Existence of Optimal Bandlimited Controls
without Convexity Condition*

N. U. Ahmed

Department of Electrical Engineering, University of Ottawa, Ottawa, Ontario

In this paper (vector valued) bandlimited functions have been used as the class of admissible controls. Essential properties of the class of admissible controls are developed. Properties of attainable sets and the set of trajectories of the dynamical system under consideration are presented. These results are proved without the convexity condition usually imposed on the "velocity field" of the dynamical system. In the final section several interesting optimal control problems are solved using the compactness properties of attainable sets and admissible trajectories.

1. INTRODUCTION

We are interested in the problem of optimal control of the system $S$ described by

$$
S: \begin{cases}
\frac{dx(t)}{dt} = f(t, x(t), u(t)) & \text{a.e. } t \in R_0 = [0, \infty) \\
x(0) = x_0, & u \in B
\end{cases}
$$

where for each $t \in R_0$, $x(t) \in E^n$, $u(t) \in E^m$, $f: R_0 \otimes E^m \otimes E^n \to E^n$ and $B$ is the class of admissible controls as defined in Section 2.

**Basic Assumptions**

Throughout the paper the function $f$ is assumed to satisfy the following properties:

$H_0$: for almost all $t \in R_0$, $f$ is continuous on $E^n \otimes E^m$ to $E^n$ and for each pair $(\xi, \eta) \in E^n \otimes E^m$, $f(\cdot, \xi, \eta)$ is a measurable function on $R_0$ to $E^n$.

* This work was supported in part by the National Research Council of Canada under grant No. A-7109.

Copyright © 1976 by Academic Press, Inc.
All rights of reproduction in any form reserved.
\( H_1 \): there exists a nonnegative, measurable and locally Lebesgue integrable function \( K \) defined on \( R_0 \) so that
\[
|f(t, x, u) - f(t, y, u)| \leq K(t) |x - y| \text{ a.e. on } R_0
\]
for all \( u \in U \subset E^n \) where \( U \) is bounded. \( K \) may depend on the set \( U \) and \( H_2 \): there exist \( \alpha, \beta \geq 0 \) so that
\[
|f(t, x, u)| \leq K(t) [x + \beta |x|] \text{ a.e. on } R_0
\]
for all \( u \in U \). The function \( K \) and the numbers \( \alpha, \beta \) may depend on the set \( U \subset E^n \).

The state space \( E^n \) is assumed to be equipped with the norm
\[
|x| = \sum_{i=1}^n |x_i|
\]
and the corresponding metric \( d \) defined by
\[
d(x, y) = \sum_{i=1}^n |x_i - y_i|.
\]
The Hausdorff distance between the set \( A, B \subset E^n \) is defined by
\[
d_H(A, B) = \max\{\sup_{x \in B} d(A, x), \sup_{x \in A} d(x, B)\},
\]
where
\[
d(A, x) = \inf_{z \in A} d(z, x).
\]

Statement of the Problem

In this paper we consider the following control problems:

(i) Find a control \( u^* \in B \) that minimizes the cost functional \( J(u) \) defined by
\[
J(u) = \int_0^{T^*} f_0(t, x_u(t), u(t)) \, dt, \quad u \in B
\]
where \( x_u \) is the response of the system \( S \) corresponding to the control \( u \in B \) and initial state \( x_u(0) = x_0 \) and \( f_0 \) is a nonnegative scalar valued function defined on \( R_0 \times E^n \times E^n \).

(ii) Find a control \( u^* \in B \) that minimizes the functional \( J(u) \) defined by
\[
J(u) = \inf\{t \geq 0: x_u(t) \in T(t)\}, \quad u \in B
\]
where \( x_u \) is the solution of the system \( S \) corresponding to the control \( u \in B \) and initial state \( x_0 \) and \( T(t), t \geq 0 \) is a moving target set in the state space \( E^n \).

(iii) Find a control \( u^* \in B \) that steers the system \( S \) from the initial state \( x_0 \in E^n \) to the set \( T(t^*) \subset E^n \) in the minimum time \( t^* \) and also minimizes the cost functional \( J(u) \) defined by
\[
J(u) = \int_0^{t^*} f_0(t, x_u(t), u(t)) \, dt, \quad u \in B
\]
where \( f_0 \) and \( T(t), t \geq 0 \) are as defined in problems (i) and (ii).
(iv) Find a control $u^* \in B$ that maximizes the functional $J(u)$ defined by

$$J(u) = \mu \{ t \geq 0 : x_u(t) \in T(t) \}, \quad u \in B$$

where $x_u$ is the response of the system $S$ corresponding to the control $u$ and initial state $x_0; T(t), t \geq 0$ is the moving target as in problem (ii) and $\mu$ is the Lebesgue measure. This is a problem of maximizing the stay of the trajectory in a given target set $T(t), t \geq 0$.

In Section 2, bandlimited (vector valued) functions are introduced for the class of admissible controls. Useful and interesting properties of the admissible class are presented. In Section 3, properties of attainable sets and trajectories of the system $S$ (corresponding to the class of admissible controls) are developed.

These results are then utilized in the final Section 4 to prove the existence of optimal controls for all the four problems as presented above. The optimization is carried out over the class of bandlimited controls.

It is important to mention that in optimal control problems with bounded measurable functions as admissible controls, it is necessary that the function $f(t, x, u)$ satisfy the convexity condition (Hermes and LaSalle, 1969, Theorem 20.1, p. 107; Oğuztöreli, 1966, Theorem 8.1, p. 184) with respect to the control parameter $u$ for each $t$ and $x$. This is illustrated well by a counterexample (Hermes and LaSalle, 1969, Example 2.1, p. 106).

In the case of bandlimited controls this condition is not necessary as demonstrated by the results (Proposition 3.3, Corollary 3.1) of this paper. This is significant since in many physical problems the convexity condition is not necessarily satisfied and further the controllers are not inertialess; that is the controls are constrained to be frequency limited. Results on the existence of optimal controls in the absence of convexity were also obtained by Neustadt (1963, pp. 110–117).

2. Admissible Controls

Let $\Omega$ be a compact subset of the real line $R$, $E^m$ the Euclidean $m$-space, $R_0 = [0, \infty)$ and $\Omega' = R \setminus \Omega$. Define $\hat{u}(\omega) = 1/(2\pi)^{1/2} \int_0^\infty u(t) e^{-i\omega t} \, dt$, $\omega \in R$ to be the Fourier transform of $u \in L_2(R_0, E^m)$, where the norm in $L_2(R_0, E^m)$ is given by

$$\| u \| = \left( \int_0^\infty | u(t) |^2 \, dt \right)^{1/2} = \left( \int_0^\infty \sum_{i=1}^m | u_i(t) |^2 \, dt \right)^{1/2}.$$
Let $B_{\Omega} = \{u \in L_2(R_0, E^m) : \hat{u}(\omega) = 0 \in E^m \text{ for almost all } \omega \in \Omega'\}$. This is the space of bandlimited vector valued functions.

**Lemma 2.1.** $B_{\Omega}$ is a closed linear subspace of $L_2(R_0, E^m)$.

**Proof.** Clearly $B_{\Omega}$ is a linear subspace of $L_2(R_0, E^m)$. For the closure let $\{u_n\} \in B_{\Omega}$ so that $u_n$ converges strongly to $u_0$.

Clearly $u_0 \in L_2(R_0, E^m)$ and consequently by Plancherel’s theorem (Wiener, 1933, Theorem 2, p. 69) $\hat{u}_0$ exists and belongs to $L_2(R, E^m)$, and $\int_{\Omega'} |u_0(t)|^2 dt = \int_R |\hat{u}_0(\omega)|^2 d\omega$. We must show that $\hat{u}_0(\omega) = 0$ a.e on $\Omega'$. By Parseval’s theorem (Wiener, 1933, Theorem 2, 3, p. 70) we have

$$\int_0^\infty |u_n(t) - u_0(t)|^2 dt = \int_R |\hat{u}_n(\omega) - \hat{u}_0(\omega)|^2 d\omega.$$  

Since $u_n$ converges strongly to $u_0$ it follows from the Parseval’s equality that

$$\lim_{n \to \infty} \int_R |\hat{u}_n(\omega) - \hat{u}_0(\omega)|^2 d\omega = 0.$$  

Consequently for any measurable set $E \subset R \lim_{n \to \infty} \int_E |\hat{u}_n(\omega) - \hat{u}_0(\omega)|^2 d\omega = 0$ also. Since, for each $n$, $\hat{u}_n(\omega) = 0$ on $\Omega'$, $\int_{\Omega'} |\hat{u}_n(\omega) - \hat{u}_0(\omega)|^2 d\omega = \int_{\Omega'} |\hat{u}_n(\omega) - \hat{u}_0(\omega)|^2 d\omega + \int_{\Omega'} |\hat{u}_0(\omega)|^2 d\omega$.

Taking the limit on either side we have $\int_{\Omega'} |\hat{u}_0(\omega)|^2 d\omega = 0$.

Thus $\hat{u}_0(\omega) = 0$ a.e. on $\Omega'$. Therefore $u_0 \in B_{\Omega}$. This completes the proof.

**Corollary 2.1.** $B_{\Omega}$ is a Banach space.

**Proof.** $B_{\Omega}$, being a closed linear subspace of the Banach space $L_2(R_0, E^m)$, is a Banach space with $L_2$ norm. This is called the space of bandlimited functions (with values in $E^m$).

**Lemma 2.2.** Let $B$ be any bounded subset of the Banach space $B_{\Omega}$. Then $B \subset B_{\Omega} \cap L_\infty(R_0, E^m)$ and is bounded as a subset of $L_\infty(R_0, E^m)$.

**Proof.** The proof is immediate.

**Remark.** The elements of $B$ are uniformly bounded on $R_0$ to $E^m$ and consequently there exists a compact set $U \subset E^m$ so that $u(t) \in U$ for all $t \in R_0$ whenever $u \in B$.

The following result is very useful in the study of optimal control problems with bandlimited controls.
PROPOSITION 2.1. The necessary and sufficient conditions for a subset $B$ of the Banach space $B_\Omega$ to be conditionally compact are that

(i) $B$ is bounded.

(ii) $\lim_{s \to \infty} \int_0^s |u(t)|^2 \, dt = 0$ uniformly with respect to $u \in B$.

Proof. Since $B_\Omega \subset L_2(L_2(\mathbb{R}^n, E^m))$, the proof of the proposition will follow from (Dunford and Schwartz, 1964, Theorem 20, p. 298) if we can establish that for $h > 0 \lim_{h \to 0} \int_0^\infty |u(t + h) - u(t)|^2 \, dt = 0$ uniformly with respect to $u \in B$. Let $u \in B$ be arbitrary and $\hat{u}$ be its $L_2$ Fourier transform then $u(t) = 1/(2\pi)^{d/2} \int_\Omega \hat{u}(\omega) e^{i\omega t} \, d\omega$ a.e. on $\mathbb{R}^n$ with $\hat{u} \in L_2(\Omega, E^m)$. For $h > 0$ it is easily shown that

$$|u(t + h) - u(t)|^2 \leq (2/\pi) \int_\Omega |\hat{u}(\omega)|^2 \, d\omega \int_\Omega \sin^2 \frac{\omega h}{2} \, d\omega.$$  

Since $B$ is bounded, it follows from Parseval’s theorem that there exists a finite positive number $b^2 > 0$ so that

$$\int_\Omega |\hat{u}(\omega)|^2 \, d\omega = \int_{\mathbb{R}^n} |u(t)|^2 \, dt \leq 2\pi b^2 \quad \text{for all } u \in B.$$  

Thus for any measurable set $E \subset \mathbb{R}^n$ with finite Lebesgue measure

$$\int_E |u(t + h) - u(t)|^2 \, dt \leq 4b^2 \mu(E) \int_\Omega \sin^2(\omega h/2) \, d\omega$$  

uniformly with respect to $u \in B$. In particular let $E = [0, T]$ for $0 \leq T < \infty$ then

$$\int_\mathbb{R}^n |u(t + h) - u(t)|^2 \, dt$$

$$= \int_0^T |u(t + h) - u(t)|^2 \, dt + \int_T^\infty |u(t + h) - u(t)|^2 \, dt$$

$$\leq 4b^2 T \int_\Omega \sin^2(\omega h/2) \, d\omega + 4 \int_T^\infty |u(t)|^2 \, dt.$$  

It is clear from the condition (ii) of the theorem that for every $\epsilon > 0$ there exists a finite number $T_\epsilon > 0$ so that $\int_T^{\infty} |u(t)|^2 \, dt < \epsilon^2/8$ independently of $u \in B$ whenever $T \geq T_\epsilon$. Further, since $\Omega$ is a closed bounded (compact) subset of the real line, $\lim_{h \to 0} \int_\Omega \sin^2(\omega h/2) \, d\omega = 0$ and therefore for the given $\epsilon > 0$ there exists an $h_\epsilon > 0$ so that $\int_\Omega \sin^2(\omega h/2) \, d\omega < \epsilon^2/(8b^2 T_\epsilon)$ whenever $0 < h < h_\epsilon$. Thus for $0 < h \leq h_\epsilon \int_\mathbb{R}^n |u(t + h) - u(t)|^2 \, dt \leq \epsilon^2$ independently of $u \in B$.  

Since \( \epsilon > 0 \) is arbitrary, \( \lim_{h \to 0} \int_{R_0} |u(t + h) - u(t)|^2 \, dt = 0 \) uniformly with respect to \( u \in B \). This completes the proof of the proposition.

Remark. If the set \( B \subset B_0 \) is assumed to be closed in addition to satisfying the conditions (i) and (ii) of Proposition 2.1 then it is also compact. Throughout this paper it will be assumed that \( B \) is closed.

DEFINITION 2.1. A measurable \( m \)-vector valued function \( u \) defined on \([0, \infty)\) is said to be an admissible control if \( u \in \mathcal{B} \subset \mathcal{B}_0 \). That is, the class of admissible controls are both bandlimited and energy limited.

PROPOSITION 2.2. Suppose \( f \) satisfies the hypotheses \( H_0, H_1, \) and \( H_2 \). Then, for every control \( u \in \mathcal{B} \), the system \( S \) has a unique solution \( x = x_u \) which is absolutely continuous and bounded on every bounded interval of \( R_0 = [0, \infty) \).

Proof. The proof is omitted.

The solution \( x \) of the system \( S \) corresponding to an admissible control \( u \) will be denoted by \( x_u \), and its values in \( E^n \), by \( x_u(t), t \in R_0 \).

DEFINITION 2.2. The set \( X = \{ x_u : u \in \mathcal{B} \} \), called the \( T \)-set, is the family of admissible trajectories of the system \( S \) corresponding to the class of admissible controls \( \mathcal{B} \).

3. Properties of the \( T \)-set \( X \)

Consider the space of continuous functions defined on \( R_0 = [0, \infty) \) with values in \( E^n \). For any finite positive integer \( s \) we define the seminorm

\[
\rho_s(x) = \sup_{0 < t < s} |x(t)|, \quad s \in \mathbb{N} \text{ (positive integers)}
\]

where \( |x| = \sum_{i=1}^{n} |x_i| \). The space \( C(R_0, E^n) \), with this family of seminorms \( \{ \rho_s, s \in \mathbb{N} \} \), is a locally convex linear topological vector space (Yosida, 1968, p. 23). We denote this space by \( C_o \). The convergence in this space is precisely the uniform convergence on every compact subset \( D \subset R_0 \).

LEMMA 3.1. The space \( C_o \) is a complete metric space with the metric \( \rho \) defined by

\[
\rho(x, y) = \sum_{s=1}^{\infty} \frac{1}{2^s} \frac{\rho_s(x - y)}{1 + \rho_s(x - y)}.
\]
Remark. The T-set $X$ of the system $S$ is a subset of the metric space $C_0$. Properties of the set $X$ useful in the study of control problems are presented in this section.

Lemma 3.2. Let $I$ be any subset of $R^n$ with finite Lebesgue measure and let $F(t,u)$ be a function defined on $I \times E^m$ to $E^n$ and suppose $F$ is measurable in $t$ on $I$ for each fixed $u \in E^m$ and continuous in $u$ on $E^m$ for almost all $t \in I$. If $\{u_n\}$ is a sequence of measurable functions on $I$ to $E^m$ and if $u_n \to u_0$ in measure then $F(t, u_n(t)) \to F(t, u_0(t))$ in measure.

Proof. The proof of this lemma is a generalization of Nemytskii's lemma (Krasnoselskii, 1964, Lemma 2.1, p. 20) as given in (Ahmed, Lemma 3, TR 70-8).

Definition 3.1. The T-set $X$ of a dynamical system is said to be closed if and only if every limit point of the set is a trajectory of the system corresponding to an admissible control.

Proposition 3.1. The T-set $X$ of the system $S$ is a closed subset of the metric space $C_0$.

Proof. Let $x_0$ be a limit point of $X$ and suppose $\{x^k\} \in X$ so that $x_n \to x_0$ (in the sense of the metric $\rho$). It is required to show that $x_0 \in X$. If there is a control $u^0 \in B$ so that $x_0 = x_{u^0}$. Since $\{x_n\} \in X$ there is a sequence $\{u_n\} \in B$ so that $x_n = x_{u_n}$. Further, $B$ being compact, there exists a subsequence $\{u_{n_k}\}$ of the sequence $\{u_n\}$ so that $u_{n_k} \to u^*$ (in the $L^2$ sense) and $u^* \in B$. Define $x^* = x_{u^*}$. Clearly $x^* \in X$ and consequently it is sufficient to show that $x^0 = x^*$. Since $x_n \to x^0$ and $\{x_{n_k}\}$ is a subsequence of the sequence $\{x_n\}$ corresponding to the controls $\{u_{n_k}\}$, $x_{n_k} \to x^0$ also. Thus, for every finite positive integer $k$, it follows from the equality

$$x^0(t) - x^*(t) = x^0(t) - x_{n_k}(t) + x_{n_k}(t) - x^*(t), \quad t \geq 0$$

that

$$\rho(x^0 - x^*) \leq \lim_{k \to \infty} \rho(x_{n_k} - x^*).$$

Further, by definition of the sequence $\{x_{n_k}, u_{n_k}\}$ and the elements $(x^*, u^*)$ we have

$$|x_{n_k}(t) - x^*(t)| \leq \int_0^t |f(\tau, x_{n_k}(\tau), u_{n_k}(\tau)) - f(\tau, x^*(\tau), u_{n_k}(\tau))| \, d\tau$$

$$+ \int_0^t |f(\tau, x^*(\tau), u_{n_k}(\tau)) - f(\tau, x^*(\tau), u^*(\tau))| \, d\tau.$$
It follows from this inequality and the hypothesis $H_1$ that for $0 \leq t \leq s < \infty$

$$|x_{n_k}(t) - x^*(t)| \leq a_{n_k}(s) + \int_0^t K(\tau) |x_{n_k}(\tau) - x^*(\tau)| \, d\tau$$

where

$$a_{n_k}(s) = \int_0^s |f(\tau, x^*(\tau), u_{n_k}(\tau)) - f(\tau, x^*(\tau), u^*(\tau))| \, d\tau.$$

Thus $\rho_\delta(x_{n_k} - x^*) \leq a_{n_k}(s) \exp(\|K\|_\infty)$ for every finite positive integer $s$. We show that, for any fixed positive integer $s \in N$, $\lim_{k \to \infty} a_{n_k}(s) = 0$ and consequently $\lim_{s \to \infty} \rho_\delta(x_{n_k} - x^*) = 0$. For the fixed $x^* \in X$ define $F(t, u) = f(t, x^*(t), u)$. Clearly,

$$a_{n_k}(s) = \int_0^s |F(t, u_{n_k}(t)) - F(t, u^*(t))| \, dt.$$

Since $u_{n_k} \to u^*$ in the $L_2$ sense and convergence in the mean always implies convergence in measure, $u_{n_k} \to u^*$ in measure. Thus by Lemma 3.2

(i) $F(t, u_{n_k}(t)) \to F(t, u^*(t))$ in measure on $[0, s]$.

Further it follows from the hypothesis $H_2$ and the Lemma 2.2 (essential boundedness of $u_{n_k}$ and $u^*$) that $|F(t, u_{n_k}(t)) - F(t, u^*(t))| \leq 2K(t) [s + \beta |x^*(t)|]$ for $t \in R_0$. Since by assumption $K$ is locally Lebesgue integrable and by Proposition 2.2 $\rho_\delta(x^*)$ is finite for every finite number $s \geq 0$, we may conclude that for any measurable set $E \subset [0, s]$

(ii) $\lim_{k \to \infty} \int_E |F(t, u_{n_k}(t)) - F(t, u^*(t))| \, dt = 0$ uniformly with respect to the index $k$.

Conditions (i) and (ii) together imply (Halmos, 1964, Theorem C, p. 108) that $F(t, u_{n_k}(t)) \to F(t, u^*(t))$ in the mean on every finite interval of $R_0$. Therefore $\lim_{k \to \infty} a_{n_k}(s) = 0$ for every finite positive number $s$. Hence $\lim_{k \to \infty} \rho_\delta(x_{n_k} - x^*) = 0$ and in turn $\rho_\delta(x^0 - x^*) = 0$, that is $x^0 = x^*$ on every finite interval $[0, s] \subset R_0$. This implies that $\rho(x^0, x^*) = 0$ and, since $x^* \in X$, $x^0 \in X$. Thus the control $u^*$ may be taken for the control $u^0$. This completes the proof of the proposition.

**Proposition 3.2.** The $T$-set $X$ of the system $S$ is a sequentially compact subset of the metric space $C_0$.

**Proof.** In a complete metric space a totally bounded subset is sequentially compact and conversely (Kantorovich and Akilov, 1964, Theorem 2, p. 19).
Note that in Russian literature (Kantorovich, p. 19) by the expression compactness is meant sequential compactness.

Since by Lemma 3.1 \( C_\rho \) is a complete metric space it is sufficient to show that the set \( X \) is totally bounded. Therefore we show that for every \( \delta > 0 \) there is a finite number of elements \( \{x_i, i = 1, 2, \ldots, n(\delta)\} \subset X \) so that

\[
X \subset \bigcup_{i=1}^{n(\delta)} M_\delta(x_i)
\]

where

\[
M_\delta(x_i) = \{ x \in X : \rho(x, x_i) < \delta \}
\]

for \( i = 1, 2, \ldots, n(\delta) \). For the given \( \delta > 0 \) we choose a number \( m_0 \in N \) so that \( 2^{-m_0} < \delta/2 \). Since \( B \subset B_2 \) is compact for every \( \epsilon_0 = \epsilon(\delta) > 0 \) there exists a finite set \( \{u_i, i = 1, 2, \ldots, n(\epsilon_0) = n_0\} \subset B \) with the property

\[
B \subset \bigcup_{i=1}^{n_0} N_{\epsilon_0}(u_i)
\]

for each \( i, N_{\epsilon_0}(u_i) = \{ u \in B : (\int_0^{t_2} |u(t) - u_i(t)|^2 dt)^{1/2} < \epsilon_0 \} \). Further, since \( f \) satisfies the hypotheses \( H_0 \) and \( H_2 \), we can choose \( \epsilon_0(\delta) \) sufficiently small and consequently \( n_0 \) sufficiently large so that

\[
C(m_0, u, u_i) \triangleq \int_0^{m_0} \left| f(t, x_i(t), u(t)) - f(t, x_i(t), u_i(t)) \right| dt
\]

\[
\leq \delta/2 \exp(-\|K\|_{m_0})
\]

for all \( u \in N_{\epsilon_0}(u_i) \) where \( \|K\|_2 = \int_0^{t_2} |K(t)| \, dt \), \( s \)-finite, and \( x_i \triangleq x_{u_i} \). This last result follows from the fact that if \( v_n \rightarrow u_i \) in the \( L_2 \) sense then

\[
\lim_{n \rightarrow \infty} C(m_0, v_n, u_i) = 0
\]

as shown in the proof of the Proposition 3.1. By use of Gronwall’s lemma it is easily verified from the expression

\[
[x_u(t) - x_i(t)] = \int_0^t \left[ f(\tau, x_u(\tau), u(\tau)) - f(\tau, x_i(\tau), u(\tau)) \right] d\tau
\]

\[
+ \int_0^t \left[ f(\tau, x_i(\tau), u(\tau)) - f(\tau, x_i(\tau), u_i(\tau)) \right] d\tau
\]

that

\[
\rho_s(x_u - x_i) \leq C(s, u, u_i) \exp(\|K\|_2) \quad \text{for all } s \geq 0.
\]

Thus for \( u \in N_{\epsilon_0}(u_i) \),

\[
\rho_s(x_u - x_i) \leq \delta/2
\]

for all \( s \in [0, m_0] \) and consequently \( \rho(x_u, x_i) < \delta \). Denote the map \( u \rightarrow x_u \) by \( G \). Clearly \( \bigcup_{i=1}^{n_0} G(N_{\epsilon_0}(u_i)) = \bigcup_{i=1}^{n_0} M_\delta(x_i) \). Since

\[
\bigcup_{i=1}^{n_0} G(N_{\epsilon_0}(u_i)) = G \left( \bigcup_{i=1}^{n_0} N_{\epsilon_0}(u_i) \right) \supseteq G(B) = X,
\]

we have

\[
X \subset \bigcup_{i=1}^{n_0} M_\delta(x_i).
\]

This completes the proof of the proposition.

**Proposition 3.3.** The \( T \)-set \( X \) of the system \( S \) is a compact subset of the metric space \( C_\rho \).

**Proof.** In a metric space a sequentially compact set is conditionally compact (Dunford and Schwartz, 1964, Theorem 15, p. 22). Thus it follows
from Proposition 3.2 that \( X \) is conditionally compact that is, \( \overline{X} \) is compact. By Proposition 3.1, \( X \) is closed. Therefore \( X \) is compact. This completes the proof of the proposition.

**Corollary 3.1.** For each \( t \in R_0 \) the set \( A(t) = \{x(t); x \in X\} \) called the attainable set of the system \( S \) is a compact subset of \( E^n \) and it is continuous on \( R_0 \) in the Hausdorff metric \( d_H \).

**Proof.** The first assertion follows from Proposition 3.3 and the second is proved as follows. By use of Proposition 2.2 and the well-known Gronwall's lemma it is easily verified that, for every finite number \( T > 0 \), \( \rho_T(x_u) \leq a(T) \exp(\beta \|K\|_T) \) independently of \( u \in B \) where \( a(T) = (|x_0| + \alpha \|K\|_T) \) and \( \|K\|_T \triangleq \int_0^T \|K(t)\| \, dt \). Further for \( 0 \leq t_1, t_2 \leq T < \infty \), \( |x_u(t_2) - x_u(t_1)| \leq \int_0^T |K(t) \alpha + \beta |x_u(t)|\, dt \). Thus for \( t_1, t_2 \in [0, T] \) there exists a finite number \( b > 0 \) independent of \( u \in B \) so that \( |x_u(t_2) - x_u(t_1)| \leq b \int_0^T |K(t)| \, dt \).

Since \( K \) is locally Lebesgue integrable this implies that for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that \( \int_0^T |K(t)| \, dt < \epsilon/b \) whenever \( t_1, t_2 \in [0, T] \) and \( |t_1 - t_2| < \delta \) and consequently \( |x_u(t_2) - x_u(t_1)| < \epsilon \) for all \( u \in B \). For \( t_1, t_2 \in [0, T] \), consider the attainable sets \( A(t_1) \) and \( A(t_2) \). Clearly for every \( z \in A(t_1) \) there exists a control \( u^* \in B \) so that \( x_u(t_1) = z \). Since \( x_u(t_2) \in A(t_2) \) we have, from the above discussion, \( |x_u(t_2) - z| < \epsilon \) whenever \( |t_1 - t_2| < \delta \) and \( t_1, t_2 \in [0, T] \). Thus for every \( z \in A(t_1) \) there exists a \( y_z \in A(t_2) \) such that \( |z - y_z| < \epsilon \) yielding the relation \( A(t_1) \subset A(t_2) \triangleq \{y \in E^n: d(y, A(t_2)) < \epsilon\} \) whenever \( |t_1 - t_2| < \delta \) and \( t_1, t_2 \in [0, T] \). Similarly it can be shown that \( A(t_2) \subset A(t_1) \) for \( t_1, t_2 \in [0, T] \) and \( |t_1 - t_2| < \delta \). Combining these results we have \( d_H(A(t_1), A(t_2)) < \epsilon, t_1, t_2 \in [0, T] \), and \( |t_1 - t_2| < \delta \). Since \( T \) is finite but arbitrary this proves the continuity of the attainable function \( A(t) \) on \( R_0 = \{t: t \geq 0\} \).

4. Applications (Existence of Optimal Controls)

We are now prepared to solve the control problems stated in the introduction. Problem (i) is solved in the following

**Proposition 4.1.** The system \( S \) is given along with the cost functional \( J_u \triangleq \int_0^\infty f^0(t, x_u(t), u(t)) \, dt, u \in B \) where \( f^0 \) is a nonnegative real valued function defined on \( R_0 \otimes E^n \otimes E^m \) and \( x_u \) is the trajectory of the system \( S \) corresponding to the control \( u \in B \) and initial condition \( x_u(0) = x_0 \). If for almost
all \( t \in R_0 \), \( f^0 \) is lower-semicontinuous on \( E^n \otimes E^n \) then there is a control \( u^* \in B \) at which \( J \) attains its minimum.

**Proof.** It is easily shown by use of Fatou's lemma and the hypothesis on \( f^0 \) that the functional \( J \) is lower-semicontinuous on \( B \). Since \( B \) is compact and a lower semicontinuous functional bounded from below attains its lower bound on a compact set we have the result. The problems (ii) and (iii) as stated in the introduction are solved in the following.

**Proposition 4.2.** Let \( T(t), t \geq 0 \) be a continuous set valued function with values in the metric space of compact sets of \( E^n \). Then if there is a \( t' \in R_0 \) so that \( T(t') \cap A(t') \neq \emptyset \) then there is an optimal \( u^* \in B \) that steers the system \( S \) from the initial state \( x_0 \) to the target \( T(t^*) \) in the minimum time \( t^* \) and also minimizes the cost functional \( J(u) = \int_0^{t^*} f^0(t, x_u(t), u(t)) \, dt \), over all possible minimal time controls within the admissible class provided \( f^0 \) satisfies the hypothesis of Proposition 4.1.

**Proof.** By hypothesis the set \( I \triangleq \{ t \in [0, t']: T(t) \cap A(t) \neq \emptyset \} \) is non-empty (where \( \emptyset \) is the empty set). Let \( t^* = \inf I \). Then there is a non-increasing sequence \( \{ t_n \} \in I \) so that \( t_n \rightarrow t^* \) and a corresponding sequence of trajectories \( \{ x_n \} \in X \) so that \( x_n(t_n) \in T(t_n) \). Since the \( T \)-set \( X \) of the system \( S \) is compact (Proposition 3.3) there is a subsequence \( \{ x_{n_k} \} \) of the sequence \( \{ x_n \} \) and \( x^0 \in X \) so that \( x_{n_k} \rightarrow x^0 \). Therefore for every \( \varepsilon > 0 \) we can find a number \( k_0(\varepsilon) \) so that \( x_{n_k}(t) \in N_{\varepsilon/3}(x^0(t)) \) for each \( t \in [0, t'] \) and for all \( k > k_0 \) (where \( N_\delta(z) \) is the \( \delta \) neighborhood of the point \( z \) in \( E^n \)). Since the set valued function \( T(\cdot) \) is continuous on \( R_0 \) and \( t_{n_k} \rightarrow t^* \), for every \( \varepsilon > 0 \) there exists a number \( k_1(\varepsilon) \) so that \( T(t_{n_k}) \subseteq T^{\varepsilon/3}(t^*) \) for all \( k > k_1 \) where \( T^{\varepsilon/3}(t^*) = N_{\varepsilon/3}(T(t^*)) \). Clearly \( x^0(t) \in N_{\varepsilon/3}(x_{n_k}(t)) \), \( t \in [0, t'] \) for all \( k > k_0 \) and \( x_{n_k}(t_{n_k}) \in T(t_{n_k}) \subseteq T^{\varepsilon/3}(t^*) \) for all \( k > k_1 \). Consequently

\[
x^0(t_{n_k}) \in N_{\varepsilon/3}(x_{n_k}(t_{n_k})) \subseteq T^{\varepsilon/3}(t_{n_k}) \subseteq T^{2\varepsilon/3}(t^*) \quad \text{for all} \ k > \max\{ k_0, k_1 \}.
\]

Since \( x^0 \) is continuous, for every \( \varepsilon > 0 \) there exists a number \( k_2(\varepsilon) \) so that

\[
x^0(t_{n_k}) \in N_{\varepsilon/3}(x^0(t^*)) \quad \text{or equivalently} \quad x^0(t^*) \in N_{\varepsilon/3}(x^0(t_{n_k})) \quad \text{for all} \ k > k_2.
\]

Combining the above results we have \( x^0(t^*) \in T(t^*) \). Since \( \varepsilon > 0 \) is arbitrary and \( T(t^*) \) is closed \( x^0(t^*) \in T(t^*) \). But \( x^0 \in X \) implies \( x^0(t^*) \in A(t^*) \) and therefore there is a control \( u_0 \in B \) that steers the system from the initial state to the target \( T(t^*) \) in minimum time \( t^* \).

If \( u_0 \) is the only minimal time control then the second assertion does not require any proof. Consider the set \( B_0 \triangleq \{ u \in B: x_u(t^*) \in T(t^*) \} \), clearly \( B_0 \) is nonempty. We show that \( B_0 \) is closed. Let \( \{ u_n \} \in B_0 \) and suppose \( u_n \xrightarrow{\text{a. s.}} u \).
clearly \( \omega \in B \) and \( x_{u_n}(t^*) \in T(t^*) \cap A(t^*) \) for each \( n = 1, 2, \ldots \). By use of the hypothesis \( H_1 \) it is easily shown that \( |x_{u_n}(t^*) - x_{u}(t^*)| \leq a_n \exp\{ \|K\|_t^* \} \) where

\[
a_n = \int_0^{t^*} |f(\tau, x_\omega(\tau), u_n(\tau)) - f(\tau, x_\omega(\tau), \omega(\tau))| \, d\tau.
\]

As in the proof of Proposition 3.1 it can be shown that \( a_n \to 0 \) and consequently \( x_{u_n}(t^*) \to x_\omega(t^*) \). Since \( T(t^*) \cap A(t^*) \) is compact and \( x_{u_n}(t^*) \in T(t^*) \cap A(t^*) \) for each \( n = 1, 2, \ldots \), we have \( x_\omega(t^*) \in T(t^*) \cap A(t^*) \). Therefore \( \omega \in B_0 \) and hence \( B_0 \) is a closed subset of \( B \). Being a closed subset of a compact set, \( B_0 \) is compact. Thus the cost functional \( J(u) = \int_0^{t^*} f(t, x_u(t), u(t)) \, dt \), being lower semicontinuous (Proposition 4.1) and bounded from below, attains its lower bound on the compact set \( B_0 \). This leads to the proof of existence of the optimal control \( u^* \).

In certain problems it is required to regulate the dynamical system so that the period of stay of its trajectory inside a desired region in the state space is maximized. This is to be achieved by choosing a control from the admissible class \( B \). The problem of existence of a control optimal in the above sense is solved in the following

**Proposition 4.3.** Let \( X \) be the \( T \)-set of the system \( S \), \( T(t), t \geq 0 \) a continuous set valued function with values in the metric space of compact sets with Hausdorff metric, and \( \mu \) the Lebesgue measure on \( R_0 = [0, \infty) \). Define \( L(x, \theta) = \mu\{t \in [0, \theta]: x(t) \in T(t)\} \), \( x \in X \), \( \theta \in R_0 \). Then

(i) \( 0 \leq L(x, \theta) \leq \theta \) for all \( (x, \theta) \in X \times R_0 \).

(ii) \( L(x, \theta_1) \leq L(x, \theta_2) \) for \( \theta_1 < \theta_2 \) and \( x \in X \).

(iii) For each \( x^* \in X \), \( L(x^*, \cdot) \) is a continuous function on \( R_0 \).

(iv) For each fixed \( \theta^* \in R_0 \), \( L(\cdot, \theta^*) \) is a continuous functional on \( X \).

(v) For each \( \theta^* \in R_0 \) there exists a control \( u^* \in B \) so that \( L(x_{u^*}, \theta^*) \geq L(x_u, \theta^*) \) for all \( u \in B \) where \( x_u \) is the trajectory of the system \( S \) corresponding to the control \( \omega \in B \).

(vi) For every nonnegative measurable functions \( h \) satisfying the property \( \int_0^\infty th(t) \, dt < \infty \), the function \( l \) defined by \( l(x) = \int_0^\infty L(x, t) h(t) \, dt \), \( x \in X \) attains its extremum on \( X \).

**Proof.** The results (i), (ii), and (iii) follow immediately from the definition of the function \( L \) and the property of Lebesgue measure \( \mu \). For the result (iv)
let \( \theta^* \in \mathbb{R}_0 \) be fixed and \( \{x_n\} \in X \) so that \( x_n \rightarrow x_0 \). Define the sequence \( \{A_n\} \), by \( A_n \triangleq \{t \in [0, \theta^*] : x_n(t) \in T(t)\} \), and the set \( A_0 \) by \( A_0 \triangleq \{t \in [0, \theta^*] : x_0(t) \in T(t)\} \). Since for each \( t \in \mathbb{R}_0 \), \( T(t) \) is a compact subset of \( E^n \) and \( \{x_n, x_0\} \) are continuous, the sets \( \{A_n, A_0\} \) are Lebesgue measurable and \( \lim_{n \to \infty} A_n = A_0 \). Further \( \mu(\bigcup_{n=1}^{\infty} A_n) < \infty \) whenever \( \theta^* < \infty \). Thus (Munroe, 1959, Theorem 10.8, Corollary 10.8.1-2, p. 83) \( \mu(\lim_n A_n) \leq \lim_n \mu(A_n) \) and \( \mu(\limsup A_n) \geq \lim_n \mu(A_n) \) and consequently \( \mu(A_0) = \lim_n \mu(A_n) \). Therefore it follows from the definition of the function \( L \) that \( L(x_0, \theta^*) = \lim_n L(x_n, \theta^*) \). This proves the continuity of the function \( L(\cdot, \theta^*) \) on \( X \) for each finite \( \theta^* \in \mathbb{R}_0 \). Since a continuous function on a compact set attains both its maximum and minimum, the result (v) follows from (iv) and the fact that the set \( X \) is a compact subset of \( C_\theta \) (Proposition 3.3). For the result (vi) let \( \{x_n\} \) be a sequence from \( X \) and suppose \( x_n \rightarrow x_0 \). Then it follows from (iv) that \( L(x_n, t) \rightarrow L(x_0, t) \) for each \( t \in \mathbb{R}_0 \) and consequently \( L(x_n, t) h(t) \rightarrow L(x_0, t) h(t) \) a.e. on \( \mathbb{R}_0 \). Furthermore \( L(x_n, t) h(t) \leq \theta h(t) \), \( t \in \mathbb{R}_0 \) for all \( n \). Thus it follows from the property of \( h \) and the Lebesgue dominated convergence theorem that

\[
\lim_n \int_{\theta}^{\infty} L(x_n, t) h(t) \, dt = \int_{\theta}^{\infty} L(x_0, t) h(t) \, dt,
\]

that is, \( \lim_n l(x_n) = l(x_0) \). This proves the continuity of \( l \) on \( X \) and consequently \( l \) attains its extremum on \( X \) which is compact. The existence of an extremum of the function \( l \) on \( X \) implies the existence of an extremal control in \( B \). This completes the proof of the proposition.

Remark. It is interesting to note that the results of this paper can be generalized without much difficulty to the case of Hereditary differential systems described by

\[
\dot{x}(t) = f(t, \pi_t x, \pi_t u) \quad \text{a.e.} \quad t \geq 0.
\]

where \( \pi_t x = \{x(\tau) : -\alpha_0 \leq \tau \leq t\} \), \( \pi_t u = \{u(\tau) : -\alpha_0 \leq \tau \leq t\} \) for each \( t \in \mathbb{R}_0 \). It is necessary to specify the initial data over the interval \([-\alpha_0, 0]\), \( x(t) = \hat{x}(t) \) and \( u(t) = \hat{u}(t) \) for \( t \in [-\alpha_0, 0] \) where \( \hat{x} \) is continuous and \( \hat{u} \) is bounded measurable. The hypotheses \( H_0, H_1, H_2 \) are replaced by \( H_0', H_1', H_2' \) where

\[
H_0' : \text{For each } x \in C_\theta \text{ and } u \in Bf \text{ is measurable in } t \text{ on } \mathbb{R}_0 \text{ and for almost all } t \in \mathbb{R}_0 f \text{ is continuous on } C_\theta \otimes B,
\]
$H_1'$: There exists a measurable function $R(\cdot, \cdot)$ locally Lebesgue integrable on the plane so that for all $u \in B$

$$|f(t, \pi_t x, \pi_t u) - f(t, \pi_t y, \pi_t u)| \leq \int_{-\alpha_0}^{t} |R(t, \tau)| |x(\tau) - y(\tau)| d\tau \quad \text{a.e.},$$

and

$H_2'$: There exist $\alpha, \beta \geq 0$ so that for all $u \in B$

$$|f(t, \pi_t x, \pi_t u)| \leq \int_{-\alpha_0}^{t} |R(t, \tau)| |x(\tau)| \, d\tau \quad \text{a.e.}$$

CONCLUSION

In the majority of engineering control problems the control signals that can be practically synthesized are the bandlimited ones. In practice it is almost impossible to generate control signals containing discontinuities (bang bang controls). This may be either due to frequency limitation of the device generating the control signals or the bandlimited property of the channel transmitting the signals.

Furthermore the function $f$ appearing in the dynamical system, $dx/dt = f(t, x, u)$, can not always be expected to satisfy the usual convexity condition (Hermes and LaSalle, 1969, Theorem 20.1, p. 107; Oguztoreli, 1966, Theorem 8.1, p. 184) with respect to the control variable $u$ for each point $(t, x)$ in the phase space.

In this paper requirement of the convexity condition is removed by constraining the control signals to be bandlimited—a more practical situation.

It should be mentioned that Pontryagin's maximum principle can not be directly used to derive the necessary conditions for optimality if the control signals are bandlimited. How ever it is possible to approximate by bandlimited functions the bounded measurable controls obtained by the maximum principle.

Received: November 21, 1972

REFERENCES

EXISTENCE OF OPTIMAL CONTROLS WITHOUT CONVEXITY