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$\mathbb{Z}$-cyclic ordered triplewhist tournaments on p elements, where $p \equiv 5 \pmod 8$

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Abstract

We construct new families of whist tournaments that are at the same time $\mathbb{Z}$-cyclic, ordered and triplewhist. In particular, we construct such a design on p elements, $p \geqslant 29$, where $p \equiv 5 \pmod 8$ is prime.

Keywords: Whist tournaments; $\mathbb{Z}$-cyclic tournaments; Ordered tournaments; Triplewhist tournaments

1. Introduction

A whist tournament $\mathrm{Wh}(4m+1)$ for $4m+1$ players is a schedule of games (or tables) (a, b, c, d) involving two players a, c opposing two other players b, d such that

i. the games are arranged into $4m+1$ rounds each of m games;
ii. each player plays in exactly one game in all but one round;
iii. each player partners every other player exactly once;
iv. each player opposes every other player exactly twice.

We shall be concerned with two refinements of the structure, called triplewhist tournaments and ordered triplewhist tournaments. Call the pairs $\{a, b\}$ and $\{c, d\}$ pairs of *opponents of the first kind*, and call the pairs $\{a, d\}$ and $\{b, c\}$ pairs of *opponents of the second kind*. We

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also say that a and c are partners of the first kind while b and d are partners of the second kind. Then a *triplewhist tournament* $\mathrm{TWh}(4m+1)$ is a $\mathrm{Wh}(4m+1)$ in which every player is an opponent of the first (respectively, second) kind exactly once with every other player; and an *ordered whist tournament* $\mathrm{OWh}(4m+1)$ is a $\mathrm{Wh}(4m+1)$ in which each player opposes every other player exactly once while being a partner of the first (respectively, second) kind. If the players are elements of $\mathbb{Z}_{4m+1}$, and if the ith round is obtained from the initial (first) round by adding $i-1$ to each element $(\bmod\, 4m+1)$, then we say that the tournament is $\mathbb{Z}$-*cyclic*. By convention we always take the initial round to be the round from which 0 is absent. The games (tables)

$$(a_1, b_1, c_1, d_1), \ldots, (a_m, b_m, c_m, d_m)$$

form the initial round of a $\mathbb{Z}$-cyclic triplewhist tournament if

$$\bigcup_{i=1}^{m} \{a_i, b_i, c_i, d_i\} = \mathbb{Z}_{4m+1} \backslash \{0\}, \tag{1}$$

$$\bigcup_{i=1}^{m} \{\pm(a_i - c_i), \pm(b_i - d_i)\} = \mathbb{Z}_{4m+1} \backslash \{0\}, \tag{2}$$

$$\bigcup_{i=1}^{m} \{\pm(a_i - b_i), \pm(c_i - d_i)\} = \mathbb{Z}_{4m+1} \backslash \{0\}, \tag{3}$$

$$\bigcup_{i=1}^{m} \{\pm(a_i - d_i), \pm(b_i - c_i)\} = \mathbb{Z}_{4m+1} \backslash \{0\}. \tag{4}$$

Eqs. (1) and (2) show that the partner pairs form a starter [1, p. 136]. Similarly for (1) and (3) with the first opponent pairs, and (1) and (4) with the second opponent pairs. These games form a $\mathbb{Z}$-cyclic ordered whist tournament if, in addition to forming the initial round of a $\mathrm{Wh}(4m+1)$,

$$\bigcup_{i=1}^{m} \{(a_i - b_i), (a_i - d_i), (c_i - b_i), (c_i - d_i)\} = \mathbb{Z}_{4m+1} \backslash \{0\}. \tag{5}$$

Now we shall look at whist tournaments which are simultaneously both triplewhist and ordered tournaments. Such designs will be called *ordered triplewhist tournaments* and will be denoted by $\mathrm{OTWh}(v)$. We shall show that an $\mathrm{OTWh}(v)$ exists for all v whenever v is a prime $p \equiv 5 \pmod 8$, and $p \geqslant 29$. Finizio [5] has verified that there is no $\mathbb{Z}$-cyclic $\mathrm{TWh}(p)$ for primes $p < 29$.

Example 1.1. A $\mathbb{Z}$-cyclic OTWh(29) is given by the initial round $(1, 3, 26, 13) \times 1, 3^4, \ldots, 3^{24}$.

The original proof by Anderson et al. [2], which dealt with the existence of $\mathbb{Z}$-cyclic $\mathrm{TWh}(p)$ with $p = 8n+5$ prime, contained a requirement that certain elements be primitive roots of $\mathbb{Z}_p$. This requirement was shown by Buratti in [4] to be an additional, but not necessary one. The elements in question need only be non-square over $\mathbb{Z}_p$, and a less difficult proof is the result. The theorem of Weil on multiplicative character sums [6, Theorem 5.41, p. 225] is used in the proof which follows. Here is the statement of Weil's theorem, in which the convention is understood that if ψ is a multiplicative character of $\mathrm{GF}(q)$, then $\psi(0)=0$. Adopting this convention, we have $\psi(xy)=\psi(x)\psi(y)$ for all $(x, y) \in \mathrm{GF}(q) \times \mathrm{GF}(q)$.

Theorem 1.1. *Let ψ be a multiplicative character of order $m>1$ of the finite $\mathrm{GF}(q)$. Let f be a polynomial of $\mathrm{GF}(q)[x]$ which is not of the form kg^m for some $k \in \mathrm{GF}(q)$ and some $g \in \mathrm{GF}(q)[x]$. Then we have*

$$\left|\sum_{x\in \mathrm{GF}(q)} \psi(f(x))\right| \leqslant (d-1)\sqrt{q},$$

where d is the number of distinct roots of f in its splitting field over $\mathrm{GF}(q)$.

2. The existence theorem

We now take a closer look at some constructions which were presented by Anderson and Finizio [3], and find the conditions which must be satisfied in order for them to produce a $\mathbb{Z}$-cyclic OTWh(p) for primes $p \equiv 5 \pmod 8$.

So let $p=8t+5$ be prime, let x be a non-square element of $\mathbb{Z}_p$, and let θ be a primitive root of p. We now present six constructions.

Construction 1. $(1, x, -1, x^3) \times 1, \theta^4, \ldots, \theta^{8t}$. First we find the conditions under which this forms a TWh(p). The partner differences are pairs $\pm 2, \pm x(x^2-1) \times 1, \theta^4, \ldots, \theta^{8t}$, and so the partner pairs form a starter provided $2x(x^2-1)$ is not a square. Similarly, the first kind opponent pairs form a starter provided $(x-1)(x^3+1)$ is not a square, and the second kind opponent pairs form a starter provided $(x+1)(x^3-1)$ is not a square. We now use the fact that 2 is a non-square since $p \equiv 5 \pmod 8$. So Construction 1 yields a $\mathbb{Z}$-cyclic TWh(p) provided x^2-1 is not a square, $x^2 \pm x+1$ are squares.

Now, we find the conditions under which this also forms an OWh(p).

Let $a=-(x-1),\ b=-(x^2+x+1)(x-1),\ c=-(x+1),\ d=-(x^2-x+1)(x+1)$. We require that *a, b, c, d* lie in distinct cyclotomic classes of index 4. Since a/c is not a square, we require b/d to be a non-square, and, to guarantee that the two squares (non-squares) lie in distinct cyclotomic classes, we also require that each of $x^2 \pm x+1$, although squares, are not fourth powers.

So, Construction 1 gives the initial round tables of a TWh(p) provided x^2-1 is not a square, $x^2 \pm x+1$ are squares. They also yield an OWh(p) provided $x^2 \pm x+1$ are both not fourth powers.

Construction 2. $(1, x^3, x^2, -x^3) \times 1, \theta^4, \ldots, \theta^{8t}$. These are the initial round tables of a TWh(p) provided x^2-1 is not a square, $x^2 \pm x+1$ are squares. They also yield an OWh(p) provided $x^2 \pm x+1$ are both fourth powers.

Construction 3. $(1, x^3, -x^4, -x^3) \times 1, \theta^4, \ldots, \theta^{8t}$. These are the initial round tables of a TWh(p) provided x^4+1 is not a square, $x^2 \pm x + 1$ are squares. We also get an OWh(p) provided $(x-1)/(x+1)$ is a square but not a fourth power and exactly one of $x^2 \pm x + 1$ is a fourth power.

Construction 4. $(1, x, -x^4, -x) \times 1, \theta^4, \ldots, \theta^{8t}$. These are the initial round tables of a TWh(p) provided x^4+1 is not a square, $x^2 \pm x + 1$ are both squares. We also get an OWh(p) provided $(x-1)/(x+1)$ is a fourth power, and exactly one of $x^2 \pm x + 1$ is a fourth power.

Construction 5. $(1, x, -x^4, x^3) \times\ 1, \theta^4, \ldots, \theta^{8t}$. For a TWh$(p)$, we require x^2-1 is a square, x^4+1 is a square, $(x^2+x+1)\ (x^2-x+1)$ is a square. We also get an OWh(p) provided x^2+x+1 is not a fourth power, but x^2-x+1 is.

Construction 6. $(1, -x, -x^4, -x^3) \times 1, \theta^4, \ldots, \theta^{8t}$. For a TWh$(p)$, we require x^2-1 is a square, x^4+1 is a square, $(x^2+x+1)\ (x^2-x+1)$ is a square. We also get an OWh(p) provided x^2-x+1 is not a fourth power, but x^2+x+1 is.

Theorem 2.1. *Let $p=8t+5$ be prime. If there exists a non-square element x of $\mathbb{Z}_p$ such that $x^2 \pm x+1$ are both squares and either x^2-1 is not a square and $(x^2+x+1)(x^2-x+1)$ is a fourth power, or x^2-1 is a square and $(x^2+x+1)(x^2-x+1)$ is not a fourth power, then a $\mathbb{Z}$-*cyclic OTWh (p) *exists.*

Proof. Suppose there exists such a non-square x. If it happens that x^2-1 is not a square, use Construction 2 if both $x^2 \pm x+1$ are fourth powers and use Construction 1 otherwise. So, now suppose that x^2-1 is a square, i.e., $(x-1)/(x+1)$ is a square. Next suppose x^4+1 is not a square. Since exactly one of $x^2 \pm x+1$ is a fourth power, we can use Construction 4 if $(x-1)/(x+1)$ is a fourth power and Construction 3 otherwise. Finally, if x^2-1 is a square and x^4+1 is a square, use Construction 6 if x^2+x+1 is a fourth power and Construction 5 otherwise. □

It therefore remains to show that a non-square x satisfying the conditions of Theorem 2.1 can be obtained.

Let λ denote the quadratic character mod p, so that $\lambda(y)=-1$ if y is not a square. Let ψ be any fixed character of order 4 exactly; then $\psi(y)=1$ if y is a fourth power, and $\psi(y)=-1$ if y is a square but not a fourth power. Let

$$S = \sum_{x\in \mathrm{GF}(p)} (1-\lambda(x))(\lambda(x^2-x+1)+1)(\lambda(x^2+x+1)+1) \times (1-\psi((x^2+x+1)(x^2-x+1)(x^2-1)^2)).$$

Then $S=16|A|$ where A is the set of non-square elements of $\mathbb{Z}_p$ satisfying the conditions of Theorem 2.1.

Since $\lambda(x)=\psi(x^2)$,

$$S=\sum_{x\in \mathrm{GF}(p)}(1-\psi(x^2))(\psi((x^2-x+1)^2)+1)(\psi((x^2+x+1)^2)+1)$$
$$\times\,(1-\psi((x^2+x+1)(x^2-x+1)(x-1)^2(x+1)^2)).$$

Thus,

$$\begin{aligned}S\geqslant p&-\left|\sum_{x\in \mathrm{GF}(p)}\psi((x^2-x+1)^2(x^2+x+1)^2)\right|\\&-\left|\sum_{x\in \mathrm{GF}(p)}\psi((x^2-x+1)^2)\right|-\left|\sum_{x\in \mathrm{GF}(p)}\psi((x^2+x+1)^2)\right|\\&-\left|\sum_{x\in \mathrm{GF}(p)}\psi((x^2-x+1)^3(x^2+x+1)^3(x-1)^2(x+1)^2)\right|\\&-\left|\sum_{x\in \mathrm{GF}(p)}\psi((x^2-x+1)^3(x^2+x+1)(x-1)^2(x+1)^2)\right|\\&-\left|\sum_{x\in \mathrm{GF}(p)}\psi((x^2-x+1)(x^2+x+1)^3(x-1)^2(x+1)^2)\right|\\&-\left|\sum_{x\in \mathrm{GF}(p)}\psi((x^2-x+1)(x^2+x+1)(x-1)^2(x+1)^2)\right|\\&-\left|\sum_{x\in \mathrm{GF}(p)}(\psi(x^2))(\psi((x^2-x+1)^2)+1)(\psi((x^2+x+1)^2)+1)\right.\\&\left.\quad\times\,(1-\psi((x^2+x+1)(x^2-x+1)(x-1)^2(x+1)^2))\right|.\end{aligned}$$

After multiplying this out fully and making the appropriate substitutions (using Theorem 1.1), it can be seen that

$$S\geqslant p-(25\sqrt{p}+32\sqrt{p}),\quad \text{i.e. } S\geqslant p-57\sqrt{p}.$$

Thus,

$$\begin{aligned}&S=16|A|\geqslant p-57\sqrt{p}>0\quad \text{if } p\geqslant 57\sqrt{p},\\&\quad \text{i.e. if } \sqrt{p}\geqslant 57,\\&\quad \text{i.e. if } p>3249.\end{aligned}$$

It was then checked by computer that appropriate values of x existed for all primes $29 \leqslant p < 3249$ where $p \equiv 5 \pmod 8$, excluding $p = 29$. But an OTWh(p) has already been constructed for this value of p in Section 1. Here, we list (p, x_p) where p is the prime and x_p is the smallest suitable value of x for that prime.

(37, 2), (53, 14), (61, 8), (101, 32), (109, 14), (149, 34), (157, 32), (173, 7), (181, 22), (197, 12), (229, 21), (269, 29), (277, 2), (293, 8), (317, 8), (349, 8), (373, 18), (389, 3), (397, 6), (421, 2), (461, 10), (509, 7), (541, 2), (557, 11), (613, 2), (653, 12), (661, 6), (677, 12), (701, 3), (709, 22), (733, 8), (757, 24), (773, 12), (797, 7), (821, 12), (829, 40), (853, 6), (877, 2), (941, 7), (997, 44), (1013, 41), (1021, 43), (1061, 14), (1069, 26), (1093, 22), (1109, 42), (1117, 2), (1181, 15), (1213, 5), (1229, 17), (1237, 15), (1277, 28), (1301, 39), (1373, 12), (1381, 10), (1429, 2), (1453, 18), (1493, 11), (1549, 40), (1597, 2), (1613, 57), (1621, 18), (1637, 41), (1669, 10), (1693, 11), (1709, 40), (1733, 32), (1741, 6), (1789, 37), (1861, 39), (1877, 52), (1901, 10), (1933, 14), (1949, 27), (1973, 26), (1997, 20), (2029, 24), (2053, 5), (2069, 15), (2141, 8), (2213, 18), (2221, 2), (2237, 20), (2269, 2), (2293, 24), (2309, 8), (2333, 8), (2341, 54), (2357, 5), (2381, 47), (2389, 23), (2437, 5), (2477, 5), (2549, 8),(2557, 2), (2621, 7), (2677, 79), (2693, 27), (2741, 18), (2749, 10), (2789, 13), (2797, 2), (2837, 3), (2861, 26), (2909, 10), (2917, 52), (2957, 61), (3037, 22), (3061, 21), (3109, 2), (3181, 28), (3221, 8), (3229, 33).

Thus the following theorem is established.

Theorem 2.2. *A $\mathbb{Z}$-cyclic* OTWh(p) *exists for all primes* $p \equiv 5 \pmod 8$, $p \geqslant 29$.

Example 2.1. The initial round games of a $\mathbb{Z}$-cyclic OTWh(37).

(1, 8, 21, 29), (7, 19, 36, 18), (9, 35, 4, 2), (10, 6, 25, 31), (12, 22, 30, 15), (16, 17, 3, 20), (26, 23, 28, 14), (33, 5, 27, 32), (34, 13, 11, 24).

Example 2.2. The suitable values of x when $p = 37$ are 2, 18, 19, 35.

We are now in a position to state the following, as similarly given for DTWh(v) in [3]. The existence of an OTWh(29) and an OTWh(37) is enough to guarantee the existence of an OTWh(v) for all sufficiently large $v \equiv 1 \pmod 4$.

Theorem 2.3. *An* OTWh(v) *exists for all sufficiently large* $v \equiv 1 \pmod 4$.

Proof. It follows from the results of Wilson [7] that a pairwise balanced design, PBD$(\{29, 37\}, v)$ exists for all sufficiently large $v \equiv 1 \pmod 4$. On each block B of this PBD, form an OTWh$(|B|)$. Then for each x, take the blocks B containing x and, for each such B, take all the tables of the round of the OTWh on B in which x sits out. These games will form a round, which we label as round x, of the required OTWh(v). It is clear that the triplewhist and ordered whist properties are preserved by this construction. □

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