Metric dimension of symplectic dual polar graphs and symmetric bilinear forms graphs

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\textbf{A R T I C L E I N F O}

Article history:
Received 24 February 2012
Received in revised form 25 September 2012
Accepted 27 September 2012
Available online 25 October 2012

Keywords:
Metric dimension
Symplectic dual polar graph
Symmetric bilinear forms graph

\textbf{A B S T R A C T}

In this paper, by constructing resolving sets of symplectic dual polar graphs and symmetric bilinear forms graphs, we obtain upper bounds on their metric dimension.

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\section{1. Introduction}

Suppose $\Gamma$ denotes a connected graph. For any two vertices $u$ and $v$ of $\Gamma$, let $d(u, v)$ denote the distance between $u$ and $v$ in $\Gamma$. A \textit{resolving set} of $\Gamma$ is a set of vertices $S = \{v_1, \ldots, v_k\}$ such that, for each vertex $w$ of $\Gamma$, the ordered list of distances $D(w|S) = (d(w, v_1), \ldots, d(w, v_k))$ uniquely determines $w$. That is, $S$ is a resolving set of $\Gamma$ if $D(u|S) \neq D(v|S)$ for any two distinct vertices $u$ and $v$ of $\Gamma$. The \textit{metric dimension} of $\Gamma$ is the smallest size of a resolving set of $\Gamma$.

Metric dimension was introduced independently by Harary and Melter\cite{9}, and Slater\cite{11}. As a graph parameter it has been heavily studied; see\cite{1,4,5,10,12,17} for a number of references on this topic. Recently a considerable literature has developed in distance-regular graphs. See\cite{1} for Johnson graphs,\cite{2} for Grassmann graphs,\cite{7} for bilinear forms graphs,\cite{8} for Johnson graphs, doubled Odd graphs, doubled Grassmann graphs and twisted Grassmann graphs.

In this paper, we continue this research, and obtain upper bounds on the metric dimension of symplectic dual polar graphs and symmetric bilinear forms graphs.

Let $\mathbb{F}_q^{2n}$ be the $2n$-dimensional row vector space over the finite field $\mathbb{F}_q$. The \textit{symplectic space} associated with $\mathbb{F}_q^{2n}$ carries the non-degenerate symplectic form

$$\omega(x, y) = x^t \begin{pmatrix} 0 & I^{(n)} \\ -I^{(n)} & 0 \end{pmatrix} y,$$

where $I^{(n)}$ is the identity matrix of order $n$. A subspace of $\mathbb{F}_q^{2n}$ is called \textit{isotropic} whenever the form vanishes completely on that subspace. It is well known that all the maximal isotropic subspaces have the same dimension $n$. For a given subspace
W, the subspace $W^\perp = \{ v \in \mathbb{F}_q^{2n} \mid \omega(v, w) = 0 \text{ for all } w \in W \}$ has dimension $2n - \dim W$, which is called the symplectic complement of $W$. For more information, see [13,14].

The symplectic dual polar graph $[C_n(q)]$ has as vertices all the maximal isotropic subspaces of $\mathbb{F}_q^{2n}$, and two vertices are adjacent if their intersection has dimension $n-1$. Two vertices have distance $i$ if and only if their intersection has dimension $n-i$. The graph $[C_n(q)]$ is a distance-transitive graph; see [3]. Note that $[C_1(q)]$ is the complete graph with $q+1$ vertices, whose metric dimension is $q$. The symmetric bilinear forms graph $\text{Sym}(n,q)$ has as vertices all $n \times n$ symmetric matrices over $\mathbb{F}_q$, and two vertices are adjacent if their difference has rank 1. The graph $\text{Sym}(n,q)$ is not distance-regular for $n > 3$; see also [3]. By Brouwer et al. [3, Proposition 9.5.10] $\text{Sym}(n,q)$ is isomorphic to the last subconstituent of $[C_n(q)]$. This is the reason why we consider these two families of graphs at the same time.

In this paper, we obtain the following results.

**Theorem 1.1.** Let $[C_n(q)]$ be the symplectic dual polar graph, where $n \geq 2$. Then the metric dimension of $[C_n(q)]$ is at most $(q^n + 1)(q^n + q - 2)/(q - 1)$.

**Theorem 1.2.** Let $\text{Sym}(n,q)$ be the symmetric bilinear forms graph, where $q$ is odd and $n \geq 2$. Then the metric dimension of $\text{Sym}(n,q)$ is at most $q^n(q^n + q - 2)/(q - 1)$.

### 2. Proof of main results

We begin with a useful lemma.

**Lemma 2.1** ([6]). Let $\mathbb{F}_q^{2n}$ be the $2n$-dimensional symplectic space. Then there exist maximal isotropic subspaces $V_i$, $i = 1, 2, \ldots, q^n + 1$, of $\mathbb{F}_q^{2n}$ such that

$$\mathbb{F}_q^{2n} = V_1 \cup V_2 \cup \cdots \cup V_{q^n + 1},$$

where $V_i \cap V_j = \{0\}$ for all $i \neq j$.

Let $\mathbb{F}_q = \{g_1, g_2, \ldots, g_{q^n - 1}, g_q = 0\}$. In order to prove **Theorem 1.1**, it suffices to construct a resolving set $\mathcal{M}$ with size at most $(q^n + 1)(q^n + q - 2)/(q - 1)$.

**Proof of Theorem 1.1.** Assume that (1) holds. For each $i \in \{1, 2, \ldots, q^n + 1\}$, let \[ \mathcal{M}_i = \{ M \subseteq V_i \mid \dim M = n - 1 \} = \{ M_{i1}, M_{i2}, \ldots, M_{iq^2} \}, \]
where $t = (q^n - 1)/(q - 1)$. For each $j \in \{1, 2, \ldots, t\}$, let $W_j$ be the set of vertices $W$ of $[C_n(q)]$ such that $M_{ij} \subseteq W$. By Wan [14, Theorem 3.38], one gets $|W_j| = q + 1$. For any $i$ and $j$, pick a vertex $W_{ij} \in W_j$ with $W_{ij} \neq V_i$. Write

$$\mathcal{M} = \bigcup_{i=1}^{q^n+1} \bigcup_{j=1}^{(q^n-1)/(q-1)} \{ V_i, W_{ij} \}.$$  

Observe that $|\mathcal{M}| \leq (q^n + 1)(q^n + q - 2)/(q - 1)$.

Now we only need to show that $\mathcal{M}$ is a resolving set. By definition, it suffices to show that, for any two distinct vertices $P$ and $Q$ of $[C_n(q)]$, there exists a vertex $W$ of $\mathcal{M}$ such that $\dim(P \cap W) \neq \dim(Q \cap W)$. For each $i \in \{1, 2, \ldots, q^n + 1\}$, let $A_i = P \cap V_i$ and $B_i = Q \cap V_i$. Since $P \neq Q$, there exists an $i$ such that $A_i \neq B_i$.

Case 1. $\dim A_1 \neq \dim B_1$. Then $W = V_i$ is the desired vertex.

Case 2. $\dim A_1 = \dim B_1 = r$. Then $1 \leq r < n$. Pick a basis $\{ \alpha_1, \ldots, \alpha_r \}$ for $A_1$. Since $A_1 \neq B_i$, there exists a $\beta \in B_i \setminus A_1$ such that $\{ \alpha_1, \ldots, \alpha_r, \beta \}$ is linearly independent. Extend this to a basis $\{ \alpha_1, \ldots, \alpha_r, \beta, \gamma_1, \ldots, \gamma_{n-r-1} \}$ for $V_i$. Let $M$ be the subspace spanned by $\{ \alpha_1, \ldots, \alpha_r, \gamma_1, \ldots, \gamma_{n-r-1} \}$. Then $\dim(P \cap M) = r$ and $\dim(Q \cap M) = r - 1$.

Since $V_i$ is a subspace of $M^\perp$, there exists a $\gamma \in M^\perp$ such that $\omega(\beta, \gamma) = 0$ and $\{ \alpha_1, \ldots, \alpha_r, \beta, \gamma_1, \ldots, \gamma_{n-r-1} \}$ is a basis for $M^\perp$. Let $W_i$ be the maximal isotropic subspace spanned by $\{ \alpha_1, \ldots, \alpha_r, \gamma_1, \ldots, \gamma_{n-r-1}, g_k \beta + \gamma \}$, where $k \in \{1, \ldots, q\}$. Then

$$W_i = \{ V_i, W_1, W_2, \ldots, W_{q-1}, W_q \}.$$  

For any vector $\alpha = \sum_{j=1}^{r} k_j \alpha_j + \sum_{j=1}^{n-r-1} l_j \gamma_j + s(g_k \beta + \gamma) \in Q \cap W_i$, we have $s \cdot \omega(\beta, \gamma) = \omega(\beta, \alpha) = 0$, which implies that $s = 0$ and so $\alpha \in M$. Hence $Q \cap W_k = Q \cap M$ for each $k$. By the construction of $\mathcal{M}$, there exists a $t \in \{1, 2, \ldots, q\}$ such that $W_t \in \mathcal{M}$. Therefore,

$$\dim(Q \cap W_t) = \dim(Q \cap M) < \dim(P \cap M) \leq \dim(P \cap W_t).$$

Hence $W = W_t$ is the desired vertex. \[\square\]

**Remark 1.** By using the existence of a symplectic spread of $\mathbb{F}_q^{2n}$ (Lemma 2.1), we can construct a resolving set of the symplectic dual polar graph $[C_n(q)]$. It seems to be interesting to construct resolving sets of other dual polar graphs.
A matrix representation of a subspace $P$ of $F_q^{2n}$ is an array whose rows form a basis for $P$. When there is no danger of confusion, we use the same symbol to denote a subspace and its matrix representation. Let $P_0$ be a fixed vertex of $[C_n(q)]$. By the transitivity of $[C_n(q)]$, we may pick $P_0 = (0^{(n)})^T$ and $V_1 = P_0$ as in (1). Let $\Delta$ be the last subconstituent of $[C_n(q)]$ with respect to $P_0$. That is, $\Delta$ is the subgraph induced on the set of vertices at distance $n$ with $P_0$. Then $\Delta$ consists of all subspaces $(l^{(n)})B$, where $B$ is an $n \times n$ symmetric matrix. By Wang et al. [16, Theorem 2.2] the map $f : B \mapsto (l^{(n)})B$ is an isomorphism from $\text{Sym}(n, q)$ to $\Delta$. Suppose that $q$ is odd. Then by Wan [15, Proposition 5.5], two vertices of $\text{Sym}(n, q)$ have distance $i$ if and only if their difference has rank $i$, which implies that two vertices of $\Delta$ have distance $i$ if and only if their intersection has dimension $n - i$.

**Proof of Theorem 1.2.** It suffices to construct a resolving set $\mathcal{M}$ of $\Delta$ with size at most $q^n(q^n + q - 2)/(q - 1)$. Assume that (1) holds. Pick $V_1 = P_0$. For each $i \in \{2, 3, \ldots, q^n + 1\}$, let $\mathcal{M}_i$ as in (2). For each $M_{ij} \in \mathcal{M}_i$, let $W_{ij}$ be the set of vertices $W$ of $\Delta$ such that $M_{ij} \subseteq W$.

We assert that $|W_{ij}| = q$. In fact, pick a basis $\{\alpha_1, \ldots, \alpha_{n-1}\}$ for $M_{ij}$, extend it to a basis $\{\alpha_1, \ldots, \alpha_{n-1}, \beta\}$ for $V_1$. Since $\dim P_0 = n$ and $\dim M_{ij} = n + 1$, we have $\dim(M_{ij} \cap P_0) = 1$. It follows that there exists a $\gamma \in M_{ij}^\perp \cap P_0$ such that $\{\alpha_1, \ldots, \alpha_{n-1}, \beta, \gamma\}$ is a basis for $M_{ij}^\perp$. Let $W_i$ be the maximal isotropic subspace spanned by $\{\alpha_1, \ldots, \alpha_{n-1}, g_{ij}\beta + \gamma\}$, where $k \in \{1, 2, \ldots, q^n\}$. By $V_1 \cap P_0 = \{0\}$, we have $\beta \notin M_{ij} + P_0$ and $g_{ij}\beta + \gamma \notin M_{ij} + P_0$ for each $k$. Hence $W_k \cap P_0 = \{0\}$ and $W_{ij} = \{V_1, W_1, W_2, \ldots, W_{q^n-1}\}$. Our assertion is valid.

For any $i$ and $j$, pick a vertex $W_{ij} \in W_{ij}$ with $W_{ij} \neq V_1$. Write $\mathcal{M} = \bigcup_{i=2}^{q^n+1} \left( \bigcup_{j=1}^{q^n} \{V_1, W_{ij}\} \right)$. Then $|\mathcal{M}| \leq q^n(q^n + q - 2)/(q - 1)$. The proof of Theorem 1.1 implies that $\mathcal{M}$ is a resolving set. □

**Remark 2.** The bounds for $|\mathcal{M}|$ in Theorems 1.1 and 1.2 are likely to be over-estimated if $n = 2$. In fact, for $M_{ij} \in \mathcal{M}_i$ and $M_{ij}^\perp \in \mathcal{M}_{i'}$ with $i \neq i'$, we have $\dim(M_{ij} + M_{ij}^\perp) = 2$. Then $W_{ij} = W_{ij}^\perp = M_{ij} + M_{ij}^\perp$ is possible. The bounds for $|\mathcal{M}|$ in Theorems 1.1 and 1.2 are tight if $n \geq 3$.

**Acknowledgments**

The authors wish to thank the referees for their helpful comments and suggestions. This research is supported by NSFC (11271047, 10971052), NSF of Hebei Province (A2012408003), TPF-2011-11 of Hebei Province, HESTRF of Hebei Education Department (ZH2012082), Langfang Teachers’ College (LSZB201104), SRFPD and the Fundamental Research Funds for the Central University of China.

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