is close to what was just done. The first portion terminates at $u_{(\ell+5\cdot 2^{d-1}a'+1)/4}$ instead of $u_{(\ell+5\cdot 2^{d-1}a'+3)/4}$,

and finishes with an arc of length -1 rather than length 1. The arc of length $(3\ell - 2^{d-1}a' - 1)/2$ then terminates at $u_{n/2+(\ell+3\cdot 2^{d-1}a'-3)/4}$ instead of $u_{n/2+(\ell+3\cdot 2^{d-1}a'-1)/4}$.

The rest of the completion is just as before except we finish with an arc of length 1 instead of length -1. This completes Subcase 3.2.1.

Subcase 3.2.2. Assume that F is even and $F \geqslant 4$: For $i = 1, 2, \ldots, (F-2)/2$, define P_{2i} by

$$P_{2i} : u_0, u_{i\ell+1}, u_{-1}, u_{i\ell+2}, u_{-2}, \ldots, u_{-(2^{d-1}a'-1)}, u_{i\ell+2^{d-1}a'}, u_{(b'+1)\ell/2}.$$

Now define $P_{2i+1} = \tau(P_{2i})$ for $i = 1, 2, \ldots, (F-2)/2$.

We again have two isolated paths whose definitions are given by

$$P_0 : u_0, u_1, u_{-1}, u_2, u_{-2}, \ldots, u_{-(2^{d-1}a'-1)}, u_{2^{d-1}a'}, u_{(b'+1)\ell/2}$$

and

$$P_1 : u_0, u_{-1}, u_1, u_{-2}, u_2, \ldots, u_{2^{d-1}a'-1}, u_{-2^{d-1}a'}, u_{-\ell}.$$

In the usual way, for $i = 0, 1, 2, 3, \ldots, F-1$, let $C_i : P_i \cup \rho^{\ell}(P_i) \cup \cdots \cup \rho^{(b'-1)\ell}(P_i)$, and note that C_i is a directed m-cycle. Let A denote the set of arc lengths used in C_0 and C_1 and let B denote the set of arc lengths used in $C_2, C_3, \ldots, C_{F-1}$. Thus

$$A = \pm\{1, 2, \ldots, 2^d a' - 1\} \cup \left\{ -\frac{(b'-1)\ell}{2} + 2^{d-1}a', -(\ell - 2^{d-1}a') \right\},$$

and

$$B = \pm\left\{ i\ell + 1, i\ell + 2, \ldots, i\ell + 2^d a' - 1, \frac{(b'+1-2i)\ell}{2} - 2^{d-1}a' \mid 1 \leqslant i \leqslant \frac{F-2}{2} \right\}.$$

The sum of the arc lengths in $A \cup B$ is $(b'-1)\ell/2$. We must show all the arc lengths in $A \cup B$ are distinct. Consider the set B first. If there is any overlap, then for some i and j $(1 \leqslant i, j \leqslant (F-2)/2)$, we have

$$\frac{(b'-2i+1)\ell - 2^d a'}{2} \in \{ j\ell + 1, j\ell + 2, \ldots, j\ell + 2^d a' - 1 \}.$$

But, if this happens, then $\ell < 3 \cdot 2^{d-1}a'$, producing a contradiction. Thus, all lengths in B are distinct.

Clearly, the lengths in A are distinct, and thus it remains to show that $A \cap B = \emptyset$. The shortest undirected length used in B is either $\ell + 1$ or $(b' - F + 3)\ell/2 - 2^{d-1}a'$. Now $2^d a' - 1, \ell - 2^{d-1}a' < \ell + 1$, and $F < b'$ as shown in the beginning of the proof. Thus, $(b' - F + 3)\ell/2 - 2^{d-1}a' > 2\ell - 2^{d-1}a' > 2^d a' - 1$ and $2\ell - 2^{d-1}a' > \ell - 2^{d-1}a'$.

We also must examine the long arc lengths in B. The longest undirected length in B is either $(F-2)\ell/2 + 2^d a' - 1$ or $(b'-1)\ell/2 - 2^{d-1}a'$. Since $F < b'$, the preceding two lengths cannot equal any of the lengths in A. Therefore, $A \cap B = \emptyset$ and all lengths used thus far are distinct.

The rest of the proof consists of showing that the remaining $m-2$ arc lengths can be ordered to form a directed path P' of length $m-2$ using each arc length once. We first eliminate some special cases.

If $2^d a' = \ell - 1$, then $\ell \geqslant 3 \cdot 2^{d-1} a'$ implies $\ell = 1$ or $\ell = 3$. Now, $\ell = 1$ is impossible, and thus suppose $\ell = 3$. Then $2^d a' = 2$. Let $\mathscr{L} = S \backslash (A \cup B \cup \{2\})$, and observe that the sum of the lengths in $\mathscr{L}$ is $(n-2)/2 = (3b'-1)/2$. By Lemma 4.4, there is a directed path P from u_0 to $u_{(3b'-1)/2}$ using neither $u_{(3b'+1)/2}$ nor $u_{(3b'+3)/2}$. Appending the arc from $u_{(3b'-1)/2}$ to $u_{(3b'+3)/2}$ gives a directed path P' of length $m-2$ using each of the unused arc lengths precisely once.

Thus, we may assume $2^d a' < \ell - 1$ and $2^d a' \geqslant 4$. (The latter inequality is valid because $2^d a' = 2$ together with the fact that $n < 2m$ implies $\ell = 3$ which was dealt with above.) Let

$$T = \pm(\{2^d a', 2^d a' + 1, \ldots, \ell\} \backslash \{\ell - 2^{d-1} a'\}) \cup \left\{ \ell - 2^{d-1} a', \frac{(b'-1)\ell}{2} + 2^{d-1} a', \frac{n-2}{2} \right\},$$

and let $\mathscr{L} = S \backslash (A \cup B \cup T)$. The shortest arc length in $\mathscr{L}$ is $\ell + 2^d a'$, and the sum of the lengths in $\mathscr{L}$ is $n/2$. Hence, by Lemma 4.4, there is a directed path P from u_0 to $u_{n/2}$ using all the lengths in $\mathscr{L}$ and not using any vertices of $[u_1, u_{\ell+2^d a'}) \cup (u_{n/2}, u_{n/2+\ell+2^d a'})$.

Suppose first that $d \geqslant 2$. We extend P by first appending the directed path

$$u_{n/2}, u_{n/2+\ell}, u_{n/2+1}, u_{n/2+\ell-1}, \ldots, u_{n/2+\ell-2^{d-2}a'+1}, u_{n/2+2^{d-2}a'},$$

$$u_{n/2+\ell-2^{d-2}a'-1}, u_{n/2+2^{d-2}a'+1}, \ldots, u_{(b'+1)\ell/2-2^{d-1}a'},$$

$$u_{(b'+1)\ell/2+2^{d-1}a'}, u_{(b'+3)\ell/2}, u_{(3\ell-1)/2}$$

using arcs $\ell, -(\ell-1), \ldots, -(\ell - 2^{d-1}a' + 1)$, $\ell - 2^{d-1}a' - 1$, $-(\ell - 2^{d-1}a' - 2), \ldots, 2^d a'$, $\ell - 2^{d-1}a', (n-2)/2$. Note that the vertices $u_{(b'+3)\ell/2}$ and $u_{(3\ell-1)/2}$ have not been used before. To see this, observe $(b'+3)\ell/2 > n/2 + \ell$ and since $\ell < 2^{d+1}a'$, we have $(b'+3)\ell/2 < n/2 + \ell + 2^d a'$. Thus $u_{(b'+3)\ell/2} \in (u_{n/2+\ell}, u_{n/2+\ell+2^d a'})$. In a similar manner, it is easy to see that $u_{(3\ell-1)/2} \in [u_1, u_{\ell+2^d a'})$.

Another important point to note is that the vertex $u_{(b'+1)\ell/2}$ has not been used. We use the rest of the arc lengths and finish the completion to the path P' by starting at $u_{(3\ell-1)/2}$ and using arcs of length $-\ell, \ell-1, \ldots, \ell - 2^{d-1}a' + 1, -(\ell - 2^{d-1}a' - 1), \ldots, -2^d a'$, ending at vertex $u_{\ell-2^{d-1}a'}$. Next, from vertex $u_{\ell-2^{d-1}a'}$, we append the arc of length $(b'-1)\ell/2 + 2^{d-1}a'$ and thus the last vertex of P' is $u_{(b'+1)\ell/2}$.

Suppose now that $d = 1$. Note that we may assume $2a' \geqslant 6$ because $2a' = 2$ was done above. The extension of P is a little different in this case. We start by appending the directed path

$$u_{n/2}, u_{n/2+\ell}, u_{n/2+1}, u_{n/2+\ell-1}, \ldots, u_{(n+a'-1)/2}, u_{\ell+(n-a'+1)/2}, u_{(n+a'+3)/2},$$

$$u_{\ell+(n-a'-1)/2}, \ldots, u_{(b'+1)\ell/2-a'+1}, u_{(b'+1)\ell/2+a'+1}, u_{2a'+1}, u_{\ell+a'+1}.$$

This extension uses the lengths $\ell, -(\ell-1), \ldots, \ell-a'+1, -(\ell-a'-1), \ell-a'-2, \ldots, 2a', (b'-1)\ell/2+a', \ell-a'$. Note that the vertex $u_{(b'+1)\ell/2}$ is not used since $a' \geqslant 3$.

We use the rest of the arc lengths in T and finish the completion of P' by starting at the vertex $u_{\ell+a'+1}$ and using arcs of length $-\ell, \ell-1, \ldots, -(\ell-a'+1), \ell-a'-1, -(\ell-a'-2)\ldots, -2a'$ ending at vertex $u_{(\ell+1)/2}$. Next, we append the arc from $u_{(\ell+1)/2}$ to $u_{(b'+1)\ell/2}$ which has length $(n-2)/2$ as required.

Hence, letting $C: u, P', u$, gives the decomposition $\{C, \rho(C), \ldots, \rho^{n-2}(C)\}$ of $\overrightarrow{X}(n-1; S\backslash(A\cup B)) \bowtie \{u\}$ into directed m-cycles.

This completes the proof of Theorem 4.1. □

5. The case when m is odd and n is even

We have arrived at the most difficult case. In this section, we will prove the following theorem.

5.1. Theorem. *For positive integers m and n with m odd, n even, and $3 \leqslant m < n$, the digraph K_n^* can be decomposed into directed cycles of length m if and only if $m \mid n(n-1)$ and $(n,m) \neq (6,3)$.*

There are four main pieces to the proof of Theorem 5.1. The first piece is to establish a sufficiency range for the proof. In particular, we show that Theorem 5.1 is true for all m and n satisfying the necessary conditions if it is true for all m and n satisfying the necessary conditions with $m < n < 3m$. The second piece is a proof of the theorem for the particular case of $n = 2m$. It is not completely surprising that this case would require a separate argument as $n = 6$ and $m = 3$ is the only case for which the necessary arithmetic is not sufficient. The third and fourth pieces consist of separate proofs for n in the range $m < n < 2m$ and $2m < n < 3m$, respectively.

Others have proved Theorem 5.1 for a few specific values of m. Of particular interest to us is $m = 3$. This was settled by Bermond [3] and allows us to assume $m \geqslant 5$ for the remainder of this section, thereby simplifying our proof significantly. We now proceed to tackle the first of the four pieces mentioned above. There are three preliminary propositions required for the reduction lemma.

Cycle decompositions of undirected graphs are sometimes mentioned below because we can apply the doubling lemma to cycle decompositions to obtain directed cycle decompositions. For a graph (digraph) G, the notation $\bar{G}$ denotes the complement of G.

5.2. Definition. Let the vertex set of K_{2m} or K_{2m}^* be partitioned into two sets, $U = \{u_0, u_1, \ldots, u_{m-1}\}$ and $V = \{v_0, v_1, \ldots, v_{m-1}\}$. The notation $K_{2m}\langle S, T, S'\rangle$ will denote the subgraph of K_{2m} such that $\langle U\rangle \cong X(m; S)$, where $\langle U\rangle$ denotes the subgraph induced by the vertices of U, $\langle V\rangle \cong X(m; S')$, and all edges of the form $u_i v_{i+t}$ are present, where $t \in T$ with $T \subseteq \{0, 1, \ldots, m-1\}$. Similarly, $K_{2m}^*\langle S, T, T', S'\rangle$

denotes the subdigraph with $\langle U\rangle \cong \overrightarrow{X}(m;S)$, $\langle V\rangle \cong \overrightarrow{X}(m;S')$, T describing the arcs from U to V, and T' describing the arcs from V to U.

5.3. Proposition. *Let $m\geqslant 3$ be an odd integer. Each of the following graphs has a C_m-decomposition*:

(a) $K_{2m}\langle\emptyset,\{0,1,\ldots,m-2s-1\},\emptyset\rangle \bowtie \overline{K_s}$ *for s odd with* $1\leqslant s\leqslant (m-1)/2$;

(b) $K_{2m}\langle\{\pm(m-1)/2\},\{0,1,\ldots,m-2s-2\},\emptyset\rangle \bowtie \overline{K_s}$ *for s even with* $0\leqslant s\leqslant (m-3)/2$;

(c) $K_{2m}\langle\{\pm(m-1)/2\},S\cup(S+1),\emptyset\rangle$ *where* $S\subseteq\{1,3,5,\ldots,m-2\}$; *and*

(d) $K_{2m}\langle A,\emptyset,B\rangle$ *where each of A and B is either* $\pm\{1,2,\ldots,(m-1)/2\}$ *or* $\pm\{1,2,\ldots,(m-3)/2\}$.

Proof. Parts (a)–(c) were proved in [6]. To prove part (d), note that the graph $X(m;\pm\{t,t+1\})$ has a Hamilton decomposition by the main result of [5]. Therefore, we may use pairs of successive lengths to form connected circulant graphs of degree 4 which can be decomposed into two m-cycles. If the number of available lengths is even, we are done. If the number of available lengths is odd, we start the process with length ± 2 as length ± 1 forms an m-cycle. □

5.4. Proposition. *Let $m\geqslant 3$ be odd. Each of the following digraphs has a $\overrightarrow{C_m}$-decomposition*:

(a) $K^*_{2m}\langle\emptyset,\{0\},\{0\},\emptyset\rangle \bowtie \overline{K^*_{(m-1)/2}}$; *and*

(b) $K^*_{2m}\langle\{\pm 1\},\emptyset,\emptyset,\emptyset\rangle \bowtie \overline{K^*_{(m-1)/2}}$.

Proof. Let $m=2k+1$ for some integer k. Define the permutation ω by

$$\omega=(u_0\ u_1\ \cdots\ u_{m-1})(v_0\ v_1\ \cdots\ v_{m-1}),$$

and note that ω fixes all other vertices. Let $w_0,w_1,\ldots,w_{k-1}$ denote the vertices of $\overline{K^*_k}$. For part (a), define two directed m-cycles by

$$C_1: u_0,v_0,w_0,v_1,w_1,v_2,w_2,\ldots,w_{k-2},v_{k-1},w_{k-1},u_0$$

and

$$C_2: v_0,u_0,w_{k-2},u_1,w_{k-3},u_2,\ldots,w_1,u_{k-2},w_0,u_{k-1},w_{k-1},v_0.$$

Then $\{C_1,\omega(C_1),\omega^2(C_1),\ldots,\omega^{m-1}(C_1),C_2,\omega(C_2),\ldots,\omega^{m-1}(C_2)\}$ is a $\overrightarrow{C_m}$-decomposition of $K^*_{2m}\langle\emptyset,\{0\},\{0\},\emptyset\rangle \bowtie \overline{K^*_k}$.

For part (b), define two directed m-cycles by

$$C_1: u_0,u_1,w_0,v_0,w_1,v_1,w_2,\ldots,w_{k-2},v_{k-2},w_{k-1},u_0$$

and

$$C_2: u_1,u_0,w_1,u_2,w_2,u_3,\ldots,w_{k-3},u_{k-2},w_{k-2},u_{k-1},w_{k-1},v_0,w_0,u_1.$$

As before, $\{C_1, \omega(C_1), \omega^2(C_1), \ldots, \omega^{m-1}(C_1), C_2, \omega(C_2), \ldots, \omega^{m-1}(C_2)\}$ is a $\overrightarrow{C_m}$-decomposition of $K^*_{2m}\langle\{\pm 1\}, \emptyset, \emptyset, \emptyset\rangle \bowtie \overline{K^*_k}$. □

5.5. Proposition. *For integers m and t satisfying $m = 2k+1$ and $t = qk + r$ where $0 \leqslant r \leqslant k-1$ and $1 \leqslant q \leqslant m + 2r - 1$, the digraph $K^*_{2m} \bowtie \overline{K^*_t}$ is $\overrightarrow{C_m}$-decomposable. In particular, $K^*_{2m} \bowtie \overline{K^*_t}$ is $\overrightarrow{C_m}$-decomposable whenever $(m-1)/2 \leqslant t \leqslant (m-1)^2/2$.*

Proof. Let m and t be integers with $m = 2k+1$ and $t = qk + r$ where $0 \leqslant r \leqslant k-1$ and $1 \leqslant q \leqslant m + 2r - 1$. Suppose first that r odd. Partition the vertex set of $\overline{K^*_t}$ into sets $B_0, B_1, \ldots, B_q$ such that $|B_0| = r$ and $|B_1| = |B_2| = \cdots = |B_q| = k$. We then obtain a decomposition of $K_{2m} \bowtie \overline{K_r}$, where the vertices of $\overline{K_r}$ are those of B_0, into the following graphs:

$$K_{2m}\langle\emptyset, \{0, 1, \ldots, m-2r-1\}, \emptyset\rangle \bowtie \overline{K_r},$$

$$K_{2m}\langle\emptyset, \{m-2r, \ldots, m-1\}, \{\pm k\}\rangle, \text{ and}$$

$$K_{2m}\langle\pm\{1, \ldots, k\}, \emptyset, \pm\{1, \ldots, k-1\}\rangle.$$

Each of these three graphs has a C_m-decomposition by parts (a), (c) and (d), respectively, of Proposition 5.3. Orienting all cycles in each possible direction gives a $\overrightarrow{C_m}$-decomposition of $K^*_{2m} \bowtie \overline{K^*_r}$.

Observe that none of the arcs between K^*_{2m} and B_i, for each i with $1 \leqslant i \leqslant q$, have been used. Counting the number of undirected m-cycles in the C_m-decomposition of the graphs in parts (c) and (d) above, we find there are $m + 2r - 1$ m-cycles; that is, there are at least q undirected m-cycles. Let $R_1, R_2, \ldots, R_q$ be q such undirected m-cycles. Then for each i with $1 \leqslant i \leqslant q$, orient R_i in each possible direction and form a digraph by adding the vertices of B_i and all arcs between B_i and vertices of R_i. The resulting digraph is isomorphic to $K^*_{2m}\langle\pm\{1\}, \emptyset, \emptyset, \emptyset\rangle \bowtie \overline{K^*_k}$ which has a $\overrightarrow{C_m}$-decomposition by Proposition 5.4.

Now assume r is even. The proof in this case is similar to the case for r odd, and we use the same notation for the vertex partition. The three graphs

$$K_{2m}\langle\pm\{k\}, \{0, 1, \ldots, m-2r-2\}, \emptyset\rangle \bowtie \overline{K_r},$$

$$K_{2m}\langle\emptyset, \{m-2r, \ldots, m-1\}, \{\pm k\}\rangle, \text{ and}$$

$$K_{2m}\langle\pm\{1, \ldots, k-1\}, \emptyset, \pm\{1, \ldots, k-1\}\rangle$$

use all edges of $K_{2m} \bowtie \overline{K_r}$ except for a perfect matching corresponding to edge length $m - 2r - 1$. By parts (b)–(d) of Proposition 5.3, each graph above has a C_m-decomposition. Observe that the number of undirected m-cycles in the C_m-decompositions of the graphs for parts (c) and (d) above is $m + 2r - 2$; that is, there are at least $q - 1$ undirected m-cycles. We set aside $q - 1$ of these undirected cycles and orient each of the remaining cycles in both possible directions.

Since $q \geqslant 1$, there is at least one part of cardinality k available. Now $\overrightarrow{K^*_{2m}}\langle \emptyset, \{m-2r-1\}, \{2r+1\}, \emptyset \rangle \bowtie \overline{K^*_k}$ has a $\overrightarrow{C_m}$-decomposition by part (a) of Proposition 5.4, uses the arcs corresponding to the omitted perfect matching, and all those between B_1 and the vertices of K^*_{2m}. The remaining arcs between $B_2, B_3, \ldots, B_q$ and the vertices of K^*_{2m} are partitioned into directed m-cycles using the $q-1$ undirected cycles we have set aside as was done in the previous case. □

5.6. Lemma. *Let $m \geqslant 5$ be an odd integer. If K^*_n is $\overrightarrow{C_m}$-decomposable for all even n satisfying $m < n < 3m$ with $m \mid n(n-1)$, then K^*_n is $\overrightarrow{C_m}$-decomposable for all even $n > m$ satisfying $m \mid n(n-1)$.*

Proof. Let n' be an even integer satisfying $n' > m$ and $m \mid n'(n'-1)$. Clearly, we may write $n' = n + 2mq$, where $q \geqslant 0$, $m < n < 3m$, and $m \mid n(n-1)$. Partition the vertex set of $K^*_{n'}$ into one set with n vertices, and q sets with $2m$ vertices. We denote the subdigraphs induced by various sets in the obvious way.

If $q \geqslant 3$, the complete multipartite graph $K_{q(2m)}$, with q parts each of cardinality $2m$, has a decomposition into undirected m-cycles by [6]. Orienting all cycles in both directions gives $\overrightarrow{C_m} \mid K^*_{q(2m)}$. We have $\overrightarrow{C_m} \mid K^*_n$ by supposition. Thus, if $\overrightarrow{C_m} \mid (K^*_{2m} \bowtie \overline{K^*_n})$, then the $q \geqslant 3$ case and the $q = 1$ case will be complete. In the $q = 2$ case, it remains to show that $\overrightarrow{C_m} \mid (K^*_{2m} \bowtie \overline{K^*_{2m+n}})$.

For $m \geqslant 13$, we have $5m \leqslant (m-1)^2/2$, and both $K^*_{2m} \bowtie \overline{K^*_n}$ and $K^*_{2m} \bowtie \overline{K^*_{2m+n}}$ are $\overrightarrow{C_m}$-decomposable by Proposition 5.5. For $m \leqslant 11$, then $m \mid n(n-1)$ and $m < n < 3m$ imply $n = m+1$ or $n = 2m$. Thus we need $\overrightarrow{C_m}$-decompositions of $K^*_{2m} \bowtie \overline{K^*_{m+1}}$, $K^*_{2m} \bowtie \overline{K^*_{3m+1}}$, and $K^*_{2m} \bowtie \overline{K^*_{2m}}$. For $m \geqslant 9$, we have $3m+1 \leqslant (m-1)^2/2$ so that each of these digraphs has a $\overrightarrow{C_m}$-decomposition by Proposition 5.5. Thus, we may assume $m \leqslant 7$.

For $m = 7$, Proposition 5.5 gives $\overrightarrow{C_7} \mid K^*_{14} \bowtie \overline{K^*_8}$ and $\overrightarrow{C_7} \mid K^*_{14} \bowtie \overline{K^*_{14}}$ as $14 < (m-1)^2/2 = 18$. For $K^*_{14} \bowtie \overline{K^*_{22}}$, partition the vertices of $\overline{K^*_{22}}$ into six sets of three vertices each and a single vertex u. Take seven digraphs isomorphic to the digraph of part (a) in Proposition 5.4. This leaves two copies of K^*_8 having the vertex u in common. By supposition, $\overrightarrow{C_7} \mid K^*_8$.

For $m = 5$, we have $\overrightarrow{C_5} \mid K^*_{10} \bowtie \overline{K^*_6}$ by Proposition 5.4. For $K^*_{10} \bowtie \overline{K^*_{10}}$, take five isomorphic copies of the digraph from part (a) of Proposition 5.4. This leaves two vertex-disjoint copies of K^*_5 which are $\overrightarrow{C_5}$-decomposable. Finally, for $K^*_{10} \bowtie \overline{K^*_{16}}$, take five isomorphic copies of the digraph from part (a) and three isomorphic copies of the digraph from part (b) of Proposition 5.4. This leaves two directed 5-cycles in one part of the bipartition. □

We now handle the special cases of $n = m+1$ and $n = 2m$.

5.7. Lemma. *For each odd integer $m \geqslant 5$, the complete directed graph K^*_{m+1} can be decomposed into directed m-cycles.*

Proof. Let $m \geqslant 5$ be an integer and let the vertices of K^*_{m+1} be labelled with $u, u_0, u_1, \ldots, u_{m-1}$. Let $K^*_{m+1} = \overrightarrow{X}(m; S) \bowtie \{u\}$ where $S = \pm\{1, 2, \ldots, (m-1)/2\}$. Define the permutation ρ as before, that is, $\rho = (u)(u_0\ u_1 \cdots u_{m-1})$. Let

$$C : u_0, u_1, u_{-1}, u_2, \ldots, u_{(m-1)/4}, u_{-(m-1)/4}, u_{(m+7)/4}, u_{-(m+3)/4},$$

$$\ldots, u_{(m+3)/2}, u_{(m+1)/2}, u, u_0$$

if $m \equiv 1 \pmod 4$, and

$$C : u_0, u_1, u_{-1}, u_2, \ldots, u_{(m-3)/4}, u_{-(m-3)/4}, u_{(m+5)/4}, u_{-(m+1)/4}, u_{(m+9)/4},$$

$$\ldots, u_{(m+3)/2}, u_{(m+1)/2}, u, u_0$$

if $m \equiv 3 \pmod 4$. Note that C uses an arc of each length in S except length $(m-1)/2$. Thus $\{C, \rho(C), \rho^2(C), \ldots, \rho^{m-1}(C)\}$ together with the directed m-cycle $u_0, u_{(m-1)/2}, u_{m-1}, u_{(m-3)/2}, \ldots, u_{(m+1)/2}, u_0$ is a partition of the arc set of K^*_{m+1} into directed m-cycles. □

5.8. Lemma. *For each odd integer $m \geqslant 5$, the complete directed graph K^*_{2m} can be decomposed into directed m-cycles.*

Proof. Let $m \geqslant 5$ be an odd integer, say $m = 2k+1$ for some integer k. Partition the vertex set of K^*_{2m} into two sets U and V of m vertices each, say $U = \{u_0, u_1, \ldots, u_{m-1}\}$ and $V = \{v_0, v_1, \ldots, v_{m-1}\}$.

Define the path P_1 by

$$P_1 : u_0, v_{m-1}, u_1, v_{m-2}, \ldots, v_{k+1}, u_k$$

and let P_2 be obtained from P_1 by reversing the direction of each arc in P_1. Each directed path has length $m-1$. Complete P_1 to a directed m-cycle C_1 by inserting the arc (u_k, u_0), and complete P_2 to a directed m-cycle C_2 by inserting the arc (u_0, u_k).

Define the permutation $\omega = (u_0\ u_1\ \cdots\ u_{m-1})(v_0\ v_1\ \cdots\ v_{m-1})$, as in Proposition 5.4. The directed m-cycles $C_1, \omega(C_1), \ldots, \omega^{m-1}(C_1)$, $C_2, \omega(C_2), \ldots, \omega^{m-1}(C_2)$ use all arcs between U and V, except those of the forms (u_i, v_i) and (v_i, u_i) for $i = 0, 1, \ldots, m-1$. In addition, they use the arcs of length $\pm k$ in U.

If $m \geqslant 7$, following the proof of Lemma 3.4, we can use arc lengths $\pm 2, \pm 3, \ldots, \pm k$ to form a directed cycle of length $2k-2$ in V starting with the arc (v_0, v_2) so that the vertex v_1 is not used. Remove the arc (v_0, v_2) and replace it with the directed path v_0, u_0, u_1, v_1, v_2 giving us a directed cycle C_3 of length $2k+1 = m$. The directed m-cycles $C_3, \omega(C_3), \ldots, \omega^{m-1}(C_3)$ use the remaining arcs between U and V, leave the arcs of length -1 and 2 in V, and use the arcs of length 1 in U.

The arcs of length -1 in V form a directed m-cycle as do the arcs of length 2, accounting for all arcs in V. All arc lengths in U are unused except 1 and $\pm k$. The arcs of length -1 form a directed m-cycle. If $k \equiv 0 \pmod 2$, then take the undirected

lengths in pairs 2,3 and 4,5 and so on until reaching $k-2, k-1$. By the main result of [5], each pair of lengths gives rise to a circulant graph that can be decomposed into two undirected Hamilton cycles. Orient both undirected cycles in the two possible directions to obtain four directed m-cycles. If $k \equiv 1 \pmod 2$, then pair the undirected lengths 3,4 and 5,6 and so on until reaching $k-2, k-1$. Repeat what was just done above for these pairs of undirected lengths. The two lengths ± 2 each produce a directed m-cycle and hence all arc lengths in U have been used.

The special case of $m = 5$ is easily done in a similar way because all arc lengths in U and V generate directed 5-cycles. □

We now present the proof of Theorem 5.1.

Proof of Theorem 5.1. Let m and n be positive integers with m odd, n even, $5 \leqslant m < n$, and $m \mid n(n-1)$. By Lemmas 5.6–5.8, we may assume that $m+2 < n < 2m$ or $2m < n < 3m$. The proof now breaks into these two cases.

Case 1. Suppose $m+2 < n < 2m$: Let $n = 2^e a$ with a odd, and let $m = a'b'$, where $a' \mid a$ and $b' \mid (2^e a - 1)$. Note that b' is odd and $b' = 1$ is impossible because $n < 2m$. Hence, $b' \geqslant 3$ holds in all cases. Also, $a' = 1$ implies $m \mid (n-1)$ and $n < 2m$ gives $m = n-1$. Therefore, by Lemma 5.7, we may assume that $a' \geqslant 3$.

We follow the strategy and notation used at the beginning of the proof of Theorem 4.1, that is, let $K_n^* = \overrightarrow{X}(n-1; S) \bowtie \{u\}$ where $S = \pm\{1, 2, \ldots, (n-2)/2\}$ and let the vertices of K_n^* be denoted by $u_0, u_1, \ldots, u_{n-2}, u$. Also ρ is the same permutation defined there and we will create one directed m-cycle by taking a directed $(m-2)$-path, formed from $m-2$ arcs of distinct length, together with arcs through the central vertex u. This directed m-cycle is rotated through all $n-1$ positions, using ρ and using all arcs incident with u and all arcs of the $m-2$ distinct lengths. The number of unused arc lengths is again

$$(n-2) - (m-2) = n - m = 2^e a - a'b'.$$

We partition the vertices $u_0, u_1, \ldots, u_{n-2}$ into b' segments, each of which has $\ell = (2^e a - 1)/b'$ vertices. Each segment will contribute a' arcs toward a directed m-cycle. Since $n > m+2$, it is easy to see that $a' \leqslant \ell - 2$ since a' and ℓ are both odd.

The number of distinct families of directed m-cycles we need to construct is

$$F = \frac{2^e a - a'b'}{a'}$$

which is odd. Since $n < 2m$ and since F and b' are both odd, we have that $F \leqslant b' - 2$. As before, we construct these families first and show that the remaining $m-2$ lengths may be used to form a directed path.

Define the directed path P by

$$P : u_0, u_{\ell-a'+1}, u_{-1}, u_{\ell-a'+2}, u_{-2}, \ldots u_{\ell-(a'+1)/2}, u_{-(a'-1)/2}, u_{(b'-1)\ell/2}.$$

Then

$$C = P \cup \rho^{\ell}(P) \cup \cdots \cup \rho^{(b'-1)\ell}(P)$$

is a directed m-cycle since $\ell-1<((b'-1)\ell+a'-1)/2<(n-2)/2$ as $b'\geqslant 3$ and $a'<\ell$. Thus the collection $\{C,\rho(C),\ldots,\rho^{\ell-1}(C)\}$ is a decomposition of $\overrightarrow{X}(n-1;B)$ into directed m-cycles where

$$B=\left\{\ell-a'+1,-(\ell-a'+2),\ldots,\ell-2,-(\ell-1),\frac{(b'-1)\ell+a'-1}{2}\right\}.$$

If $F=1$, then we need only show that the remaining arc lengths and arcs through the central vertex can be used to form another directed m-cycle. However, if $F\geqslant 3$, we must form more families of directed cycles before using the central vertex. Let $c=(F-1)/2$ so that $c\leqslant(b'-3)/2$.

Suppose first that $a'\equiv 3\pmod 4$. For each i with $1\leqslant i\leqslant c$, let $P_{i,0}$ be the directed path defined by

$$P_{i,0}:u_0,u_{i\ell+1},u_{-1},u_{i\ell+2},u_{-2},\ldots u_{i\ell+(a'-3)/4},u_{-(a'-3)/4},$$
$$u_{i\ell+(a'+5)/4},u_{-(a'+1)/4},\ldots,u_{-(a'-1)/2}.$$

Rather than introducing additional subcases, we explain another feature of the directed paths just defined. The jump over the vertex $u_{i\ell+(a'+1)/4}$ has been chosen carefully. Making this jump means we do not use arc length $i\ell+(a'-1)/2$. As the directed path $P_{i,0}$ continues, we may come to an arc of length $(i+1)\ell-(a'-1)/2$ depending on the size relationship between a' and ℓ. If we do encounter this length, we do so on an arc coming into a vertex of the interval $(u_{i\ell},u_{(i+1)\ell})$. So we insert another jump here. This requires at least $(a'+3)/2$ vertices in the open interval and $a'<\ell$ implies this to be the case.

The directed paths $P_{1,0},P_{2,0},\ldots,P_{c,0}$ have the following two important features: they all start with u_0 and end with $u_{-(a'-1)/2}$ and they use no arcs whose lengths are congruent to $\pm(a'-1)/2$ modulo ℓ. The only occurrence of arcs with lengths in these congruence classes are in the cycle C, where lengths $-(\ell-(a'-1)/2)$ and $((b'-1)\ell+a'-1)/2$ are used.

For each i with $1\leqslant i\leqslant c$ and for each j with $1\leqslant j\leqslant b'-1$, define $P_{i,j}$ to be the directed path $\rho^{j\ell}(P_{i,0})$, and define the family $\mathscr{P}_i$ to be $\{P_{i,j}\mid 0\leqslant j\leqslant b'-1\}$. For each i with $1\leqslant i\leqslant c$, the paths in $\mathscr{P}_i$ are vertex-disjoint because

$$\frac{a'-1}{2}+\frac{a'+3}{2}=a'+1\leqslant\ell-1.$$

We are going to use arcs whose lengths are in the congruence classes $\pm(a'-1)/2$ modulo ℓ to link the families of directed paths into directed cycles. We must be careful and avoid using either of the two lengths in C that belong to these congruence classes.

Write $F=8q+r$, where $1\leqslant r\leqslant 7$ and r is odd. We now explain the reason 8 arises in the expression for F. We use an auxiliary circulant graph of degree 4 to provide the linking scheme for families of the paths. Consider the circulant graph $X(b';\pm\{t,t+1\})$ with vertex set $v_0,v_1,\ldots,v_{b'-1}$. It may be decomposed into two

Hamilton cycles H_1, H_2 by applying the main result of [5]. Orient each cycle in both directions to obtain four directed cycles $\vec{H}_1, \vec{H}_2, \vec{H}_3, \vec{H}_4$ of length b'.

We use each $\vec{H}_k$ as a schematic for linking the directed paths of one of the families $\mathscr{P}_i$ into a directed m-cycle in the following manner. If there is an arc from v_{j_1} to v_{j_2} in $\vec{H}_k$, then we connect the terminal vertex of P_{i,j_1} to the initial vertex of P_{i,j_2}, using an arc of length $(j_2 - j_1)\ell + (a' - 1)/2$. In this way, each $\vec{H}_k$ links the directed paths in one family forming a directed cycle of length m. Also, when using the reversal of $\vec{H}_k$ to link another family $\mathscr{P}_j$ of directed paths, we will use an arc of length $(j_1 - j_2)\ell + (a' - 1)/2$. Since these two lengths sum to $a' - 1$, it follows that neither is the negative of the other. Thus, we can reverse the direction of each directed m-cycle created from a family of directed paths to create another directed m-cycle.

We now describe the F directed m-cycles. If $r = 1$, then the directed cycle C is the required directed m-cycle. If $r = 3$, join the terminal vertex of $P_{1,j}$ to the initial vertex of $P_{1,j+1}$, where the second subscript is evaluated modulo b', to obtain a directed m-cycle C_1. Reverse C_1 to obtain a third directed m-cycle C_1'.

For $r = 5$, join the terminal vertex of $P_{2,j}$ to the initial vertex of $P_{2,j+2}$ with the usual modulo b' arithmetic on the second coordinate. Call the resulting directed m-cycle C_2 and its reversal C_2'.

Finally, for $r = 7$, we obtain two more directed m-cycles by joining the terminal vertex of $P_{3,j}$ to the initial vertex of $P_{3,j-2}$ and taking the reversal of the resulting directed m-cycle.

If $q \geqslant 1$, we get eight directed m-cycles using the auxiliary circulant graph $X(b'; \pm\{3,4\})$. We get eight more by using the auxiliary circulant graph $X(b'; \pm\{5,6\})$. We continue in this way q times, ending with the circulant $X(b'; \pm\{2q+1, 2q+2\})$. The longest undirected length used in this linking process is $(2q+2)\ell + (a' - 1)/2$, which is clearly less than $(b' - 1)\ell/2 + (a' - 1)/2$ since $F \leqslant b' - 2$.

When $a' \equiv 1 \pmod 4$, we do close to what was just done so we need only describe the differences. One difference is that we use lengths congruent to $\pm(a' + 3)/2$ modulo ℓ as the linking arcs for the families of directed paths. The two values are not equal modulo ℓ because ℓ is odd.

The second difference, using $P_{1,0}$ to illustrate, is that we insert two jumps in the interval $(u_{-(a'+3)/2}, u_0)$ and have at most one jump in the interval $[u_{\ell+1}, u_{\ell+(a'-1)/2})$. The reason is that the arc with length $-(\ell + (a' + 3)/2)$ is coming into the interval $(u_{-(a'+3)/2}, u_0)$. We arbitrarily insert a second jump so that the last arc of $P_{1,0}$ is $u_{\ell+(a'-1)/2}u_{-(a'+3)/2}$. If we encounter an arc with length $2\ell - (a' + 3)/2$, we insert an additional jump in the interval $[u_{\ell+1}, u_{\ell+(a'-1)/2})$, thus ending the directed path with the arc $u_{\ell+(a'+1)/2}u_{-(a'+3)/2}$. All the other directed paths have the same modification with respect to insertion of jumps.

This construction works whenever $\ell > a' + 2$. If $\ell = a' + 2$, however, we do not have a sufficient number of vertices in a segment. In this case, we can proceed as follows. First notice that we may assume $a' \geqslant 5$ and that $F = (2b' - 1)/a'$. For

each i with $i = 1, \ldots, (F-1)/2$, define the directed paths $P_{i,0}$ as

$$P_{i,0} : u_0, u_{1+2i\ell}, u_{-1}, u_{2+2i\ell}, u_{-2}, \ldots, u_{-(\ell+1)/4}, u_{(\ell+9)/4+2i\ell}, u_{-(\ell+5)/4},$$
$$u_{(\ell+13)/4+2i\ell}, \ldots, u_{(\ell-1)/2+2i\ell}, u_{-(\ell-3)/2}$$

using arc lengths $1+2i\ell, -(2+2i\ell), 3+2i\ell, \ldots, -((\ell+1)/2+2i\ell), (\ell+5)/2+2i\ell, -((\ell+7)/2+2i\ell), \ldots, -(\ell-2+2i\ell)$. Notice that arcs of length $\pm(\pm(\ell-3)/2+(2i+1)\ell)$ have not been used in the directed paths. We can therefore use auxiliary circulant graphs of order b' with connection sets $\pm\{3,5\}$, $\pm\{7,9\}$, ... to link the families of directed paths into directed cycles, which are then reversed as before. If $F = 8q+r$, where $1 \leqslant r \leqslant 7$, this takes care of $8q$ directed cycles. The upper bound for F guarantees that the length of the longest undirected edge thus used will be less than $(b'-1)\ell/2 + (a'-1)/2$. The additional r cycles are handled in a way similar to the case $a' \equiv 3 \pmod 4$. For example, if $r = 7$, then arc lengths $\pm(\ell + (\ell-3)/2)$, $\pm(2\ell - (\ell-3)/2)$, and $\pm(4\ell - (\ell-3)/2)$ are used to link families $\mathscr{P}_{1,0}$, $\mathscr{P}_{2,0}$, $\mathscr{P}_{3,0}$, and their reversed copies, while the last remaining directed cycle is C.

We obtain the collection $\mathscr{C}$ of directed m-cycles by taking $D, \rho(D), \ldots, \rho^{\ell-1}(D)$, where D runs over all the cycles $C_1, C_1', C_2, C_2', \ldots$ constructed above for $a' \equiv 1 \pmod 4$ or $a' \equiv 3 \pmod 4$. The most important feature of the cycles in $\mathscr{C}$ is that if length d is used then so is $-d$, and all lengths used have undirected length in $(\ell, (n-2)/2)$.

We complete Case 1 by showing the remaining $m-2$ arc lengths can be used to form a directed path of length $m-2$ using each length once. Let L be the set of lengths from

$$\pm\left\{\ell, \ell+1, \ldots, \frac{(b'-1)\ell + a' - 3}{2}, \frac{(b'-1)\ell + a' + 1}{2}, \ldots, \frac{n-4}{2}\right\}$$

not used in the construction of the directed cycles in $\mathscr{C}$. Note that the lengths $\pm\ell$ have not been used.

By Lemma 4.4, there is a directed path Q from u_0 to $u_{n/2}$ using lengths in $L \cup \{n/2\}$, and not using any vertices of $(u_0, u_\ell) \cup (u_{n/2}, u_{n/2+\ell})$. Extend Q by adding the directed path

$$u_{n/2}, u_{(n-2)/2+\ell}, u_{\ell-1}, u_1, u_{\ell-2}, u_2, \ldots, u_{(\ell+1)/2}, u_{(\ell-1)/2}.$$

The extension uses arc lengths $\ell - 1$, $-n/2 = (n-2)/2, -(\ell-2), \ell-3, \ldots, 2, -1$.

Continue from $u_{(\ell-1)/2}$ to $u_{(n-a'-1)/2+\ell}$ which has length $(\ell + n - a')/2 = -((b'-1)\ell + a' - 1)/2$. From $u_{(n-a'-1)/2+\ell}$, we finish with arcs having lengths $-(\ell - a'), \ell - a' - 1, -(\ell - a' - 2), \ldots, -2, 1$. These arcs do not encounter vertices already used because $a' \geqslant 3$. We now use the central vertex u to give us a directed cycle of length m completing this case.

Case 2. Suppose $2m < n < 3m$: Let $n = 2^e a$ with a odd, and let $m = a'b'$, where $a' \mid a$ and $b' \mid (2^e a - 1)$. Both a' and b' are odd, and $n > 2m$ implies that $a' \geqslant 3$ and $b' \geqslant 3$ both hold. The notation remains the same as in Case 1. Note that $b'\ell = n-1$ and ℓ odd imply $2a' < \ell < 3a'$. Recall $c = (F-1)/2$ where $F = (n-m)/a'$ is the number of

families of directed m-cycles that need to be constructed. It is then easy to verify that $c = (b'(\ell - a') - a' + 1)/2a'$ so that $c \leqslant b' - 1$.

Before continuing, it is helpful to consider some specific arithmetic conditions. Suppose that $\ell = 2a' + 1$. Then, $n = 2m + b' + 1$ which in turn implies that a' is an odd divisor of $b' + 1$. Thus, $b' = 3$ and $b' = 7$ cannot occur when $\ell = 2a' + 1$. The case $b' = 5$ implies that $a' = 3$ giving $m = 15$ and $n = 36$. This special case is handled at the end of the proof. Hence, we may assume $b' \geqslant 9$. This fact together with the fact that $\ell = 2a' + 1$ imply that $c \leqslant b' - 3$. Therefore, if $\ell = 2a' + 1$, we may assume that $c \leqslant b' - 3$ and if $c' = b' - 1$, then $\ell > 2a' + 1$. However, ℓ odd gives $\ell \geqslant 2a' + 3$ in this case.

We make a small modification to the directed m-cycle C used in Case 1 to avoid using the length $\ell - (a' - 1)/2$. Assume $a' \equiv 3 \pmod 4$. Let P be the directed path

$$P : u_0, u_{\ell - a'}, u_{-1}, u_{\ell - a' + 1}, u_{-2}, \ldots, u_{\ell - (a' + 1)/2}, u_{-(a' - 1)/2}, u_{(b' - 1)\ell/2},$$

where we skip a vertex in the interval $(u_{\ell - a'}, u_{\ell - (a' + 1)/2})$ when we come to the arc whose length is $\ell - (a' - 1)/2$. The paths

$$P, \rho^{\ell}(P), \rho^{2\ell}(P), \ldots, \rho^{(b' - 1)\ell}(P)$$

have no internal vertices in common so that their union forms a directed m-cycle C. Then the collection $\{C, \rho(C), \rho^2(C), \ldots, \rho^{\ell - 1}(C)\}$ is a decomposition of $\overrightarrow{X}(n - 1; B)$ where

$$B = \left\{ \ell - a', -(\ell - a' + 1), \ldots, -\left(\ell - \frac{a' + 1}{2}\right), \ell - \frac{a' - 3}{2}, \right.$$
$$\left. \ldots, \ell - 2, -(\ell - 1), \frac{(b' - 1)\ell + a' - 1}{2} \right\}.$$

For much of what follows, it is a little simpler to construct undirected m-cycles and obtain two directed m-cycles from each cycle by giving it the two possible cyclic orientations. Thus, we emphasize that when the word path or cycle is used without the preceding adjective 'directed' in what follows, we are talking about undirected paths and cycles.

Observe that $F \geqslant 5$ so that $c \geqslant 2$. We now define c families of m-cycles. Let

$$P_0 : u_0, u_{\ell - 2a' + 1}, u_{-1}, u_{\ell - 2a' + 2}, u_{-2}, \ldots, u_{-(a' - 1)/2};$$

let

$$P_i : u_0, u_{i\ell + 1}, u_{-1}, u_{i\ell + 2}, u_{-2}, \ldots, u_{-(a' - 3)/4}, u_{i\ell + (a' + 5)/4},$$
$$u_{-(a' + 1)/4}, \ldots u_{-(a' - 1)/2}$$

for $i = 1, 2, \ldots, \lceil (c - 1)/2 \rceil$; and let

$$Q_i : u_0, u_{i\ell + a' + 1}, u_{-1}, u_{i\ell + a' + 2}, u_{-2}, \ldots, u_{-(a' - 1)/2}$$

for $i = 1, 2, \ldots, \lfloor (c - 1)/2 \rfloor$, where we skip a vertex in the interval $(u_{i\ell + a' + 1}, u_{i\ell + (3a' - 1)/2}]$ if the edge length $(i + 1)\ell - (a' - 1)/2$ is encountered.

There are some features of these paths worth noting. All of them have length $a' - 1$, all of them have u_0 and $u_{-(a' - 1)/2}$ as end vertices, P_0 uses edges of lengths

$\ell - 2a' + 1$ through $\ell - a' - 1$, and P_i uses edges of lengths $i\ell + 1$ through $i\ell + a'$ except for omitting length $i\ell + (a' - 1)/2$. The longest edge length appearing in any P_i is $\lceil (c-1)/2 \rceil \ell + a'$. If $c = b' - 1$, then $\ell > 2a' + 1$ so that $\lceil (c-1)/2 \rceil \ell + a' < (n-2)/2$. Otherwise $c < b' - 1$ so that $\lceil (c-1)/2 \rceil \ell + a' < (b'-1)\ell/2 + a' \leqslant (n-2)/2$.

The lengths used by Q_j start with $j\ell + a' + 1$ and go through $j\ell + 2a' - 1$ if we skip no vertex. Otherwise, the lengths go through $j\ell + 2a'$ with $(j+1)\ell - (a'-1)/2$ omitted. Since $c \leqslant b' - 1$, the largest possible value of j is $(b'-3)/2$. Thus, the longest edge length appearing in any Q_j is at most $(b'-3)\ell/2 + 2a'$ which is smaller than $(n-2)/2$. So all edge lengths are smaller than $(n-2)/2$.

Since the paths, other than P and P_0, use no edges with lengths congruent to $\pm(a'-1)/2$ modulo ℓ, these are the edges we use to link the paths and form m-cycles. Since the linking scheme has been described in detail earlier and the mechanics of linking remain the same, we need only verify that there are enough lengths available to perform the linking. The available lengths are

$$\ell \pm \frac{a'-1}{2}, 2\ell \pm \frac{a'-1}{2}, \ldots, \frac{(b'-3)\ell}{2} \pm \frac{a'-1}{2}, \frac{(b'-1)\ell}{2} - \frac{a'-1}{2}.$$

The $b' - 2$ potential linking lengths are distinct and bounded above by $(b'-1)\ell/2 - (a'-1)/2$ which is smaller than $(n-2)/2$. Since $c \leqslant b' - 1$, it is clear we can form m-cycles using these paths for all values of c other than $c = b' - 1$. We consider that case separately.

After linking the paths to form c m-cycles, we obtain $2c$ directed m-cycles by orienting all the m-cycles in both possible directions. We are left with $m - 2$ unused arc lengths. If we can produce a directed path using each arc length once, we have a decomposition into directed m-cycles.

It is easy to guarantee that the length $\ell - (a'-1)/2$ actually is used in the linking scheme. Since that length is used, the unused arc lengths are

$$\pm\{1, 2, \ldots, \ell - 2a'\} \cup \left\{ -(\ell - a'), \ell - a' + 1, \ldots, \ell - \frac{a'+1}{2}, -\left(\ell - \frac{a'-3}{2}\right), \right.$$
$$\left. \ldots, -(\ell - 2), \ell - 1 \right\} \cup L \cup \left\{ -\frac{(b'-1)\ell + a' - 1}{2}, \pm\frac{n-2}{2} \right\},$$

where L is a subset of $\pm\{\ell, \ell+1, \ell+2, \ldots, (n-4)/2\}$.

By Lemma 4.4, there is a directed path Q from u_0 to $u_{n/2}$ using each arc length in $L \cup \{n/2\}$ once, and not using any vertices of (u_0, u_ℓ) or $(u_{n/2}, u_{n/2+\ell})$. We extend Q by adding the directed path

$$u_{n/2}, u_{n/2+\ell-1}, u_{\ell-1}, u_1, u_{\ell-2}, u_2, \ldots, u_{(a'-3)/4}$$

which uses arc lengths $\ell - 1, -n/2, -(\ell-2), \ell - 3, -(\ell-4), \ldots, -(\ell - (a'-3)/2)$.

We continue the extension with the directed path

$$u_{(a'-3)/4}, u_{\ell-(a'+5)/4}, u_{(a'+1)/4}, \ldots, u_{\ell-(a'+1)/2}, u_{(a'-1)/2}$$

using arc lengths $\ell - (a'+1)/2, -(\ell - (a'+3)/2), \ell - (a'+5)/2, \ldots, -(\ell - a')$.

We continue the extension with the directed path

$$u_{(a'-1)/2}, u_{\ell-(3a'+1)/2}, u_{(a'+1)/2}, \ldots, u_{(\ell-a'-2)/2}, u_{(\ell-a')/2}$$

using arc lengths $\ell-2a', -(\ell-2a'-1), \ell-2a'-2, \ldots, 1$.

We add the arc from $u_{(\ell-a')/2}$ to $u_{n/2+\ell-a'}$ which has length

$$-\frac{(b'-1)\ell+a'-1}{2}.$$

We then complete the directed $(m-2)$-path with the extension

$$u_{n/2+\ell-a'}, u_{n/2+a'}, u_{n/2+\ell-a'-1}, \ldots, u_{(n+\ell-1)/2}$$

which uses the remaining unused lengths.

This leaves us with the special case of $c=b'-1$. We have already seen that when $c=b'-1$, we may assume $\ell>2a'+1$. Since ℓ is odd, we have $\ell\geqslant 2a'+3$ and this gives enough room to modify our construction slightly so that we obtain a decomposition. The last edge of the path P_1 is $u_{\ell+(a'+1)/2}u_{-(a'-1)/2}$ of length $\ell+a'$. Change the edge to $u_{\ell+(a'+1)/2}u_{-(a'+1)/2}$ which has length $\ell+a'+1$. If the path Q_1 exists (depending on the value of c), then this is the length of the first edge of Q_1. So we also must modify Q_1 so it no longer uses an edge of length $\ell+a'+1$. What we do is add 2 to the subscript of every vertex of Q_1 in the interval $(u_{\ell+a'}, u_{2\ell})$ taking care that we skip the appropriate vertex so as not to use an edge of length $2\ell-(a'-1)/2$.

Now we use the edge from the terminal vertex $u_{-(a'+1)/2}$ of the modified path P_1 to $u_{(b'-1)\ell/2}$ to obtain path of length a' which leads to an m-cycle. However, the latter edge we are using for linking the family of paths arising from P_1 has length $(b'-1)\ell/2+(a'+1)/2$ and this length is being used in the current path $P_{(b'-1)/2}$. So we modify the latter path by skipping over two consecutive vertices in the interval $(u_{(b'-1)\ell/2}, u_{(b'-1)\ell/2+(a'+1)/2}]$ instead of the single vertex that avoids using an edge of length $(b'-1)\ell/2+(a'+1)/2$. This modification avoids both lengths $(b'-1)\ell/2+(a'-1)/2$ and $(b'-1)\ell/2+(a'+1)/2$. The price we pay is that the longest length being used in the modified path $P_{(b'-1)/2}$ is $(b'-1)\ell/2+a'+1$ rather than $(b'-1)\ell/2+a'$. But this longest length is smaller than $(n-2)/2$ provided that $\ell\geqslant 2a'+5$, and everything works. On the other hand, if $\ell=2a'+3$, then it is not difficult to see that $c=b'-1$ forces $a'=5$, a contradiction. This leaves us only $b'-2$ families of paths to link together and that is precisely the number of linking edges of different lengths we have available.

The directed $(m-2)$-path for the central vertex is constructed as before. This completes the subcase of $a'\equiv 3 \pmod 4$. In the special case that $a'=3$, the preceding construction works, but many of the paths become trivial.

The last general subcase we need to consider is $a'\equiv 1 \pmod 4$. There is considerable similarity with the preceding case so that we may outline what is done for this subcase. The directed path of length a' on which the directed cycle C is based is

$$u_0, u_{\ell-a'-1}, u_{-1}, \ldots, u_{\ell-(a'+3)/2}, u_{-(a'-1)/2}, u_{(b'-1)\ell/2},$$

where we skip the vertex in the interval $(u_{\ell-a'}, u_{\ell-(a'+3)/2})$ giving rise to the arc length $\ell-(a'-1)/2$. This accounts for arc lengths

$$\ell-a'-1, -(\ell-a'), \ldots, -(\ell-)(\frac{a'+1}{2}, \ell-\frac{a'-3}{2}, \ldots, -(\ell-2), \frac{(b'-1)\ell}{2}+\frac{a'-1}{2})).$$

The path P_0 is given by

$$P_0 : u_0, u_{\ell-2a'}, u_{-1}, \ldots, u_{\ell-(3a'+3)/2}, u_{-(a'-1)/2}$$

using edges of lengths $\ell-2a', \ell-2a'+1, \ldots, \ell-a'-2$. For $i=1,2,\ldots,\lceil (c-1)/2 \rceil$, define the path P_i by

$$P_i : u_0, u_{i\ell+2}, u_{-1}, \ldots, u_{i\ell+(a'+3)/2}, u_{-(a'-1)/2},$$

where we skip the vertex in the interval $(u_{i\ell+1}, u_{i\ell+(a'+3)/2})$ giving rise to the edge of length $i\ell+(a'-1)/2$. The edge lengths being used are

$$i\ell+2, i\ell+3, \ldots, i\ell+(a'-3)/2, i\ell+(a'+1)/2, \ldots, i\ell+a'+1.$$

For $i=1,2,\ldots,\lfloor (c-1)/2 \rfloor$, define the path Q_i by

$$Q_i : u_0, u_{i\ell+a'+2}, u_{-1}, \ldots, u_{-(a'-1)/2},$$

where the vertex preceding $u_{-(a'-1)/2}$ is $u_{i\ell+(3a'+1)/2}$ if the edge length $(i+1)\ell-(a'-1)/2$ is not encountered, and is $u_{i\ell+(3a'+3)/2}$ if the edge length $(i+1)\ell-(a'-1)/2$ is encountered thereby forcing us to skip a vertex. The edge lengths being used here start with $i\ell+a'+2$ and go to either $i\ell+2a'$ or $i\ell+2a'+1$, depending on whether or not $(i+1)\ell-(a'-1)/2$ was encountered. One important fact about both P_i and Q_i is that we need $\ell \geqslant 2a'+3$ in order to have the distinct paths used to link together to form an m-cycle be vertex-disjoint.

As before, we have $b'-2$ lengths congruent to $\pm(a'-1)/2$ modulo ℓ available to link families of paths. So as long as $c \leqslant b'-2$ we can complete the decomposition when $\ell \geqslant 2a'+3$ via the same method used in the preceding case.

We now are left with special cases. One special case is $\ell=2a'+1$ and the other is $c=b'-1$. As we saw earlier, when $\ell=2a'+1$, we may assume $b' \geqslant 9$ and this, in turn, implies $c \leqslant b'-3$. Thus, there are two cases to consider: $\ell=2a'+1$ and $c \leqslant b'-3$, and $c=b'-1$ and $\ell \geqslant 2a'+3$.

When $c=b'-1$ and $\ell \geqslant 2a'+3$, modify P_1 by removing the last edge joining $u_{\ell+(a'+3)/2}$ and $u_{-(a'-1)/2}$, and replacing it with the edge joining $u_{\ell+(a'+3)/2}$ and $u_{\ell+1}$. The latter edge has length $(a'+1)/2$. We use edges of length 1 (which have not been used) as the linking edges for the modified P_1. However, P_0 may use an edge of length $(a'+1)/2$. If it does not, no further modifications are necessary. If it does, then start P_0 with $u_0, u_{\ell-2a'-2}$ which is possible without repetition of edge lengths whenever $\ell \geqslant 2a'+5$. In that case, continue as before and skip the vertex in the interval $(u_{\ell-2a'-2}, u_{\ell-(3a'+3)/2})$ with length $(a'+1)/2$. If $\ell=2a'+3$, however, it is not difficult to see that $a'=5$ and $b'=3$, which yields a special case $m=15$, $n=40$ to be handled at the end of the proof. No other modifications are necessary.

We now have one less family of paths to link together and there are now sufficiently many edges whose lengths are congruent to $\pm(a'-1)/2$ modulo ℓ to do the job. The unused arc lengths have been changed slightly, however, the construction of the directed $(m-2)$-path for the central vertex remains essentially the same as before.

We have left $\ell = 2a'+1$ for last. Because the proof is intricate, we are going to present it carefully. The directed path P of length a' we use to obtain the directed m-cycle C is given by

$$P: u_0, u_{a'+2}, u_{-1}, \ldots, u_{(3a'+1)/2}, u_{-(a'-1)/2}, u_{(b'-1)\ell/2}.$$

It uses arc lengths

$$a'+2, -(a'+3), \ldots, -(\ell-1), \frac{(b-1)\ell + a' - 1}{2}.$$

Define the path P_0 by

$$P_0: u_0, u_2, u_{-2}, u_3, u_{-3}, \ldots, u_{(a'+1)/2}, u_{-(a'+1)/2}$$

using edges with lengths $2, 4, 5, \ldots, a'+1$.

The segment length $\ell = 2a'+1$ is not quite enough to fit two paths per segment. What we do is construct three paths for every pair of successive segments and show this gives us enough paths.

Define the path P_1 by

$$P_1: u_0, u_{\ell+2}, u_{-1}, u_{\ell+3}, u_{-2}, \ldots, u_{\ell+(a'+1)/2}, u_{-(a'+1)/2},$$

where we omit the vertex in the interval $(u_{-(a'+1)/2}, u_0)$ which would have given rise to an edge of length $\ell + (a'+1)/2$. Note that P_1 uses edges of lengths $\ell+2$ through $\ell + a' + 1$ with $\ell + (a'+1)/2$ omitted.

The path Q_1 is the one requiring the most care. We start the path as

$$u_0, u_{\ell+a'+2}, u_{-1}, \ldots, u_{-(a'-5)/4}, u_{\ell+a'+(a'+3)/4}, u_{-(a'+3)/4}$$

noting we have omitted the vertex $u_{-(a'-1)/4}$ so that the edge length $2\ell - (a'+1)/2$ is omitted. As we continue the path, no more vertices will be omitted from the interval $(u_{-\ell}, u_0)$ so that the vertex of Q_1 in that interval furthest from u_0 is going to be $u_{-(a'+1)/2}$. There has to be room for those vertices to fit under powers of ρ^ℓ. This means the last successive vertex we can use in the interval $(u_\ell, u_{2\ell})$ is $u_{\ell+a'+(a'-1)/2}$. But there are only three vertices to follow this in the path. The next edge is $u_{\ell+a'+(a'-1)/2}u_{-(a'-1)/2}$. However, under rotation the vertex $u_{2\ell-(a'-1)/4}$ is available. We use it to complete the path. It is then clear we do not use any lengths congruent to $\pm(a'+1)/2$ modulo ℓ.

We define the path R_1 to be

$$u_0, u_{2\ell+(a'-1)/2}, u_{-2}, \ldots, u_{2\ell+a'-2}, u_{-(a'+1)/2}$$

using edge lengths

$$2\ell + \frac{a'-1}{2}, 2\ell + \frac{a'+3}{2}, \ldots, 2\ell + \frac{3a'-3}{2},$$

where the last length is smaller than $3\ell - (a'+1)/2$. If $a' \geqslant 9$, the edge lengths used in the paths P_0, P_1, Q_1, and R_1 are pairwise distinct. If $a' = 5$, however, to avoid duplication of edge lengths, we define R_1 as $u_0, u_{2\ell+4}, u_{-1}, u_{2\ell+6}, u_{-3}$, thus using edge lengths $2\ell+4, 2\ell+5, 2\ell+7$, and $2\ell+9$, none of which is congruent to $\pm(a'+1)/2$ modulo ℓ.

There are sufficiently many edges of length congruent to $\pm(a'+1)/2$ to link together the paths to form cycles of length m. The question is whether or not we are able to construct c paths using the scheme just described. We now address this issue.

We form P_2, Q_2, R_2 by adding 2ℓ to the subscripts of all the vertices of P_1, Q_1, R_1, respectively, not in the interval $(u_{-\ell}, u_0]$. This increases the lengths of all edges by 2ℓ. We form P_3, Q_3, R_3 by adding 4ℓ and so on. The number of paths we can obtain this way clearly is limited by the fact we does not want any edge lengths to exceed $(n-4)/2$.

We have $c = (b'-1)/2 + (b'+1)/(2a')$, $a' \equiv 1 \pmod 4$, and $a' \mid (b'+1)$. Let us look at two examples. The smallest value of b' for which the preceding conditions can be satisfied is $b' = 9$ with $a' = 5$. We then have $c = 5$ and we can construct the paths P_0, P_1, Q_1, R_1, P_2 without reaching length $(n-2)/2$. The next viable value of b' is $b' = 17$ which implies $c = 9$. We can then construct the paths $P_0, P_1, Q_1, R_1, P_2, Q_2, R_2, P_3, Q_3, R_3, P_4, Q_4$.

It is easy to see that we need $2(c-i)/3 + i$ intervals of length ℓ for the edge lengths used in the paths $P_0, P_1, Q_1, R_1, P_2, \ldots$ where $c \equiv i \pmod 3$ for $i \in \{1,2,3\}$, and that $2(c-i)/3 + i \leqslant (b'-1)/2$ since $b' \geqslant 9$. Thus, at least c paths can be constructed for all viable values of b'. This takes care of the case $\ell = 2a' + 1$.

The last two steps required to complete the proof of Theorem 5.1 are solutions for $m = 15$ and $n = 36$ or $n = 40$. As usual, we will use a central vertex. Assume first that $n = 36$. The directed path $P: u_0, u_5, u_{34}, u_{14}$ generates a directed cycle C of length 15 where $C = P \cup \rho^7(P) \cup \rho^{14}(P) \cup \rho^{21}(P) \cup \rho^{28}(P)$. The three paths u_0, u_1, u_{34}, u_7; u_0, u_3, u_{34}, u_{21}; and u_0, u_9, u_{33}, u_{21} lead to six different directed cycles of length 15. Finally, the directed 13-path

$$u_0, u_7, u_{32}, u_{11}, u_{30}, u_{12}, u_{28}, u_{14}, u_{24}, u_{17}, u_{23}, u_6, u_1, u_{21}$$

gives the last remaining directed 15-cycle using the central vertex.

In the case $n = 40$, the directed cycle $P \cup \rho^{13}(P) \cup \rho^{26}(P)$ is generated by the directed path $P: u_0, u_9, u_{38}, u_{10}, u_{37}, u_{13}$. Four directed cycles are generated by paths $u_0, u_1, u_{36}, u_2, u_{34}, u_{13}$ and $u_0, u_2, u_{38}, u_5, u_{36}, u_{13}$. Finally, the remaining arc lengths are used in the directed cycle that connects the central vertex with the directed 13-path

$$u_0, u_{13}, u_{38}, u_{16}, u_{36}, u_{19}, u_{33}, u_{20}, u_{32}, u_{12}, u_1, u_{11}, u_2, u_{26}.$$

This now completes the proof of Theorem 5.1. □

Theorems 3.1, 4.1, 5.1 together with the observations following the proof of Lemma 1.2 for m and n both being odd serve to prove the main result, Theorem 1.1.

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