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# Non-rational centers of log canonical singularities

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ABSTRACT

We show that if $(X, B)$ is a log canonical pair with $\dim X \geqslant d+2$, whose non-klt centers have dimension $\geqslant d$, then $X$ has depth $\geqslant d+2$ at every closed point.



**Contents**



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## 1. Introduction

The main classes of singularities in the Minimal Model Program are: terminal, canonical, log terminal, and log canonical. It is well known that the singularities in the first three classes are Cohen–Macaulay and rational, and in the last class they are neither, in general. The main aim of this paper is to establish several general tools for measuring how far a particular log canonical singularity is from being CM (Cohen–Macaulay) or rational. (We note that [KK10] takes a different approach, and proves that all log canonical singularities are Du Bois.)

Let $X$ be an algebraic variety over an algebraically closed field of characteristic zero and let $f: Y \to X$ be a resolution of singularities. It is well known that the sheaves $R^i f_* \mathcal{O}_Y$ are coherent on $X$ and do not depend on the choice of the resolution. We make the following definition.

**Definition 1.1.** Let $X$ be a normal algebraic variety. The **centers of non-rational singularities** of $X$ (or simply **non-rational centers**) are the subvarieties $Z_i$ defined by the associated primes of the sheaves $R^i f_* \mathcal{O}_Y$, $i > 0$.

Thus, each non-rational center $Z_i$ is an irreducible subvariety of $X$, by definition.

Let $(X, B)$ be a klt pair (see Section 2 below for all standard definitions). Then $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$ and $X$ has rational singularities, hence $X$ is CM. As mentioned above, log canonical does not imply rational or CM: the simplest example is provided by the cone over an abelian surface in which case $R^1 f_* \mathcal{O}_Y$ is supported at the vertex and $X$ is also not CM. Our first main theorem is the following:

**Theorem 1.2.** *Let $(X, B)$ be a log canonical pair. Then every non-rational center of $X$ is a non-klt center of $(X, B)$.*

**Remark 1.3.** A similar result was independently proven in [Kov11]. Note that in [Kov11] the terminology “irrational centers” is used instead of non-rational centers.

Note that the closed set of non-rational singularities is a subset of the closed set of non-klt singularities, but (1.2) is far from being obvious.

It is natural to assume that the failure of log canonical pairs to be CM can be described in terms of the non-rational centers. There are several ways to measure this failure. A variety is CM if and only if it satisfies Serre’s condition $S_{\dim X}$ or, equivalently, if and only if it is $S_{\dim X}$ *at every closed point.* Thus, the conditions $S_n$ generalize the CM property. But there is another logical generalization:

**Definition 1.4.** We say that a coherent sheaf $F$ on a scheme $X$ satisfies condition $C_n$ (or simply is $C_n$) if for every *closed point* $x \in \operatorname{Supp} F$ one has $\operatorname{depth} F_x \geqslant n$. (Here, “C” stands for a “closed point”.) If $F = 0$ then we say that $F$ is $C_n$ for all $n$. We say that $X$ is $C_n$ if so is $\mathcal{O}_X$.

It turns out that for projective varieties the $C_n$ condition is frequently easier to work with than the $S_n$ condition because it admits a simple cohomological criterion, see Lemma 2.3. Our second main result is the following theorem, generalizing the $C_3$ case contained in [Ale08, 3.1 and 3.2]. (Recently, O. Fujino communicated to us another proof of the $C_3$ case [Fuj09, 4.21 and 4.27].)

**Theorem 1.5.** *Let $X$ be a normal variety of $\dim X \geqslant d + 2$. Assume that the pair $(X, B)$ is log canonical and that every non-klt center of $(X, B)$ has dimension $\geqslant d$. Then:*

(1) *For each $i > 0$, the sheaf $R^i f_* \mathcal{O}_Y$ is $C_{d+1-i}$.*
(2) *$X$ is $C_{d+2}$.*

**Remark 1.6.** It would be interesting to know if a similar result is true when “non-klt centers” are replaced by “non-rational centers”.

## 2. Preliminaries

We work over the field of complex numbers $\mathbb{C}$.

### 2.1. Singularities of the MMP and non-klt centers

Let $X$ be a normal variety. A **boundary** is a $\mathbb{Q}$-divisor $B = \sum b_i B_i$ on $X$ such that $0 \leqslant b_i \leqslant 1$. If $K_X + B$ is $\mathbb{Q}$-Cartier, then $(X, B)$ is a **log pair**. A **log resolution** of a log pair $(X, B)$ is a projective birational morphism $f : Y \to X$ such that $Y$ is smooth and $\operatorname{Exc}(f) \cup f^{-1}(B)$ is a divisor with simple normal crossings. We may then uniquely write

$$K_Y + f_*^{-1} B \equiv f^*(K_X + B) + \sum a_E(X, B) E$$

where $E \subset Y$ are all the $f$-exceptional divisors. The numbers $a_E(X, B)$ are the **discrepancies** of $(X, B)$ along $E$. They do not depend on the choice of a log resolution $f$. A pair $(X, B)$ is

(1) **log canonical**, abbreviated **lc**, if $b_i \leqslant 1$ and $a_E(X, B) \geqslant -1$,
(2) **kawamata log terminal (klt)** if $b_i < 1$ and $a_E(X, B) > -1$

for some (or equivalently for all) log resolution.

A **non-klt place** is a component of $\llcorner B \lrcorner$ or a divisor of discrepancy $a_E(X, B) \leqslant -1$. A **non-klt center** is the image in $X$ of a non-klt place (note that often in the literature non-klt places and centers are called lc places and centers).

A pair $(X, B)$ such that $b_i \leqslant 1$ is **divisorially log terminal** (dlt for short) if there is a log resolution $f : Y \to X$ such that $a_E(X, B) > -1$ for any $f$-exceptional divisor $E \subset Y$. Equivalently, by [Sza94] $(X, B)$ is dlt if there is an open subset $U \subset X$ such that $U$ is smooth, $B|_U$ has simple normal crossings and $a_E(X, B) > -1$ for any divisor $E$ over $X$ with center contained in $X - U$.

### 2.2. CM, $S_n$, and $C_n$

**Definition 2.1.** A coherent sheaf $F$ on a Noetherian scheme $X$ is Cohen–Macaulay if for every scheme point $x \in \operatorname{Supp} F$ one has $\operatorname{depth} F_x = \dim_x \operatorname{Supp} F$. The sheaf $F$ satisfies Serre's condition $S_n$ if one has $\operatorname{depth} F_x \geqslant \min(n, \dim_x \operatorname{Supp} F)$.

Note that some authors (e.g. [KM98]), define $S_n$ by the condition $\operatorname{depth} F_x \geqslant \min(n, \dim_x X)$. We follow [EGA4, 5.7.1 and 5.7.2]. This should not lead to confusion in the settings of this paper.

Compare the condition $S_n$ with our condition $C_n$. One obvious difference is that in Definition 1.4 we did not ask for $\min(n, \dim_x \operatorname{Supp} F)$. Hence, a CM variety satisfies $S_{\dim X+1}$ but not $C_{\dim X+1}$. A more subtle difference is provided by the following example.

**Example 2.2.** Let $Y$ be the cone over an abelian surface and let $X = Y \times \mathbb{P}^1$. Then $X$ is not $S_3$ at the generic point of $Z = (\text{vertex}) \times \mathbb{P}^1$. However, $X$ is $S_3$ at every closed point $x \in X$: $\mathcal{O}_{X,x}$ is $S_3$ if and only if the hyperplane section $\mathcal{O}_{Y,x}$ is $S_2$, which is true since $Y$ is normal. Thus $X$ is $C_3$ but not $S_3$.

Thus, assuming $n \leqslant \dim X$, the property $S_n$ is stronger than $C_n$. On the other hand, knowing that a certain class of varieties satisfies $C_n$ allows to conclude the property $S_n$ for this class indirectly, as follows. (See e.g. [Ale08] for an application of this principle.) If $X$ is not $S_n$ at the generic point of a subvariety $Z$ then a general hyperplane section $H$ is not $S_n$ at the generic points of $Z \cap H$. Cutting down this way, we get to a variety which is not $C_n$, and this process preserves other nice properties of $X$, such as being log canonical.

The reason why the condition $C_n$ is so convenient to work with for projective varieties is the following simple cohomological criterion:

**Lemma 2.3.** *Let $F$ be a coherent sheaf on a projective scheme $X$ with an ample invertible sheaf $L$, over an algebraically closed field. Then $F$ is $C_n$ if and only if $H^i(X, F(-sL)) = 0$ for any $i < n$ and $s \gg 0$.*

**Proof.** This is basically proved in [Har77, III.7.6], although it is not stated there in this way. We give the proof here for clarity.

We embed $X$ into $P = \mathbb{P}^N$ by some power of $L$. Then the cohomology group $H^i(X, F(-q)) = H^i(P, F(-q))$ is dual to the group $\operatorname{Ext}_P^{N-i}(F, \omega_P(q))$ which equals $\Gamma(P, \underline{Ext}_P^{N-i}(F, \omega_P(q)))$ if $q \gg 0$ (where $\omega_P = \mathcal{O}_P(-N-1)$ is the dualizing sheaf). Thus, one has $H^i(X, F(-sL)) = 0$ for $i < n$ and $s \gg 0$ if and only if the sheaf $\underline{Ext}_P^{N-i}(F, \omega_P)$ is zero for $i < n$ or, equivalently, if $\underline{Ext}_P^{N-i}(F, \mathcal{O}_P) = 0$ for $i < n$.

This sheaf is zero if and only if its stalks are zero at every closed point $x \in \operatorname{Supp} F$. Denote $A = \mathcal{O}_{P,x}$ for short, it is a regular local ring. The stalk at $x$ is $\operatorname{Ext}_A^{N-i}(F_x, A)$ and it is zero for $i < n$ if and only if the projective dimension $\operatorname{pd}_A F_x \leqslant N - n$ (see [Har77, Ex. 6.6]). The latter is equivalent to $\operatorname{depth} F_x \geqslant n$ since $\operatorname{pd}_A F_x + \operatorname{depth} F_x = \dim A = N$ (cf. [Har77, 6.12A]). $\square$

**Example 2.4.** Let $Y$ be a projective klt variety such that $NK_Y \sim 0$ for some $N \in \mathbb{N}$. Let $L$ be an ample invertible sheaf on $Y$. Then the cone $X = \operatorname{Spec} \bigoplus_{k \geqslant 0} H^0(Y, L^k)$ is lc, and its vertex $P$ is the unique non-klt center. Indeed, for the blowup $f : X' \to X$ at $P$ which inserts an exceptional divisor $E = Y$, one has $K_{X'} + E = f^* K_X$, $(X', E)$ is dlt, and so $E$ is the only divisor with discrepancy $a_E(X, B) = -1$.

Assuming $n \leqslant \dim X$, the cone $X$ satisfies condition $C_n$ if and only if $H^i(\mathcal{O}_Y) = 0$ for $0 < i \leqslant n - 2$. This follows from the interpretation of the depth in terms of the local cohomology groups $H_P^i(\mathcal{O}_X) = \bigoplus_{k \in \mathbb{Z}} H^i(\mathcal{O}_Y(L^k))$ and the fact that $H^i(\mathcal{O}_Y(L^k)) = 0$ for $k \neq 0$ by the Kawamata–Viehweg vanishing and Serre duality.

In particular, if $Y$ is an abelian surface then $X$ is not $C_3$ at the vertex $P$, but if $Y$ is a K3 or Enriques surface then $Y$ *is* $C_3$. Since for $i > 0$ one has $R^i f_* \mathcal{O}_{X'} = H^i(\mathcal{O}_Y)$, we see that $P$ is a non-rational center if $Y$ is abelian or K3, and there are no non-rational centers if $Y$ is an Enriques surface.

By a theorem of Kempf, a variety $X$ has rational singularities if and only if $X$ is CM and $f_* \omega_Y = \omega_X$ (see e.g. [KM98, 5.12]). We conclude this section with the following result which we will need below.

**Theorem 2.5.** *Let $(X, B)$ be a dlt pair, then $X$ has rational singularities.*

**Proof.** See [KM98, 5.22]. $\square$

### *2.3. Ambro's and Fujino's results on quasi log varieties*

In this section we recall some definitions and results concerning quasi-log-canonical pairs (qlc pairs for short).

**Definition 2.6.** Let $Y \subset M$ be a simple normal crossings (reduced) divisor $Y$ on a smooth variety and $D$ be a divisor on $M$ whose support contains no components of $Y$ and such that $D + Y$ has simple normal crossings support. The pair $(Y, B = D|_Y)$ is a **global embedded simple normal crossings pair**. Let $\nu : Y^\nu \to Y$ be the normalization and $K_{Y^\nu} + \Theta = \nu^*(K_Y + D)$. A **stratum** of $(Y, B)$ is a component of $Y$ or the image of a non-klt center of $(Y^\nu, \Theta)$. Thus, the strata of $(Y, B)$ are irreducible by definition.

We have the following torsion-freeness and vanishing theorems of F. Ambro cf. [Fuj09, 2.47]:

**Theorem 2.7.** *Let $(Y, B)$ be a global embedded simple normal crossings pair. Assume that $B$ is a boundary $\mathbb{Q}$-divisor $L$ is a Cartier divisor and $f : Y \to X$ is a proper morphism. Then*

(1) *If $L-(K_Y+B)$ is semiample over $X$, then every non-zero local section of $R^q f_*\mathcal{O}_Y(L)$ contains in its support the image of some stratum of $(Y,B)$.*
(2) *If $\pi: X\to Z$ is a projective morphism and there is a $\mathbb{Q}$-Cartier divisor $H$ on $X$ such that $f^*H\sim_{\mathbb{Q}} L-(K_Y+B)$, and such that $H$ is big and nef on the image of every stratum of $(Y,B)$, then $R^p\pi_*(R^q f_*\mathcal{O}_Y(L))=0$ for any $p>0$ and $q\geqslant 0$.*

**Proof.** See [Amb03, 3.2], [Fuj09, 2.47]. □

**Corollary 2.8.** *Let $\mathcal{O}_Y(L)$ be as in part (1) of the above Theorem 2.7. Then for any $q\geqslant 0$, each associated prime of the sheaf $R^q f_*\mathcal{O}_Y(L)$ is the generic point of the image of a stratum of $(Y,B)$.*

**Proof.** Since the question is local on the base, we can assume that $X=\operatorname{Spec} R$ is affine. Let $P\subset R$ be an associated prime of $R^q f_*\mathcal{O}_Y(L)$. By definition, there exists a non-zero section $s\in\Gamma(X, R^q f_*\mathcal{O}_Y(L))$ whose support is $Z(P)$.

We claim that $Z(P)$ is the image of a stratum of $(Y,B)$. Suppose it is not. Let $Z(Q_i)$ be all the (finitely many) images of strata of $(Y,B)$ that are contained in $Z(P)$. We have $Z(Q_i)\neq Z(P)$. Over the open subset $U=X\setminus\bigcup Z(Q_i)$ the support of the section $s|_U$ is $Z(P)\cap U\neq\emptyset$. But by (2.7)(1) we must have $s|_U=0$. Contradiction. □

We will also need the following weak form of the above theorem:

**Theorem 2.9.** *In the settings of the above theorem, in part (2) assume instead that $H$ is nef and that there exists an ample divisor $M$ on $X$ such that for every image $V$ of a stratum of $(Y,B)$ with $\dim V\geqslant c$, one has $V\cdot M^c\cdot H^{\dim V-c}>0$.*

*Then $R^p\pi_*(R^q f_*\mathcal{O}_Y(L))=0$ for any $p>c$ and $q\geqslant 0$.*

**Proof.** Let $D_X$ be a general element in the linear system $|nM|$, with $nM$ very ample. Denote $D_Y:=f^{-1}D_X$ and $L_{D_Y}=(L+D_Y)|_{D_Y}$. Consider the short exact sequence

$$0\to\mathcal{O}_Y(L)\to\mathcal{O}_Y(L+D_Y)\to\mathcal{O}_{D_Y}(L_{D_Y})\to 0.$$

In this sequence the line bundle $L_{D_Y}$ is $\mathbb{Q}$-linearly equivalent to $(K_Y+D_Y+B+f^*H)|_{D_Y}=K_{D_Y}+B\cap D_Y+f^*H|_{D_Y}$, and so is of the same nature as $L$ but for the smaller global embedded simple normal crossing pair $(D_Y, B\cap D_Y)$.

The images of the strata of $(D_Y, B\cap D_Y)$ are strictly smaller than the images of the strata of $(Y,B)$. Part (1) of the above Theorem 2.7 implies that in the long exact sequence the connecting homomorphisms $R^q f_*\mathcal{O}_{D_Y}(L_{D_Y})\to R^{q+1}f_*\mathcal{O}_Y(L)$ are zero. Indeed, the sections in the image are generically zero at the image of every stratum of $(Y,B)$. Thus, for every $q$ we have a short exact sequence

$$0\to R^q f_*\mathcal{O}_Y(L)\to R^q f_*\mathcal{O}_Y(L+D_Y)\to R^q f_*\mathcal{O}_{D_Y}(L_{D_Y})\to 0.$$

Since $H+D_X$ is ample, for the middle term we have $R^p\pi_*(R^q f_*\mathcal{O}_Y(L+D_Y))=0$ for $p>0$ by (2.7)(2). Thus, $R^p\pi_*(R^q f_*\mathcal{O}_{D_Y}(L_{D_Y}))=0$ for $p>k$ implies $R^p\pi_*(R^q f_*\mathcal{O}_Y(L))=0$ for $p>k+1$.

Cutting $c$ times by general divisors in $|nM|$, we arrive at the situation of (2.7)(2), which gives vanishing of $R^p\pi_*$ for $p>0$. For the original sheaves $R^q f_*\mathcal{O}_Y(L)$, this gives vanishing of $R^p\pi_*$ for $p>c$. □

**Definition 2.10.** A **qlc** variety is a variety $X$, a $\mathbb{Q}$-Cartier divisor $\omega$ on $X$ and a finite collection $\{C\}$ of irreducible reduced subvarieties of $X$ such that there is a proper morphism $f: Y\to X$ from a global embedded simple normal crossings pair $(Y,B)$ such that:

(1) $f^*\omega \sim_{\mathbb{Q}} K_Y + B$ and $B$ is a boundary divisor,
(2) $\mathcal{O}_X \cong f_*\mathcal{O}_Y(\ulcorner -(B_Y^{<1})\urcorner)$,
(3) $\{C\}$ is given by the images of the strata of $(Y, \llcorner B \lrcorner)$.

The elements of $\{C\}$ are the **qlc centers** of the **qlc pair** $[X, \omega]$.

**Proposition 2.11.** *If $(X, B)$ is a lc pair, then $X$ is a qlc pair with $\omega = K_X + B$ and $\{C\}$ is given by $X$ and the non-klt centers of $(X, B)$.*

**Proof.** See [Fuj09, 3.31]. □

**Proposition 2.12.** *Let $[X, \omega]$ be a qlc pair, $X'$ be a union of qlc centers of $[X, \omega]$ with the reduced scheme structure and $\mathcal{I}_{X'} \subset \mathcal{O}_X$ the corresponding ideal. Then*

(1) *$[X', \omega' = \omega|_{X'}]$ is a qlc pair whose centers are the centers of $[X, \omega]$ contained in $X'$.*
(2) *If $X$ is projective and $L$ is a Cartier divisor on $X$ such that $L - \omega$ is ample, then*

$$H^q\big(\mathcal{O}_X(L)\big) = H^q\big(\mathcal{I}_{X'} \otimes \mathcal{O}_X(L)\big) = 0 \quad \textit{for } q > 0.$$

*In particular $H^q(\mathcal{O}_{X'}(L)) = 0$ for $q > 0$ as $[X', \omega']$ is a qlc pair and $L|_{X'} - \omega'$ is ample.*

**Proof.** See [Amb03, 4.4], [Fuj09, 3.39]. □

## 3. Non-rational centers

### 3.1. The "Get rid of the A" trick

Let $f : Y \to X$ be a log resolution of $(X, B)$. We write

$$K_Y + E - A + \Delta = f^*(K_X + B)$$

where $E \geqslant 0$ is reduced, $A \geqslant 0$ is integral and exceptional, and $\llcorner \Delta \lrcorner = 0$. Here, $E$ has no common components with either $A$ or $\Delta$, but $A$ and $\Delta$ may have common components.

**Theorem 3.1.** *Let $(X, B)$ be a lc pair. Then there exist morphisms $f' : Y' \to X$ and $\nu : Y \to Y'$ such that*:

(1) *$f = f' \circ \nu$ is a log resolution of $(X, B)$,*
(2) *$Y'$ is normal, $E' = \nu_* E$, $\Delta' = \nu_* \Delta$, and $A' = \nu_* A = 0$,*
(3) *$(Y', E' + \Delta')$ is dlt,*
(4) *$\nu$ is an isomorphism at the generic point of each non-klt center of $(Y, E + \Delta)$ and in particular there is a bijection between the non-klt centers of $(Y, E + \Delta)$ and the non-klt centers of $(Y', E' + \Delta')$.*

**Proof.**

**Claim 3.2.** *For any resolution $Y_1 \to X$, there exists a rational map $\alpha : Y_1 \dashrightarrow Y'$ and a morphism $f' : Y' \to X$ such that $\alpha_* A_1 = 0$, $K_{Y'} + \nu_*(E_1 + \Delta_1)$ is dlt and $K_{Y'} + \nu_*(E_1 + \Delta_1) = (f')^*(K_X + B)$.*

**Proof.** By [BCHM] we may run a $(K_{Y_1} + \Delta_1)$-MMP over $X$ say $\beta : Y_1 \dashrightarrow Y_2$. Let $f_2 : Y_2 \to X$ be the corresponding morphism, $E_2 = \beta_* E_1$, $A_2 = \beta_* A_1$ and $\Delta_2 = \beta_* \Delta_1$. Since

$$E_2 - A_2 + (K_{Y_2} + \Delta_2) = (f_2)^*(K_X + B),$$

$K_{Y_2} + \Delta_2$ is nef over $X$ and $(f_2)_*(E_2 - A_2) \geqslant 0$, by the Negativity Lemma, $E_2 - A_2 \geqslant 0$. As $A_2$ and $E_2$ have no common components, $A_2 = 0$. Therefore

$$K_{Y_2} + E_2 + \Delta_2 = (f_2)^*(K_X + B),$$

$(Y_2, E_2 + \Delta_2)$ is lc with the same non-klt places as $(X, B)$ and $Y_2$ is $\mathbb{Q}$-factorial.

Let $\gamma : Y_3 \to Y_2$ be a log resolution of $(Y_2, E_2 + \Delta_2)$. We write

$$K_{Y_3} + \Gamma = \gamma^*(K_{Y_2} + E_2 + \Delta_2) + F$$

where $\Gamma$ and $F$ are effective with no common components, $\gamma_*(\Gamma) = E_2 + \Delta_2$ and $\gamma_* F = 0$. Let $C$ be an effective $\gamma$-exceptional divisor such that $-C$ is ample over $Y_2$ and $\|C\| \ll 1$. Let $H \sim_{\mathbb{Q}, Y_2} -C$ be a general ample $\mathbb{Q}$-divisor such that $K_{Y_3} + \Gamma + H$ is dlt. We have that

$$\Gamma + H \sim_{\mathbb{Q}, Y_2} \Gamma - C \sim_{\mathbb{Q}, Y_2} \Xi$$

where $(Y_3, \Xi)$ is klt. After running a $(K_{Y_3} + \Xi)$-MMP over $Y_2$, we obtain $\mu : Y' \to Y_2$ such that $K_{Y'} + \Xi'$ is $\gamma$-nef, where $\Xi'$ denotes the strict transform of $\Xi$ on $Y'$ and similarly for other divisors. Since

$$C' - F' + \left(K_{Y'} + \Xi'\right) \sim_{\mathbb{Q}, Y_2} 0$$

and $\mu_*(C' - F') \geqslant 0$, then by the Negativity Lemma, we have that $C' - F' \geqslant 0$. Since $\|C\| \ll 1$, this implies that $F' = 0$ so that $K_{Y'} + \Gamma' = \mu^*(K_{Y_2} + E_2 + \Delta_2)$. Since a $(K_{Y_3} + \Xi)$-MMP over $Y_2$ is automatically a $(K_{Y_3} + \Gamma + H)$-MMP over $Y_2$, it follows that $K_{Y'} + \Gamma' + H'$ is dlt. In particular $K_{Y'} + \Gamma'$ is dlt. $\square$

**Claim 3.3.** *There exists a resolution $\nu : Y \to Y'$ which is an isomorphism at the generic point of any non-klt center of $(Y, E + \Delta)$.*

**Proof.** This follows from the characterization of dlt singularities given in [Sza94]. $\square$

Theorem 3.1 now follows. $\square$

**Remark 3.4.** Note that variants of (3.2) appear in [KK10, 3.1] and [Fuj11, 10.4], where they are referred to as an unpublished theorem of Hacon.

**Corollary 3.5.** *Let $Y$ be as in Theorem 3.1. Then one has $R^i f_* \mathcal{O}_Y = R^i f_* \mathcal{O}_Y(A)$ for all $i \geqslant 0$.*

**Proof.** We have $\nu_* \mathcal{O}_Y(A) = \mathcal{O}_{Y'}$, and for $i > 0$ one has $R^i \nu_* \mathcal{O}_Y(A) = 0$ at the generic point of any non-klt center of $(Y', E' + \Delta')$ and so by (2.7), $R^i \nu_* \mathcal{O}_Y(A) = 0$. Therefore, $R^\bullet \nu_* \mathcal{O}_Y(A) = \mathcal{O}_{Y'}$.

Since $(Y', B')$ is dlt, by (2.5) $Y'$ has rational singularities, so $R^\bullet \nu_* \mathcal{O}_Y = \mathcal{O}_{Y'}$. It follows that $R^\bullet f_* \mathcal{O}_Y(A) = R^\bullet f_* \mathcal{O}_Y$ and so $R^i f_* \mathcal{O}_Y = R^i f_* \mathcal{O}_Y(A)$ for all $i$. $\square$

**Proof of Theorem 1.2.** Let $f : (Y, E + \Delta) \to X$ be a log resolution of $(X, B)$, as in Theorem 3.1. Thus, $(Y, E + \Delta)$ is a normal crossing pair, and the $f$-images of the strata of $(Y, E + \Delta)$ are the non-klt centers of $(X, B)$.

By (2.8), the zero locus of any associated prime of $R^i f_* \mathcal{O}_Y(A)$ is a non-klt center. We are now done by the above Corollary 3.5. $\square$

**Theorem 3.6.** *For* any *log resolution $f : Y \to X$ one has $R^\bullet f_* \mathcal{O}_Y(A) = R^\bullet f_* \mathcal{O}_Y$.*

**Proof.** Let $f_1 : Y_1 \to X$, $f_2 : Y_2 \to X$ be two log resolutions. First, we are going to construct modifications $g_1 : Y_1' \to Y_1$, $g_2 : Y_2' \to Y_2$ by blowing up *the strata of* $E_1$*, resp.* $E_2$*, only*. If $g_1$ is a sequence of such blowups then

$$R^\bullet(g_1)_*\mathcal{O}_{Y_1'}\big(A_1'\big) = \mathcal{O}_{Y_1}(A_1) \quad \text{and} \quad R^\bullet(g_1)_*\mathcal{O}_{Y_1'} = \mathcal{O}_{Y_1}.$$

The second equality follows because $Y_1$ is nonsingular. For the first equality, note that over the generic point of each stratum of $E_1$ one has $\mathcal{O}_{Y_1'}(A_1') = \mathcal{O}_{Y_1'}$. Indeed, in this case $A_1'$ is the strict preimage of $A_1$, and $E_1 \cup \operatorname{Supp} A_1$ is a normal crossing divisor. Therefore, $R^i(g_1)_*\mathcal{O}_{Y_1'}(A_1') = 0$ for $i > 0$ at the generic point of each stratum of $E_1$. Hence, by (2.7), $R^i(g_1)_*\mathcal{O}_{Y_1'}(A_1') = 0$ for $i > 0$.

By making such sequences of blowups $g_1$, $g_2$, we can assume that $Y_1'$, $Y_2'$ have the same places of non-klt singularities of $(X, B)$ and that the birational map $\phi : Y_1' \dashrightarrow Y_2'$ is an isomorphism at the generic point of each stratum of $E_1'$ and $E_2'$.

Now by Hironaka there exists a sequence of blowups $h_1 : \tilde{Y} \to Y_1'$ and a regular map $h_2 : \tilde{Y} \to Y_2'$ resolving the indeterminacies of $\phi$. If the blowups $h_1$ are performed only inside the nonregular locus of $\phi$, as it can always be done, then $h_1$, $h_2$ are isomorphisms at the generic point of each stratum of $\tilde{E}$. Applying (2.7) again, we get

$$R^\bullet(h_k)_*\mathcal{O}_{\tilde{Y}}(\tilde{A}) = \mathcal{O}_{Y_k'}\big(A_k'\big) \quad \text{and} \quad R^\bullet(h_k)_*\mathcal{O}_{\tilde{Y}} = \mathcal{O}_{Y_k'} \quad \text{for } k = 1, 2.$$

Putting this together, we get

$$R^\bullet(f_k \circ g_k \circ h_k)_*\mathcal{O}_{\tilde{Y}}(\tilde{A}) = R^\bullet(f_k)_*\mathcal{O}_{Y_k}(A_k) \quad \text{for } k = 1, 2.$$

Since $f_1 \circ g_1 \circ h_1 = f_2 \circ g_2 \circ h_2$, we get

$$R^\bullet(f_1)_*\mathcal{O}_{Y_1}(A_1) = R^\bullet(f_2)_*\mathcal{O}_{Y_2}(A_2) \quad \text{and} \quad R^\bullet(f_1)_*\mathcal{O}_{Y_1} = R^\bullet(f_2)_*\mathcal{O}_{Y_2}.$$

Now if we choose $Y_1$ to be as in Corollary 3.5 then we get the same conclusion for any other resolution $Y_2$. □

### 3.2. A resolution separated into levels

For any $l \geqslant 0$, let $E'_{\geqslant l}$ (resp. $E'_{=l}$ and $E'_{\leqslant l}$) be the sum of the components of $E'$ whose image via $f' : Y' \to X$ has dimension at least (resp. equal to and less or equal to) $l$. We use a similar notation for $E$.

**Proposition 3.7.** *In Theorem 3.1 we may assume that the dimension of the image via $f$ of any stratum of $E_{\geqslant l}$ is at least $l$.*

**Proof.** Let $\mu : \tilde{Y} \to Y$ be a log resolution of $(Y, E + \Delta)$ and of any non-klt center $V$ of $(X, B)$. We then have that the dimension of the image via $\tilde{f} = f \circ \mu$ of any stratum of $\tilde{E}_{\geqslant l}$ is at least $l$. Notice in fact that if this is not the case, then there are divisors $F_1, \dots, F_k$ given by components of $\tilde{E}_{\geqslant l}$ such that $W = F_1 \cap \dots \cap F_k$ is a non-klt center of $(\tilde{Y}, \tilde{E} + \tilde{\Delta})$ with $\dim \tilde{f}(W) < l$. But then as $\tilde{f}(W)$ is a non-klt center, $W$ is contained in a component of $\tilde{E}$ different from $F_1, \dots, F_k$. As $\tilde{E}$ has simple normal crossings, this is impossible.

We must now show that there are morphisms $\tilde{\nu} : \tilde{Y} \to \tilde{Y}'$ and $\eta : \tilde{Y}' \to Y'$ such that $\nu \circ \mu = \eta \circ \tilde{\nu}$, $\tilde{\nu}_*(\tilde{E} + \tilde{\Delta}) = \tilde{E}' + \tilde{\Delta}'$ where $(\tilde{Y}', \tilde{E}' + \tilde{\Delta}')$ is dlt, $\tilde{A}' = 0$ and that $\tilde{\nu}$ is an isomorphism at the generic point of any non-klt center of $(\tilde{Y}, \tilde{E} + \tilde{\Delta})$.

Let $U$ be an open subset of $Y'$ which is isomorphic to an open subset of $Y$ such that $(E'+\Delta')|_U$ has simple normal crossings support and for any divisor $F$ exceptional over $Y'$ with center contained in $Z = Y' - U$, we have $a(F, Y', E'+\Delta') > -1$. We may assume that if $G \subset \tilde{Y}$ is a $\mu$-exceptional divisor such that $a(G, Y, E+\Delta) > -1$, then $\mu(G) \subset Z$. For any $0 < \epsilon \ll 1$, we write $\nu^*(K_{Y'} + (1-\epsilon)(E'+\Delta')) + F = K_Y + \Gamma$ where $F$ and $\Gamma$ are effective with no common components and $\nu_*(\Gamma) = (1-\epsilon)(E'+\Delta')$ and we let $K_{\tilde{Y}} + \tilde{\Gamma} = \mu^*\nu^*((K_{Y'} + (1-\epsilon)(E'+\Delta')) + F) + \tilde{F}'$. We now run a $(K_{\tilde{Y}} + \tilde{\Gamma})$-MMP over $Y'$ to obtain $\eta : \tilde{Y}' \to Y'$.

Since over $U$, $F$ and $\tilde{F}'$ are zero, we have that $\tilde{\nu}$ is an isomorphism over $U$. There is a neighborhood of $Z$ over which $(Y', E'+\Delta')$ is klt. It follows that $(\tilde{Y}', \tilde{E}'+\tilde{\Delta}')$ is dlt. After blowing up centers over $Z$, we may assume that $\tilde{\nu}$ is a morphism. □

## 4. The sheaves $R^i f_*\mathcal{O}_Y$ and the proof of the main theorem

### 4.1. Leray spectral sequence

**Lemma 4.1.** *Let $X$ be a projective normal variety of dimension $\geqslant n$ and $f : Y \to X$ a resolution of singularities. Assume that the sheaves $R^i f_*\mathcal{O}_Y$, $i > 0$, are $C_{n-1-i}$. Then $X$ is $C_n$.*

**Proof.** Let $L$ be an ample sheaf on $X$. Consider the Leray spectral sequence

$$E_2^{p,q} = H^p\big(R^q f_*\mathcal{O}_Y(-sL)\big) \Rightarrow H^{p+q}\big(\mathcal{O}_Y\big(-sf^*L\big)\big), \quad \text{for some } s \gg 0.$$

Since $f^*L$ is big and nef, the limit $E_\infty^k = H^k(\mathcal{O}_Y(-sf^*L))$ is zero for $k < n \leqslant \dim X$ by the Kawamata–Viehweg vanishing theorem.

By the assumption, we have $E_2^{p,q} = 0$ for $p+q \leqslant n-2$ and $q > 0$. Inspecting this spectral sequence we easily conclude that $E_2^{p,0} = E_\infty^{p,0}$ for $p \leqslant n-1$. On the other hand, we have $E_\infty^{p,0} \subset E_\infty^p$ and the latter is zero for $p \leqslant n-1$. So $H^p(\mathcal{O}_X(-sL)) = 0$ for $p \leqslant n-1$ and $\mathcal{O}_X$ is $C_n$ by (2.3). □

**Remark 4.2.** For the $C_3$ case, [Ale08, 3.1] gives a necessary and sufficient condition: $X$ is $C_3$ $\Leftrightarrow$ $R^1 f_*\mathcal{O}_Y$ is $C_1$.

To prove Theorem 1.5, it is now sufficient to prove that $H^p(X, R^q f_*\mathcal{O}_Y(-sL)) = 0$ for $q > 0$, $p+q \leqslant d$ and $s \gg 0$. The rest of this section will be devoted to establishing this fact.

### 4.2. Vanishing theorems for unions of centers

Let $(X, B)$ be a lc pair and let $f' : Y' \to X$ and $\nu : Y \to Y'$ be as in Section 3.2. In particular, we may assume that $Y$ and $Y'$ satisfy (3.1) and (3.7).

**Lemma 4.3.** *Let $Z$ and $W$ be unions of non-klt centers of $(Y, E+\Delta)$ and let $Z' = \nu(Z)$, $W' = \nu(W)$. Then*

$$R^\bullet \nu_*\mathcal{O}_Z(-W) = R^\bullet \nu_*\mathcal{O}_Z(A-W) = \mathcal{O}_{Z'}\big(-W'\big),$$

*where $\mathcal{O}_{Z'}(-W')$ is the ideal sheaf of $W' \cap Z'$ in $Z'$.*

**Proof.** If $Z$ is irreducible, then $(Z, (K_Y+E+\Delta)|_Z)$ is a global embedded simple normal crossings pair. For $j > 0$, $R^j\nu_*\mathcal{O}_Z(A) = 0$ at the generic point of any non-klt center of $(Y', E'+\Delta')$ (cf. (3.1), (3.3)) and hence $R^j\nu_*\mathcal{O}_Z(A) = 0$ by (2.7). Since $[Z', (K_{Y'}+E'+\Delta')|_{Z'}]$ is a qlc variety cf. (2.12), we also have $\nu_*\mathcal{O}_Z(A) = \mathcal{O}_{Z'}$ cf. (2) of (2.10).

We proceed by induction on $d$ the maximum dimension of a component of $Z$, the case $d = 0$ being obvious. For a fixed $d$, we proceed by induction on the number of components of $Z$. If $Z$ is irreducible, then as $(Y', E'+\Delta')$ is dlt, $Z'$ is normal with rational singularities and $R^\bullet\nu_*\mathcal{O}_Z \cong \mathcal{O}_{Z'} \cong R^\bullet\nu_*\mathcal{O}_Z(A)$. By induction on $d$, $R^\bullet\nu_*\mathcal{O}_{Z\cap W} \cong R^\bullet\nu_*\mathcal{O}_{Z\cap W}(A) \cong \mathcal{O}_{Z'\cap W'}$. Pushing forward the short exact sequence

$$0 \to \mathcal{O}_Z(A-W) \to \mathcal{O}_Z(A) \to \mathcal{O}_{Z\cap W}(A) \to 0$$

and noticing that

$$\nu_*\mathcal{O}_Z = \nu_*\mathcal{O}_Z(A) \cong \mathcal{O}_{Z'} \to \nu_*\mathcal{O}_{Z\cap W} = \nu_*\mathcal{O}_{Z\cap W}(A) \cong \mathcal{O}_{Z'\cap W'}$$

is surjective, it follows that $R^\bullet\nu_*\mathcal{O}_Z(-W) \cong R^\bullet\nu_*\mathcal{O}_Z(A-W) \cong \mathcal{O}_{Z'}(-W')$.

If $Z$ is not irreducible, then let $Z_0$ be an irreducible component of $Z$ and write $Z = Z_0 + Z_1$ where $Z_1$ is the union of the components of $Z$ distinct from $Z_0$. Consider the short exact sequence

$$0 \to \mathcal{O}_{Z_0}(A-W-Z_1) \to \mathcal{O}_Z(A-W) \to \mathcal{O}_{Z_1}(A-W) \to 0.$$

By induction on $d$ the number of components of $Z$, we have $R^\bullet\nu_*\mathcal{O}_{Z_1}(-W) \cong R^\bullet\nu_*\mathcal{O}_{Z_1}(A-W) \cong \mathcal{O}_{Z'_1}(-W')$. By what we have shown above, $R^\bullet\nu_*\mathcal{O}_{Z_0}(-W-Z_1) \cong R^\bullet\nu_*\mathcal{O}_{Z_0}(A-W-Z_1) \cong \mathcal{O}_{Z'_0}(-W'-Z'_1)$. It follows that $R^\bullet\nu_*\mathcal{O}_Z(-W) \cong R^\bullet\nu_*\mathcal{O}_Z(A-W) \cong \mathcal{O}_{Z'}(-W')$. The assertion is proved. □

Let $L$ be an ample line bundle on $X$.

**Lemma 4.4.** *If $X$ is projective, then $H^j(\mathcal{O}_{Y'}(-s(f')^*L)) = 0$ for all $j < n$ and $s > 0$.*

**Proof.** Since $(Y', \nu_*(E+\Delta) = E'+\Delta')$ is dlt, it has rational singularities and so $R^\bullet\nu_*\mathcal{O}_Y \cong \mathcal{O}_{Y'}$. Therefore, by Serre duality,

$$H^j\big(\mathcal{O}_{Y'}\big(-s\big(f'\big)^*L\big)\big) \cong H^j\big(\mathcal{O}_Y\big(-sf^*L\big)\big) \cong H^{n-j}\big(\mathcal{O}_Y\big(K_Y+sf^*L\big)\big)^\vee$$

and the lemma follows from Kawamata–Viehweg vanishing cf. [Laz04, 4.3.7]. □

**Lemma 4.5.** *Assume that $X$ is projective. Let $0 \leqslant F \leqslant E$ be a reduced divisor such that the minimum of the dimension of the images of any stratum of $F$ in $X$ is $\geqslant k$. Let $Z$ and $W$ be unions of non-klt centers of $(Y, F)$. Then*

$$H^l\big(\mathcal{O}_Z\big(-W-sf^*L\big)\big) \cong H^l\big(\mathcal{O}_Z\big(A-W-sf^*L\big)\big) \cong H^l\big(\mathcal{O}_{Z'}\big(-W'-s\big(f'\big)^*L\big)\big) = 0$$

*for all $l \leqslant k-1$ and any $s > 0$.*

**Proof.** Since $R^\bullet\nu_*\mathcal{O}_Z(-W) \cong R^\bullet\nu_*\mathcal{O}_Z(A-W) \cong \mathcal{O}_{Z'}(-W')$ cf. (4.3), it suffices to prove that $H^l(\mathcal{O}_Z(-W-sf^*L)) = 0$. If $Z$ is irreducible, then it is a smooth variety. We have $(f^*L|_Z)^k \neq 0$ and $f^*L|_Z$ is nef so that by Kawamata–Viehweg vanishing (cf. [Laz04, 4.3.7]), we have

$$h^l\big(\mathcal{O}_Z\big(-sf^*L\big)\big) = 0 \quad \forall l \leqslant k-1.$$

In general, the proof is by induction on the maximal dimension of a component of $Z$ and on the number of components of $Z$. When $\dim Z = 0$, there is nothing to prove. If $Z$ is irreducible, then the statement follows from the short exact sequence

$$0 \to \mathcal{O}_Z(-W) \to \mathcal{O}_Z \to \mathcal{O}_{Z\cap W} \to 0$$

and induction on $\dim Z \cap W$.

If $Z$ is not irreducible, then we let $Z_0$ be an irreducible component of $Z = Z_0 \cup Z_1$ and we consider the short exact sequence

$$0 \to \mathcal{O}_{Z_0}(-W - Z_1) \to \mathcal{O}_Z(-W) \to \mathcal{O}_{Z_1}(-W) \to 0.$$

The statement now follows by induction on the number of components and what we have shown above. $\square$

**Lemma 4.6.** *If $X$ is projective and $K_X + B$ is Cartier and every non-klt center of $(X, B)$ has dimension $\geqslant d$, then $H^i(R^j f_* \mathcal{O}_{E_{=k}}(A - E + E_{\geqslant k} - sf^*L)) = 0$ for $i + j \leqslant d - 1$ and $s > 0$ (resp. for $i + j = d$, $j > 0$ and $s > 0$).*

The proof given below is fairly technical. We use the prepared resolution of Section 3.2, previously established vanishing results, and a multilevel induction. At the heart of the argument, however, is the method of Kollár introduced in [Kol87], which gives vanishing and torsion free theorems for the sheaves $R^j f_* \omega_Y$ by using variation of pure Hodge structures, and the extension of this method to variation of mixed Hodge structures in [Kaw02,Kaw09,Kaw11].

**Proof of Lemma 4.6.** We will begin by proving the required vanishing for $i + j \leqslant d - 1$ and $s > 0$. Let $V_1, \dots, V_\tau$ be the irreducible non-klt centers of $(X, B)$ of dimension $k$ and let $Z_t$ be the union of the components of $E_{=k}$ that dominate $V_t$. Note that $E - E_{\geqslant k} = E_{\leqslant k-1}$. Let $Z_{\geqslant t} = Z_t + \cdots + Z_\tau$. We have short exact sequences

$$\begin{aligned} 0 &\to \mathcal{O}_{Z_t}(A - E_{\leqslant k-1} - Z_{\geqslant t+1}) \to \mathcal{O}_{Z_{\geqslant t}}(A - E_{\leqslant k-1}) \\ &\to \mathcal{O}_{Z_{\geqslant t+1}}(A - E_{\leqslant k-1}) \to 0. \end{aligned} \tag{$\star$}$$

Note that by (3.7), $Z_t \cap Z_{\geqslant t+1} = \emptyset$, so that the sequences $(\star)$ are split and we have the equalities $(A - E_{\leqslant k-1} - Z_{\geqslant t+1})|_{Z_t} \sim_X K_{Z_t} + E_{\geqslant k+1}|_{Z_t}$. Since the above short exact sequence $(\star)$ is split, it remains exact (and split) after applying $R^j f_*$ and twisting by $-sL$. Thus, it suffices to show that $H^i(R^j f_* \mathcal{O}_{Z_t}(A - E_{\leqslant k-1} - Z_{\geqslant t+1} - sf^*L)) = 0$ for $i + j \leqslant d - 1$ and $s > 0$.

Let $Z = Z_t$ and $V = V_t$. We may assume that there are resolutions $g : \tilde{Z} \to Z$ and $h : \tilde{V} \to V$ and there is a snc divisor $\Xi$ on $\tilde{V}$ such that $\tilde{f} : \tilde{Z} \to \tilde{V}$ is a morphism which is smooth over $\tilde{V} - \Xi$ and the same holds for any stratum of $g_*^{-1}(E_{\geqslant k+1}|_Z)$ (cf. see for example the proof of [Kaw11, 2] for a similar statement). We may also assume that $g$ is an isomorphism at the generic point of the strata of $(Z, E_{\geqslant k+1}|_Z)$ so that $F := K_{\tilde{Z}} + g_*^{-1}(E_{\geqslant k+1}|_Z) - g^*(K_Z + E_{\geqslant k+1}|_Z)$ is effective and $g$-exceptional. Note also that if $W$ is any stratum of $(Z, E_{\geqslant k+1}|_Z)$, then by the same argument, $F|_{\tilde{W}}$ is $g|_{\tilde{W}}$-exceptional. We may further assume that $g$ and $h$ are given by sequences of blow ups along smooth centers and thus that they are induced by morphisms which (by abuse of notation) we also denote by $h : \tilde{X} \to X$ and $g : \tilde{Y} \to Y$. We write $K_{\tilde{Y}} + \tilde{E} - \tilde{A} = (f \circ g)^*(K_X + B)$. Note that by construction $\tilde{Y} \to \tilde{X}$ is an isomorphism over the complement of a proper closed subset of $V$ and so it is easy to see that we have $\tilde{E}_{\geqslant k+1}|_{\tilde{Z}} = g_*^{-1}(E_{\geqslant k+1}|_Z)$.

Let $M := (A - E_{\leqslant k-1})|_Z$ and $\tilde{M} = g^*M + F \sim_X K_{\tilde{Z}} + \tilde{E}_{\geqslant k+1}|_{\tilde{Z}}$. Since $F$ is $g$-exceptional, $g_*\tilde{M} = M$. By (2.7), $R^\bullet g_* \mathcal{O}_{\tilde{Z}}(\tilde{M}) \cong g_* \mathcal{O}_{\tilde{Z}}(\tilde{M}) = \mathcal{O}_Z(M)$ and so it suffices to show that $H^i(R^j(f \circ g)_* \mathcal{O}_{\tilde{Z}}(\tilde{M} - s(f \circ g)^*L)) = 0$ for $i + j \leqslant d - 1$ and $s > 0$.

By (2.7) $R^i h_* R^j \tilde{f}_* \mathcal{O}_{\tilde{Z}}(\tilde{M}) = 0$ for $i > 0$, and thus it suffices to show that $H^i(R^j \tilde{f}_* \mathcal{O}_{\tilde{Z}}(\tilde{M} - s(f \circ g)^*L)) = 0$ for $i + j \leqslant d - 1$ and $s > 0$.

To this end, let $\tilde{Z} = \tilde{Z}^1 + \cdots + \tilde{Z}^p$ and for any multi-index $I = \{i_1, \dots, i_l\} \subset \{1, \dots, p\}$ let $\tilde{Z}^I = \tilde{Z}^{i_1} \cap \cdots \cap \tilde{Z}^{i_l}$ and $\tilde{Z}[I] = \tilde{Z}^{i_1} + \cdots + \tilde{Z}^{i_l}$. Similarly let $\tilde{E}_{\geqslant k+1} = \tilde{G}_1 + \cdots + \tilde{G}_\rho$ and for any multi-index $I = \{i_1, \dots, i_l\} \subset \{1, \dots, \rho\}$ let $\tilde{G}^I = \tilde{G}_{i_1} \cap \cdots \cap \tilde{G}_{i_l}$ and $\tilde{G}[I] = \tilde{G}_{i_1} + \cdots + \tilde{G}_{i_l}$. We use a similar notation for $Z$ and $E_{\geqslant k+1}$. It suffices to show that:

(1) The sheaves $R^j\tilde{f}_*\mathcal{O}_{\tilde{Z}}(\tilde{M})$ admit filtrations whose quotients are direct sums of summands of sheaves of the form $R^j\tilde{f}_*\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}(\tilde{M}-\tilde{G}[\bar{N}]-\tilde{Z}[\bar{L}])$ where $N\subset\{1,\dots,\rho\}$, $L\subset\{1,\dots,p\}$, $\bar{N}=\{1,\dots,\rho\}\setminus N$ and $\bar{L}=\{1,\dots,p\}\setminus L$.
(2) $H^i(R^j\tilde{f}_*\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}(\tilde{M}-\tilde{G}[\bar{N}]-\tilde{Z}[\bar{L}]-s(f\circ g)^*L))=0$ for $i+j\leqslant d-1$ and $s>0$.

To see the first statement, notice that $(\tilde{M}-\tilde{G}[\bar{N}]-\tilde{Z}[\bar{L}])|_{\tilde{Z}^L\cap\tilde{G}^N}\sim_X K_{\tilde{Z}^L\cap\tilde{G}^N}$ and hence the sheaves $R^j\tilde{f}_*\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}(\tilde{M}-\tilde{G}[\bar{N}]-\tilde{Z}[\bar{L}])$ are obtained as upper canonical extensions of the bottom pieces in the Hodge filtration of a variation of *pure* Hodge structures cf. [Kol87]. Since pure Hodge structures are a semisimple category, these sheaves split as a direct sum of simple summands.

Consider now two disjoint sets $I,J\subset\{1,\dots,\rho\}$, $I\cap J=\emptyset$, and an index $\alpha\in\overline{I\cup J}$ in the complement of $I\sqcup J$. Let $I'=I\cup\alpha$ and $J'=J\cup\alpha$.

We have short exact sequences

$$0\to\mathcal{O}_{\tilde{Z}\cap\tilde{G}^I}\big(\tilde{M}-\tilde{G}\big[J'\big]\big)\to\mathcal{O}_{\tilde{Z}\cap\tilde{G}^I}\big(\tilde{M}-\tilde{G}[J]\big)\to\mathcal{O}_{\tilde{Z}\cap\tilde{G}^{I'}}\big(\tilde{M}-\tilde{G}[J]\big)\to 0. \tag{$\sharp$}$$

Proceeding by ascending induction on $|I|+|J|$ we may assume that the claim (1) holds for $R^j\tilde{f}_*$ of the right and left hand sides of the above short exact sequence. Note that $(\tilde{M}-\tilde{G}[J])|_{\tilde{Z}\cap\tilde{G}^I}\sim_X K_{\tilde{Z}\cap\tilde{G}^I}+\tilde{G}[\overline{J+I}]|_{\tilde{Z}\cap\tilde{G}^I}$, and so by [Kaw09, 5.1] (see also [Kaw11, 15]), we have that the sheaves $R^j\tilde{f}_*\mathcal{O}_{\tilde{Z}\cap\tilde{G}^I}(\tilde{M}-\tilde{G}[J])$ are obtained as upper canonical extensions of the bottom pieces in the Hodge filtration of a variation of *mixed* Hodge structures. Since pure Hodge structures are a semisimple category, a morphism of mixed Hodge structures to a simple Hodge structure is either surjective or trivial. Pushing forward ($\sharp$) and using the filtration on $R^j\tilde{f}_*\mathcal{O}_{\tilde{Z}\cap\tilde{G}^{I'}}(\tilde{M}-\tilde{G}[J])$ first and then the filtration on $R^j\tilde{f}_*\mathcal{O}_{\tilde{Z}\cap\tilde{G}^I}(\tilde{M}-\tilde{G}[J'])$ (both filtrations exist by our inductive hypothesis), we obtain the required filtration on $R^j\tilde{f}_*\mathcal{O}_{\tilde{Z}\cap\tilde{G}^I}(\tilde{M}-\tilde{G}[J])$. The claim now follows (once the base of the induction has been verified).

We must now verify the base of the induction, i.e. that the claim holds for sheaves of the form $R^j\tilde{f}_*\mathcal{O}_{\tilde{Z}\cap\tilde{G}^N}(\tilde{M}-\tilde{G}[\bar{N}])$ where $N\subset\{1,\dots,\rho\}$. Note that $\tilde{Z}=\tilde{Z}[1,2,\dots,p]$. Recall that $(\tilde{M}-\tilde{G}[\bar{N}])|_{\tilde{Z}\cap\tilde{G}^N}\sim_X K_{\tilde{Z}\cap\tilde{G}^N}$ and so $(\tilde{M}-\tilde{Z}[\bar{I}]-\tilde{G}[\bar{N}])|_{\tilde{Z}[I]\cap\tilde{G}^N}\sim_X K_{\tilde{Z}[I]\cap\tilde{G}^N}$. Consider the short exact sequences

$$0\to\mathcal{O}_V(-V\cap W)\to\mathcal{O}_{V+W}\to\mathcal{O}_W\to 0,$$
$$0\to\mathcal{O}_W\to\mathcal{O}_W(V)\to\mathcal{O}_{W\cap V}\to 0,$$

where $V=\tilde{Z}^\alpha$, $W=\tilde{Z}[I]\cap\tilde{Z}^J\cap\tilde{G}^N$, and index sets $I,J,\alpha,I',J'\subset\{1,\dots,p\}$ defined as above.

Tensoring the above sequences by the line bundles $\mathcal{O}_{\tilde{Z}}(\tilde{M}-\tilde{G}[\bar{N}]-\tilde{Z}[\overline{I'+J}])$ and $\mathcal{O}_{\tilde{Z}}(\tilde{M}-\tilde{G}[\bar{N}]-\tilde{Z}[\overline{I+J}])$, we obtain (up to linear equivalence over $X$) the following short exact sequences

$$0\to\omega_{\tilde{Z}[I]\cap\tilde{Z}^{J'}\cap\tilde{G}^N}\to\omega_{\tilde{Z}[I']\cap\tilde{Z}^J\cap\tilde{G}^N}\to\omega_{\tilde{Z}[I]\cap\tilde{Z}^J\cap\tilde{G}^N}\big(\tilde{Z}^\alpha\big)\to 0,$$
$$0\to\omega_{\tilde{Z}[I]\cap\tilde{Z}^J\cap\tilde{G}^N}\to\omega_{\tilde{Z}[I]\cap\tilde{Z}^J\cap\tilde{G}^N}\big(\tilde{Z}^\alpha\big)\to\omega_{\tilde{Z}[I]\cap\tilde{Z}^{J'}\cap\tilde{G}^N}\to 0.$$

By the above arguments, using the higher direct images of these exact sequences, and proceeding by descending induction on $|I|$, we reduce to the case that $|I|=1$ i.e. to the case $K_{\tilde{Z}^J\cap\tilde{G}^N}\sim_X(\tilde{M}-\tilde{Z}[\bar{J}]-\tilde{G}[\bar{N}])|_{\tilde{Z}^J\cap\tilde{G}^N}$ and the first statement follows.

We now prove the second statement. By (4.5) we have $H^i(\mathcal{O}_{Z^L\cap G^N}(M-G[\bar{N}]-Z[\bar{L}]-sf^*L))=0$ for $i\leqslant d-1$ and $s>0$. Notice that $g$ is an isomorphism on an open subset of $\tilde{Z}^L\cap\tilde{G}^N$, thus, by (2.7),

$$R^\bullet(g|_{\tilde{Z}^L\cap\tilde{G}^N})_*\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}\big(\tilde{M}-\tilde{G}[\bar{N}]-\tilde{Z}[\bar{L}]\big)\cong(g|_{\tilde{Z}^L\cap\tilde{G}^N})_*\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}\big(\tilde{M}-\tilde{G}[\bar{N}]-\tilde{Z}[\bar{L}]\big).$$

Moreover, since $g|_{\tilde{Z}^L\cap\tilde{G}^N}$ is birational and $F|_{\tilde{Z}^L\cap\tilde{G}^N}$ is $g|_{\tilde{Z}^L\cap\tilde{G}^N}$-exceptional, we have that

$$(g|_{\tilde{Z}^L\cap\tilde{G}^N})_*\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}\big(\tilde{M}-\tilde{G}[\bar{N}]-\tilde{Z}[\bar{L}]\big)=\mathcal{O}_{Z^L\cap G^N}\big(M-G[\bar{N}]-Z[\bar{L}]\big).$$

In particular $H^i(\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}(\tilde{M}-\tilde{G}[\bar{N}]-\tilde{Z}[\bar{L}]-s(f\circ g)^*L))=0$ for $i\leqslant d-1$ and $s>0$.

By [Kol87], it follows that

$$R^\bullet\tilde{f}_*\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}\big(\tilde{M}-\tilde{G}[\bar{N}]-\tilde{Z}[\bar{L}]\big)=\sum R^i\tilde{f}_*\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}\big(\tilde{M}-\tilde{G}[\bar{N}]-\tilde{Z}[\bar{L}]\big)[-i]$$

and so $H^i(R^j\tilde{f}_*\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}(\tilde{M}-\tilde{G}[\bar{N}]-\tilde{Z}[\bar{L}])-s(f\circ g)^*L)=0$ for $i+j\leqslant d-1$.

We will now prove the required vanishing for $i+j=d$, $k>d$ and $s>0$. Note that in this case we have a short exact sequence

$$\begin{aligned}0&\to\mathcal{O}_{Z^L\cap G^N}\big(M-G[\bar{N}]-Z[\bar{L}]\big)\to\mathcal{O}_{Z^L\cap G^N}\big(M+E_{=d}-G[\bar{N}]-Z[\bar{L}]\big)\\&\to\mathcal{O}_{E_{=d}\cap Z^L\cap G^N}\big(M+E_{=d}-G[\bar{N}]-Z[\bar{L}]\big)\to 0.\end{aligned}$$

By (4.5) we have $H^i(\mathcal{O}_{Z^L\cap G^N}(M+E_{=d}-G[\bar{N}]-Z[\bar{L}]-sf^*L))=0$ for $i\leqslant d$ and $s>0$ and $H^i(\mathcal{O}_{E_{=d}\cap Z^L\cap G^N}(M+E_{=d}-G[\bar{N}]-Z[\bar{L}]-sf^*L))=0$ for $i\leqslant d-1$ and $s>0$. It follows that $H^i(\mathcal{O}_{Z^L\cap G^N}(M-G[\bar{N}]-Z[\bar{L}]-sf^*L))=0$ for $i\leqslant d$ and $s>0$. The required vanishing now follows from the proof of the previous case $i+j\leqslant d-1$.

Finally, we will now prove the required vanishing for $i+j=d$, $k=d$ and $s>0$. It suffices to show that $H^{d-j}(R^jf_*\mathcal{O}_Z(A-sf^*L))=0$ for $j>0$ and $s>0$. Following the above arguments, it suffices to check that if $(\tilde{G}[\bar{N}]+\tilde{Z}[\bar{L}])|_{\tilde{Z}^L\cap\tilde{G}^N}\neq 0$, then

$$H^d\big(\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}\big(\tilde{M}-\tilde{G}[\bar{N}]-\tilde{Z}[\bar{L}]-s(f\circ g)^*L\big)\big)=0\quad\text{for } s>0$$

and if $(\tilde{G}[\bar{N}]+\tilde{Z[\bar{L}]})|_{\tilde{Z}^L\cap\tilde{G}^N}=\emptyset$, then

$$H^d\big(\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}\big(\tilde{M}-s(f\circ g)^*L\big)\big)\cong H^d\big(\tilde{f}_*\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}\big(\tilde{M}-s(f\circ g)^*L\big)\big)\quad\text{for } s>0.$$

Note that as $Z^L\cap G^N\cap(G[\bar{N}]+Z[\bar{L}])$ is a union of non-klt centers, it is seminormal cf. [Fuj11, 9.1]. Similarly to what we have seen above, we have

$$R^\bullet g_*\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N\cap(\tilde{G}[\bar{N}]+\tilde{Z}[\bar{L}])}(\tilde{M})=\mathcal{O}_{Z^L\cap G^N\cap(G[\bar{N}]+Z[\bar{L}])}(M)$$

and so

$$R^\bullet g_*\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}\big(\tilde{M}-\tilde{G}[\bar{N}]-\tilde{Z}[\bar{L}]\big)=\mathcal{O}_{Z^L\cap G^N}\big(M-G[\bar{N}]-Z[\bar{L}]\big).$$

We also have $R^\bullet g_*\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}(-\tilde{G}[\bar{N}]-\tilde{Z}[\bar{L}])=\mathcal{O}_{Z^L\cap G^N}(-G[\bar{N}]-Z[\bar{L}])$. Since $M=A$, by (4.3) we have

$$\begin{aligned}H^d\big(\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}\big(\tilde{M}-\tilde{G}[\bar{N}]-\tilde{Z}[\bar{L}]-s(f\circ g)^*L\big)\big)&=H^d\big(\mathcal{O}_{Z^L\cap G^N}\big(M-G[\bar{N}]-Z[\bar{L}]-sf^*L\big)\big)\\&\overset{(4.3)}{\cong}H^d\big(\mathcal{O}_{Z^L\cap G^N}\big(-G[\bar{N}]-Z[\bar{L}]-sf^*L\big)\big)\\&\cong H^d\big(\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}\big(-\tilde{G}[\bar{N}]-\tilde{Z}[\bar{L}]-s(f\circ g)^*L\big)\big).\end{aligned}$$

Let $f = \dim(\tilde{Z}^L \cap \tilde{G}^N) - \dim \tilde{V}$. We have

$$H^d\big(\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}\big(-\tilde{G}[\bar{N}] - \tilde{Z}[\bar{L}] - s(f\circ g)^*L\big)\big) \overset{\text{(Serre duality)}}{\cong} H^f\big(\omega_{\tilde{Z}^L\cap\tilde{G}^N}\big(\tilde{G}[\bar{N}] + \tilde{Z}[\bar{L}] + s(f\circ g)^*L\big)\big)^\vee$$
$$\overset{(2.7)}{\cong} H^0\big(R^f\tilde{f}_*\omega_{\tilde{Z}^L\cap\tilde{G}^N}\big(\tilde{G}[\bar{N}] + \tilde{Z}[\bar{L}]\big) \otimes \mathcal{O}_{\tilde{V}}\big(sh^*L\big)\big)^\vee.$$

If $(\tilde{G}[\bar{N}] + \tilde{Z}[\bar{L}])|_{\tilde{Z}^L\cap\tilde{G}^N} \neq 0$, then $R^f\tilde{f}_*\omega_{\tilde{Z}^L\cap\tilde{G}^N}(\tilde{G}[\bar{N}] + \tilde{Z}[\bar{L}])$ is torsion free and generically 0 and hence vanishes (cf. (2.7)). If, on the other hand, $(\tilde{G}[\bar{N}] + \tilde{Z}[\bar{L}])|_{\tilde{Z}^L\cap\tilde{G}^N} = \emptyset$, then $R^f\tilde{f}_*\omega_{\tilde{Z}^L\cap\tilde{G}^N} \cong \omega_{\tilde{V}}$ (cf. [Kol86]). By Serre duality and (2.7), we have

$$H^0\big(\omega_{\tilde{V}}\big(sh^*L\big)\big)^\vee \overset{\text{S.D.}}{\cong} H^d\big(\mathcal{O}_{\tilde{V}}\big(-sh^*L\big)\big) \overset{(2.7)}{\cong} H^d\big(\tilde{f}_*\mathcal{O}_{\tilde{Z}^L\cap\tilde{G}^N}\big(-s(f\circ g)^*L\big)\big). \qquad \square$$

*4.3. The structure of the sheaves $R^if_*\mathcal{O}_Y$*

By the Kawamata–Viehweg vanishing theorem, we have that $R^if_*\mathcal{O}_Y(A-E) = 0$ for $i > 0$ since $A - E = K_Y + \Delta - f^*(K_X + B)$. We will now build up the sheaves $R^if_*\mathcal{O}_Y \cong R^if_*\mathcal{O}_Y(A)$, going from $R^if_*\mathcal{O}_Y(A-E)$ to $R^if_*\mathcal{O}_Y(A)$ by adding the parts $E_{=l}$ defined in Section 3.2 one by one.

**Proof of (1.5).** By [HX11], we may assume that $(X, B)$ is projective. Adding a sufficiently ample divisor to $B$ we may assume that $K_X + B$ is ample, and so we may assume that $m(K_X + B) \sim H$ is a general very ample divisor (for some integer $m > 0$). Let $\nu : X' \to X$ be the corresponding normal cyclic cover (cf. [KM98, 5.20]) and $K_{X'} + B' = \nu^*(K_X + B)$. Then $(X', B')$ is log canonical, $K_{X'} + B'$ is Cartier and the non-klt centers are given by the inverse images of the non-klt centers of $(X, B)$ cf. [KM98, 5.20]. Note that if $Y' = Y \times_X X'$, then $f' : Y' \to X'$ is a resolution and $\mu : Y' \to Y$ is a finite map so that $R^i\mu_*\mathcal{O}_{Y'} = 0$ for $i > 0$ and $\mathcal{O}_Y$ is a direct summand of $\mu_*\mathcal{O}_{Y'}$. Thus it is easy to see that if $X'$ is $C_{d+2}$ then so is $X$ (similarly if $R^if'_*\mathcal{O}_{Y'}$ is $C_{d+1-i}$ for $i > 0$, then so is $R^if_*\mathcal{O}_Y$). Thus, replacing $(X, B)$ by $(X', B')$, we may assume that $K_X + B$ is Cartier.

Recall that by (3.6), we have $R^jf_*\mathcal{O}_Y \cong R^jf_*\mathcal{O}_Y(A)$. For $\dim X - 1 \geqslant k \geqslant d$, consider the short exact sequences

$$0 \to R^jf_*\mathcal{O}_Y(A - E + E_{\geqslant k+1}) \to R^jf_*\mathcal{O}_Y(A - E + E_{\geqslant k}) \to R^jf_*\mathcal{O}_{E=k}(A - E + E_{\geqslant k}) \to 0.$$

(Note that these sequences are exact by (2.7); moreover $E_{\geqslant \dim X-1} = 0$.) By (4.6) we have $H^i(R^jf_*\mathcal{O}_{E=k}(A - E + E_{\geqslant k} - sf^*L)) = 0$ for $i + j \leqslant d$, $j > 0$ and $s > 0$. Since $R^jf_*\mathcal{O}_Y(A - E) = 0$ for $j > 0$, it follows that $0 = H^i(R^jf_*\mathcal{O}_Y(A - E - sf^*L)) \to H^i(R^jf_*\mathcal{O}_Y(A - sf^*L))$ is surjective for $i + j \leqslant d$ and $j > 0$ and so $R^jf_*\mathcal{O}_Y \cong R^jf_*\mathcal{O}_Y(A)$ is $C_{d-j+1}$ for $j > 0$. By (4.1), $X$ is $C_{d+2}$. $\square$

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