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Permanence and Positive Periodic Solution for the Single-Species Nonautonomous Delay Diffusive Models

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Abstract—Single-species nonautonomous delay diffusion models with nonlinear growth rates are investigated. Some sufficient conditions are determined that guarantee the permanence of the species and the existence of a positive periodic solution which is global attractively.

Keywords—Delay, Diffusion, Ultimately-bounded domain, Permanence, Periodic solution, Global attractivity.

1. INTRODUCTION

One of the most interesting questions in mathematical biology to discuss is permanence. Takeuchi [1] showed global stability of diffusive cooperative systems under some conditions. Takeuchi [2] discussed the persistence of two species models. But for some systems, we find that they have nonlinear growth rates [3]. On the other hand, it is recognized that time delays are natural components of the dynamic processes of biology, economics, and physiology, etc. Beretta and Takeuchi [4] discussed the globally asymptotic stability of the Lotka-Voterra autonomous model with diffusion and time delay. These motivate us to consider the possible effect of both diffusion and time delay on the stability of nonautonomous systems with nonlinear growth.

The organization of this paper is as follows. In Section 2, two models, some notations and lemmas are given. In Section 2 and 3, we employ differential inequations and Lyapunov-Rzumikhin type theorems to obtain an ultimate bounded domain and establish sufficient conditions that ensure that there exists a positive periodic solution which is global attractively in each system.

2. MODELS AND LEMMAS

Let us consider the following systems:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left[\frac{k_1(t)-a_1(t)x_1(t)+b_1(t)x_1(t-\tau)}{k_1(t)+r_1(t)x_1(t)}\right] + d_1(t)\left[x_2(t)-x_1(t)\right], \\ \dot{x}_2(t) &= x_2(t)\left[\frac{k_2(t)-a_2(t)x_2(t)+b_2(t)x_2(t-\tau)}{k_2(t)+r_2(t)x_2(t)}\right] + d_2(t)\left[x_1(t)-x_2(t)\right], \end{aligned} \tag{2.1}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX and

$$\dot{x}_1(t) = x_1(t)\left[\frac{k_1(t) - a_1(t)x_1(t) - b_1(t)x_1(t-\tau)}{k_1(t) + r_1(t)x_1(t)}\right] + d_1(t)\left[x_2(t) - x_1(t)\right],$$
$$\dot{x}_2(t) = x_2(t)\frac{k_2(t) - a_2(t)x_2(t) - b_2(t)x_2(t-\tau)}{k_2(t) + r_2(t)x_2(t)} + d_2(t)\left[x_1(t) - x_2(t)\right], \tag{2.2}$$

where $x_i(t)$ is the density of species x in patch $i(i = 1,2)$, $x_i(t) = \phi_i(t) \geq 0$, $\phi_i(0) > 0$ $(i = 1,2)$, $t \in [-\tau, 0]\,\tau \geq 0$, $\phi_i \in C([-\tau,0], R)$. $k_i(t)$ and $a_i(t)$ are continuous functions which have positive upper bound and positive lower bound. $r_i(t)$, $b_i(t)$, $d_i(t)$ are nonnegative bounded continuous functions $(i = 1,2)$. We define constants a_{iM}, $a_{iL}, (i = 1,2)$, h_1, h_2, h_3, and h_4 by $a_{iM} = \max\{a_i(t), t \geq 0\}$, $a_{iL} = \min\{a_i(t), t \geq 0\} (i = 1,2)$. k_{iM}, K_{iL}, b_{iM}, b_{iL}, r_{iM}, r_{iL}, d_{iM}, $d_{iL} (i = 1,2)$ have the same definition as a_{iM}, a_{iL}.

$$h_1 = \max\left\{\frac{k_{1M}}{a_{1L} - b_{1M}}, \frac{k_{2M}}{a_{2L} - b_{2M}}\right\}, \qquad h_2 = \min\left\{\frac{k_{1L}}{a_{1M}}, \frac{k_{2L}}{a_{2M}}\right\},$$
$$h_3 = \max\left\{\frac{k_{1M}}{a_{1L}}, \frac{k_{2M}}{a_{2L}}\right\}, \qquad h_4 = \min\left\{\frac{k_{1L} - b_{1M}h_3}{a_{1M} + b_{1M}}, \frac{k_{2L} - b_{2M}h_3}{a_{2M} + b_{2M}}\right\}.$$

Denote

$$C^n = \{\phi = (\phi_1, \ldots, \phi_n) : \phi \in C([-\tau, 0], R^n)\},$$
$$C^n_{+0} = \{\phi = (\phi_1, \ldots, \phi_n) : \phi \in C([-\tau, 0], R^n_{+0})\},$$
$$C^n_{-0} = \{\phi = (\phi_1, \ldots, \phi_n) : \phi \in C([-\tau, o], R^n_{-0})\},$$
$$R^n_{+0} = \{(x_1, \ldots, x_n) : x_i \geq 0, i = 1, \ldots, n\},$$
$$R^n_{-0} = \{(x_1, \ldots, x_n) : x_i \leq 0, i = 1, \ldots, n\}.$$

Before stating our main theorem, we need the following lemmas.

LEMMA 1. *Suppose $\Omega \subset R \times C^n$ is open, $(\sigma, \phi) \in \omega$, $f \in C(\Omega, R^N)$, and x is a solution of $\dot{x}(t) = f(t, x_t)$ through (σ, ϕ) which exists and is unique on $[\sigma - r, b], b > \sigma - r$. Then, there exists positive integer M large enough, such that for $m > M$, each solution x^m through (σ, ϕ) of*

$$\dot{x}_i(t) = f_i(t, x_t) + \frac{1}{m}, \qquad i = 1, \ldots, n$$

exists on $[\sigma - r, b]$ and $x^m \to x$ uniformly on $[\sigma - r, b]$, as $m \to \infty$.

PROOF. Let $\sigma^m = \sigma$, $\phi^m = \phi$, $f^m = f + (1/m, \ldots, 1/m)^{\mathsf{T}}$. Then, applying Theorem 2.3 in [5, pp. 19–20], we can see that the conclusion of Lemma 1 is true.

LEMMA 2. *Suppose $\Omega \subset R \times C^n$ is open, $f_i \in C(\Omega, R)$, $i = 1, \ldots, n$. If*

$$f_i|_{x_i(t)=0, X_t \in C^n_{+0}} \geq 0, \qquad X_t = (x_{1t}, \ldots, x_{nt}), \quad i = 1, \ldots, n,$$

then C^n_{+0} is the invarious domain of the following equations:

$$\dot{x}_i(t) = f_i(t, X_t), \qquad t \geq \sigma, \quad i = 1, \ldots, n; \tag{2.3}$$

if $f_i|_{x_i(t)=0, X_t \in C^n_{-0}} \leq 0$, $X_t = (x_{1t}, \ldots, x_{nt})$, $i = 1, \ldots, n$, then C^n_{-0} is the invarious domain of the above equation.

PROOF. We consider the equation

$$\dot{x}_i(t) = f_i(t, X_t) + \frac{1}{m}, \qquad i = 1, \ldots, n, \tag{$*$}$$

m is any positive integer. Let $x_i(t)$ be the solution of $(*)$ and $x_i(t) \geq 0$, $t \in [\sigma - r, \sigma]$, $x_i(\sigma) > 0$, $i = 1, \dots, n$. If there is a T, $T > \sigma$, X_T not in C^n_{+0}, then there must be i and $t_0 > \sigma$ such that $x_i(t_0) = 0$, $X_{it} \geq 0$ for $t \in [\sigma, t_0]$. This implies $\dot{x}_i(t_0) \leq 0$. It contradicts $\dot{x}_i(t_0) = f_i(t_0, X_{t_0}) + 1/m > 0$. So we can say that C^n_{+0} is the invarious domain of $(*)$.

Letting $m \to \infty$, from Lemma 1, we get that C^n_{+0} is the invarious domain of (2.3).

The other conclusion can be deduced similarly.

LEMMA 3. *Domain C^2_{+0} is the invarious domain of systems (2.1) and (2.2).*

PROOF. It can be deduced from Lemma 2.

LEMMA 4. *Any of the solutions of systems (2.1) and (2.2) are positive for $t \geq 0$.*

PROOF. From Lemma 3 and (2.1), we know that for $t \geq 0$,

$$\dot{x}_i(t) \geq x_i(t) \left[\frac{k_i(t) - a_i(t)x_i(t) + b_i(t)x_i(t-\tau)}{k_i(t) + r_i(t)x_i(t)} - d_i(t) \right], \qquad i = 1, 2.$$

This implies that for $i = 1, 2$,

$$x_i(t) \geq x_i(0) \exp \left\{ \int_0^t \left(\frac{k_i(s) - a_i(s)x_i(s) + b_i(s)x_i(s-\tau)}{k_i(s) + r_i(s)x_i(s)} - d_i(s) \right) ds \right\} > 0.$$

Therefore, the conclusion of Lemma 4 is true.

The proof for system (2.2) is similar to the above.

3. PERMANENCE

DEFINITION 1. *System $\dot{X}(t) = f(t, X_t)$, $X \in R^n$ is said to be permanence, if for any solution $X = X(t, \phi)$, there exists a constant $m > 0$ and $T = T(\phi)$, such that $X(t) > m$ for all $t > T$.*

Domain $D \subset C^n$ is said to be an ultimately-bounded domain, if D is a closed, bounded subset of C^n, and there exists constant $T = T(\phi)$ such that, for $t > T$, $X_t \in D$.

THEOREM 1. *If $0 < h_1 < \infty$, then, for any $\eta_1 \geq 0$, $\eta_2 > 0$,*

$$D_{\eta_1\eta_2} = \{(x_{1t}, x_{2t}) : h_2 - \eta_2 \leq x_{it} \leq h_1 + \eta_1,\ t \geq -\tau,\ i = 1, 2\}$$

is the invarious domain of (2.1). *We can let η_2 small enough such that $h_2 - \eta_2 > 0$.*

PROOF. Let $u_i(t) = x_i(t) - h_1 - \eta_1$, $i = 1, 2$, then

$$\dot{u}_i(t) = \frac{u_i + h_1 + \eta_1}{k_i + r_i(u_i + h_1 + \eta_1)} \left[k_i - a_i u_i + b_i u_i(t-\tau) - (a_i - b_i)(h_1 + \eta_1) \right] + d_i(u_j - u_i), \qquad i, j = 1, 2, \quad i \neq j. \tag{3.1}$$

We find $\dot{u}_i(t)|_{u_i(t)=0, u_{1t} \leq 0, u_{2t} \leq 0} \leq 0$, $i = 1, 2$. According to Lemma 2, c^2_{-0} is the invarious domain of (3.1). That is to say, if $x_1(s), x_2(s) \leq h_1 + \eta_1$, $s \in [-\tau, 0]$, then $x_1(t), x_2(t) \leq h_1 + \eta_1$ for all $t \geq 0$.

Let $v_i(t) = x_i(t) - h_2 + \eta_2$, $i = 1, 2$. Then, by a similar argument as above, we can deduce that if $x_1(t), x_2(t) \leq h_1 + \eta_1$, $t \geq -\tau$ and $x_1(s), x_s \geq h_2 - \eta_2$, $s \in [-\tau, 0]$, then $x_1(t), x_2(t) \geq h_2 - \eta_2$ for all $t \geq 0$.

Taking all into account, we can say $D_{\eta_1\eta_2}$ is the invarious domain of (2.1).

THEOREM 2. *If $h_4 > 0$, then for any $\xi_1 \geq 0$, $\xi_2 > \max\{(b_{1M}\xi_1/a_{1L}), (a_{2M}\xi_1/a_{2L})\}$,*

$$D_{\xi_1\xi_2} = \{(x_{1t}, x_{2t}) : h_4 - \xi_2 \leq x_{it} \leq h_3 + \xi_1,\ i = 1, 2\}$$

is the invarious domain of (2.2).

PROOF. By a similar argument as in Theorem 1, we can prove our result.

Denote

$$\overline{\eta}_1 = \max\left\{\frac{k_{iM}+\epsilon_0}{a_{1L}-rb_{1M}}, \frac{k_{2M}+\epsilon_0}{a_{2L}-rb_{2M}}\right\} - h_1,$$

where r is a constant more than 1, $\epsilon_0 > 0$ is any constant.

THEOREM 3. *Suppose* $0 < \overline{\eta}_1$, $h_1 < \infty$; *then domain* $D_{\overline{\eta}_1\eta_2}$ *(it has a similar definition as in Theorem 1) is the ultimately-bounded domain of (2.1).*

PROOF. Suppose that $x(t) = (x_1(t), x_2(t))^\top$ is the solution of system (2.1). Then, from Lemma 4, $x_1(t), x_2(t) > 0$. Define the norm of $x(t)$ by

$$|x(t)| = \max\{|x_1(t)|, |x_2(t)|\}.$$

We will finish our proof in two aspects.

CLAIM 1. There exists $T_1 \geq 0$ such that for all $t \geq T_1$, $x_i(t) \leq h_1 + \overline{\eta}_1$, $i = 1, 2$.

Define function $V(t) = \max\{x_1(t), x_2(t)\}$. Assume first $V(T) = x_1(t)$. Then we have

$$\begin{aligned}\dot{V}(t) &= \dot{x}_1(t)\\ &\leq w(t)(k_1(t) - a_1(t)x_1(t) + b_1(t)x_1(t-\tau))\\ &\leq w(t)(k_{1M} - (a_{1L} - rb_{1M})x_1(t))\\ &\leq -w(t)\epsilon_0 \leq -\alpha,\end{aligned}$$

if $|x(t)| \geq h_1 + \overline{\eta}_1$, $V(t+s) \leq V(t)$, $t \in [-\tau, 0]$, where $w(t) = \frac{x_1(t)}{k_1(t)+r_1(t)x_1(t)}$, $\alpha = \frac{\epsilon_0(h_1+\overline{\eta}_1)}{k_{1M}+r_{1M}(h_1+\overline{\eta}_1)}$.

For $V(t) = x_2(t)$, by similar argument, we have $\dot{V}(t) \leq -\beta$, if $|x(t)| \geq h_1 + \overline{\eta}_1$, and $V(t+s) \leq V(t)$, $s \in [-\tau, 0]$, where β is positive constant.

Then, from the Razumilihin-type Theorem 6.4 in [5, pp. 19–20], we can see that Claim 1 is true.

Define function $g_1(t) = \min\{x_{1t}, x_{2t}, t \geq 0\}$.

CLAIM 2. If $g_1(0) \leq h_2 - \eta_2$, then there exists $T_2 > 0$ such that $g_1(t) \geq h_2 - \eta_2$ for $t \geq T_2$.

Otherwise, $g_1(t) < h_2 - \eta_2$, $t \geq 0$, then for $t \geq \tau$

$$\begin{aligned}\dot{g}_1(t) &= \dot{x}_{it}\\ &\geq \frac{x_{it}}{k_i(t) + r_i(t)x_{it}}(k_i(t) - a_i(t)x_{it})\\ &\geq \frac{x_{it}}{k_i(t) + r_i(t)x_{it}}(k_{iL} - a_{iM}h_2 + a_{iL}\eta_2)\\ &\geq \frac{a_{iL}\eta_2 x_{it}}{k_i(t) + r_i(t)x_i(t)} \qquad i = 1, 2.\end{aligned} \tag{3.2}$$

Denote $L_1 = \min\{\frac{a_{iL}\eta_2}{k_{iM}+r_{iM}(h_1+\overline{\eta}_1)}, i = 1, 2\}$. Then, for $t \geq T' = T_1 + \tau$, (3.2) implies $\dot{g}_1(t) \geq g_1(t)L_1$. So from Lemma 4, we have

$$g(t) \geq g(T')\, e^{L_1(t-T')} \to \infty, \qquad \text{as } t \to \infty.$$

This contradicts $g(t) < h_2 - \eta_2$, $(t \geq 0)$. So there exists $T_2 > 0$ such that $g(T_2) \geq h_2 - \eta_2$. On the other hand, for any constants $0 < x_L \leq h_2 - \eta_2$, we have

$$\begin{aligned}\dot{x}_i(t)|_{x_i(t)=x_L, x_j(t)\geq x_L} &\geq \frac{x_i(t)}{k_i(t) + r_i(t)x_i(t)}[k_i(t) - a_i(t)x_i(t)]\\ &\geq \frac{a_{iM}\eta_2 x_i(t)}{k_i(t) + r_i(t)x_i(t)}\\ &> 0, \qquad i, j = 1, 2, \quad i \neq j.\end{aligned}$$

It is easy to know that the above inequality implies that the conclusion of Claim 2 is true. Choose $T = \max\{T_1 + \tau, T_2\}$; then for $t \geq T$, $(x_{1t}, x_{2t}) \in D_{\overline{\eta}_1,\eta_2}$, we have completed the proof.

THEOREM 4. *Assume that $h_4 > 0$. Then $D_{\xi_1\xi_2}$ (its definition is in Theorem 2) is an ultimately-bounded domain of (2.2).*

PROOF. Suppose $(x_1(t), x_2(t))$ is the only solution of (2.2). Then by a similar argument in Theorem 3, we get that there exists $T_3 > 0$ such that

$$x_i(t) \leq h_3 + \xi_1, \qquad t \geq T_3, \qquad i = 1, 2.$$

Define function $g_2(t) = \min\{x_{1t}, x_{2t}, t \geq 0\}$, if $g_2(0) < h_4 - \xi_2$. We claim there exists $T_4 > 0$ such that $g_2(t) \geq h_4 - \xi_2$, for all $t \geq T_4$. Otherwise, $g_2(t) < h_4 - \xi_2 < h_4$, $t \geq 0$. Then for $t \geq \tau$

$$\begin{aligned} \dot{g}_2(t) &= \dot{x}_{it} \\ &\geq \frac{x_{it}}{k_i(t) + r_i(t)x_{it}} \left[k_{iL} - a_{iM}h_4 - b_{iM}h_4 + b_{iL}\xi_2\right] \\ &\geq \frac{x_{it}b_{iL}\xi_2}{k_i(t) + r_i(t)x_{it}}. \end{aligned} \tag{3.3}$$

Denote $L_2 = \min\{\frac{b_{iL}\xi_2}{k_{iM}+r_{iM}(h_3+\xi_1)}, i = 1, 2\}$, Then, for $t > T'' = T_3 + \tau$, (3.3) implies $\dot{g}_2(t) \geq L_2 g_2(t)$. Therefore, from Lemma 4, we have

$$g_2(t) \geq g_2(T'')\, e^{L_2(t-T'')} \to \infty, \qquad \text{as } t \to \infty.$$

This contradicts $g_2(t) < h_4 - \xi_2$, $t \geq 0$. So there exists $T_4 > 0$ such that $g(T_4) \geq h_4 - \xi_2$. On the other hand, for $t \geq T_3 + \tau$ and any constants $0 < x_L \leq h_4 - \xi_2$, we have

$$\begin{aligned} \dot{x}_i(t)|_{x_i(t)=x_L, x_j(t)\geq x_L} &\geq \frac{x_i(t)}{k_i(t) + r_i(t)x_i(t)}[k_i(t) - a_i(t)x_i(t) - b_i(t)(h_3 + \xi_1)] \\ &\geq \frac{(a_{iL}\xi_2 - b_{iM}\xi_1)x_i(t)}{k_i(t) + r_i(t)x_i(t)} \\ &> 0, \qquad i, j = 1, 2, \quad i \neq j. \end{aligned}$$

It is easy to know that the above inequality implies that the claim is true.

Choosing $T = \max\{T_3 + \tau, T_4\}$, then for $t \geq T$, $(x_{1t}, x_{2t}) \in D_{\xi_1,\xi_2}$, we have completed the proof.

From Theorems 3 and 4, we immediately get the following theorem.

THEOREM 5. *If $0 < \overline{\eta_1}$, $h_1 < \infty$, then system (2.1) is permanent.*

THEOREM 6. *If $h_4 > 0$, then system (2.2) is permanent.*

4. EXISTENCE AND GLOBAL ATTRACTIVITY OF POSITIVE PERIODIC SOLUTIONS

Throughout this section, we assume that for $i = 1, 2$, $a_i(t)$, $b_i(t)$, $d_i(t)$, $k_i(t)$, and $r_i(t)$ are ω-period functions, i.e., systems (2.1) and (2.2) are ω-period systems. We define the norm of C^n by $\|\phi\| = \max\{|\phi_i|, i = 1, \dots n\}$, where $\phi \in C^n$. Then $D_{\overline{\eta}_1\eta_2}, D_{\xi_1\xi_2}$ (their definition is in Theorems 3 and 4) are closed, convex and bounded subset of C^2.

THEOREM 7. *If $0 < \overline{\eta_1}$, $h_1 < \infty$. Then system (2.1) has a positive ω-period solution.*

PROOF. We define a mapping P by $P : C^2 \to C^2$, $P(\phi) = X_\omega(\phi)$, $\phi \in C^2$ where $X_\omega(\phi)$ represents the solution of system (2.1) with initial function ϕ. P is continuous and P maps domain $D_{\overline{\eta}_1\eta_2}$ into itself. Then by the Brouwer fixed point theorem, P has a fixed point ϕ_0 in $D_{\overline{\eta}_1\eta_2}$. Since system (2.1) is a ω-period system, we can say that $X_t(\phi_0)$ is a positive ω-period solution of system (2.1). We have completed the proof of Theorem 7.

By a similar argument as in Theorem 7, we can prove the following theorem.

THEOREM 8. *If $h_4 > 0$, then system (2.2) has a positive ω-period solution.*

Denote $N = \max\{(b_{1M}n/k_{1L}), (b_{2M}n/k_{2L})\}$; n is a constant more than 1. Suppose $X(t) = (x_1(t), x_2(t))$ is any nonnegative solution of system (2.1). $U(t) = (u_1(t), u_2(t))$ is a positive ω-period solution of (2.1) in $D_{\overline{\eta}_1\eta_2}$. We introduce transform

$$\begin{aligned} x_i(t) &= \exp\{\overline{x}_i(t)\}, \qquad i = 1, 2, \\ u_i(t) &= \exp\{\overline{u}_i(t)\}, \qquad i = 1, 2. \end{aligned}$$

Then, we have

$$\begin{aligned} \dot{\overline{x}}_i(t) - \dot{\overline{u}}_i(t) = d_i(t)\left(\frac{x_j(t)}{x_i(t)} - \frac{u_j(t)}{u_i(t)}\right) &+ \frac{k_i}{k_i + r_i x_i(t)} - \frac{k_i}{k_i + r_i u_i(t)} - \frac{a_i \exp\{\overline{x}_i(t)\}}{k_i + r_i x_i(t)} \\ + \frac{a_i \exp\{\overline{u}_i(t)\}}{k_i + r_i x_i(t)} &+ \frac{b_i \exp\{\overline{x}_i(t-\tau)\}}{k_i + r_i x_i(t)} - \frac{b_i \exp\{\overline{u}_i(t-\tau)\}}{k_i + r_i x_i(t)}, \\ i, j = 1, 2, \quad i \neq j. \end{aligned} \tag{4.1}$$

Denote $h_5 = \max\{\frac{k_{1M}}{a_{1L} - rb_{1M}}, \frac{k_{2M}}{a_{2L} - rb_{2M}}\}$, where r is a constant more than 1.

THEOREM 9. *Assume $0 < h_1$, $h_5 < \infty$. If, in addition,*

$$\frac{a_{iL}k_{iL}}{k_{iM} + r_{iM}(h_1 + h_5)} - N - \frac{r_{iM}}{k_{iL}} - \frac{d_{jM}}{h_2} > \alpha, \qquad i, j = 1, 2, \quad i \neq j,$$

then system (2.1) has a unique positive ω-period solution which has global attractivity, where α is a positive constant.

PROOF. From the assumption of the theorem, it is easy to know that there exist positive constants ϵ_0 and η_2 which are small enough such that $h_2 - \eta_2 > 0$, and

$$\frac{a_{iL}k_{iL}}{k_{iM} + r_{iM}(h_1 + \overline{\eta}_1)} - N - \frac{r_{iM}}{k_{iL}} - \frac{d_{jM}}{h_2 - \eta_2} > \frac{\alpha}{2} \qquad i, j = 1, 2, \quad i \neq j,$$

where $\overline{\eta}_1 = \max\{\frac{k_{1M} + \epsilon_0}{a_{1L} - rb_{iM}}, i = 1, 2\}$; then $D_{\overline{\eta}_1\eta_2}$ (its definition is in Theorem 3) is the ultimately-bounded domain of system (2.1).

From Theorem 7, system (2.1) has a positive periodic solution $U(t)$ in $D_{\overline{\eta}_1\eta_2}$. Define function $W(t)$ by

$$W(t) = \sum_{i=1}^{2} |\overline{x}_i(t) - \overline{u}_i(t)|.$$

From Theorem 3, there exists T such that for all $t > T$, $h_2 - \eta_2 \leq x_i(t) \leq h_1 + \overline{\eta}_1$, $i = 1, 2$. Then the upper derivative of $W(t)$ along system (2.1) satisfies, for $t > T$,

$$D^+W(t) \leq -\sum_{i=1}^{2} \frac{k_{iL}a_{iL}|x_i(t) - u_i(t)|}{k_{iM} + r_{iM}(h_1 + \overline{\eta}_1)} + \sum_{i=1}^{2} \frac{r_{iM}}{k_{iL}}|x_i(t) - u_i(t)| + \sum_{i=1}^{2} A_i + \sum_{i=1}^{2} B_i,$$

where

$$\begin{aligned} A_i &= \operatorname{sgn}(x_i(t) - u_i(t)) \left[\frac{b_i x_i(t-\tau)}{k_i + r_i x_i(t)} - \frac{b_i u_i(t-\tau)}{k_i + r_i u_i(t)}\right], \\ B_i &= \operatorname{sgn}(x_i(t) - u_i(t)) d_i \left(\frac{x_j(t)}{x_i(t)} - \frac{u_j(t)}{u_i(t)}\right), \\ i, j &= 1, 2, \qquad i \neq j. \end{aligned}$$

If $x_i(t) \geq u_i(t)$, then

$$A_i \leq \frac{b_i(x_i(t-\tau)-u_i(t-\tau))}{k_i+r_iu_i(t)} \leq \frac{b_{iM}}{k_{iL}}|x_i(t-\tau)-u_i(t-\tau)|,$$
$$B_i \leq \frac{d_i(x_i(t)-u_i(t))}{u_i(t)} \leq \frac{d_{iM}}{h_2-\eta_2}|x_j(t)-u_j(t)|,$$
$$i=1,2, \qquad i \neq j.$$

If $x_i(t) < u_i(t)$, then

$$A_i \leq \frac{b_i(u_i(t-\tau)-x_i(t-\tau))}{k_i+r_iu_i(t)} \leq \frac{b_{iM}}{k_{iL}}|x_i(t-\tau)-u_i(t-\tau)|,$$
$$B_i \leq \frac{d_i(u_j(t)-x_j(t))}{x_i(t)} \leq \frac{d_{iM}}{h_2-\eta_2}|x_j(t)-u_j(t)|,$$
$$i,j=1,2, \qquad i \neq j.$$

Then we have

$$\begin{aligned} D^+W(t) &\leq -\sum_{i=1}^{2}\sum_{j=1,j\neq i}^{2}\left[\frac{a_{iL}k_{iL}}{k_{iM}+r_{iM}(h_1+\overline{\eta}_1)}+\left(N+\frac{r_{iM}}{k_{iL}}+\frac{d_{jM}}{h_2-\eta_2}\right)\right]|x_i(t)-u_i(t)| \\ &\leq -\frac{\alpha}{2}\sum_{i=1}^{2}|x_i(t)-u_i(t)|, \end{aligned}$$

if $t > T$, and $W(t+s) \leq nW(t)$, $s \in [-\tau, 0]$.

By the value theorem, we have

$$m|\overline{x}_i(t)-\overline{u}_i(t)| \leq |x_i(t)-u_i(t)| \leq M|\overline{x}_i(t)-\overline{u}_i(t)|,$$

where

$$t > T, \qquad m = h_2-\eta_2, \qquad M = h_1+\overline{\eta}_1.$$

Hence,

$$D^+W(t) \leq -mW(t)\frac{\alpha}{2},$$

for

$$t > T, \quad W(t+s) \leq nW(t), \qquad s \in [-\tau, 0].$$

Then, by the Razumikhin-type Theorem 6.1 [5, pp. 38–46], we know

$$\lim_{t\to\infty} |\overline{x}_i(t)-\overline{u}_i(t)| = 0, \qquad i=1,2.$$

So we can get

$$0 \leq \lim_{t\to\infty} |x_i(t)-u_i(t)| \leq \lim_{t\to\infty} M|\overline{x}_i(t)-\overline{u}_i(t)| = 0, \qquad i=1,2,$$

which implies that the conclusion of Theorem 9 holds. We have completed the proof.

THEOREM 10. *Suppose $h_4 > 0$. For a given positive constant ξ, assume*

$$\frac{a_{iL}k_{iL}}{k_{iM}+r_{iM}(h_3+\xi)} - N - \frac{r_{iM}}{k_{iL}} - \frac{d_{jM}}{h_4} > \alpha,$$
$$i,j=1,2, \qquad i \neq j,$$

where α is a positive constant. Then system (2.2) has a unique positive ω-period solution which is global attractivity.

PROOF. By the assumption of the theorem, we can choose ξ_1, ξ_2 small enough such that $\xi_1 > 0$, $h_4 - \xi_2 > 0$ and

$$\frac{a_{iL}k_{iL}}{k_{iM} + r_{iM}(h_3 + \xi_1)} - N - \frac{r_{iM}}{k_{iL}} - \frac{d_{jM}}{h_4 - \xi_2} \geq \frac{\alpha}{2}, \quad i, j = 1, 2, \quad i \neq j;$$

ξ_2 is satisfied by

$$\xi_2 > \max\left\{\frac{b_{iM}\xi_1}{a_{iL}}, i = 1, 2\right\}.$$

Then $D_{\xi_1\xi_2}$ (the definition is in Theorem 4) is the ultimately-bounded domain of system (2.2). In the following example, we can finish our proof by a similar argument as that in Theorem 9. Therefore, we will omit it.

EXAMPLE. Consider the following systems:

$$\dot{x}_1(t) = \frac{x_1(t)}{10 + (2 + \sin t)x_1(t)}(10 - 20x_1(t) + 5x_1(t - \tau)) + \frac{x_2(t) - x_1(t)}{20},$$
$$\dot{x}_2(t) = \frac{x_2(t)}{10 + 2x_2(t)}(10 + \sin t - 35x_2(t) + 2x_2(t - \tau)) + \frac{x_1(t) - x_2(t)}{40},$$
$$\dot{x}_1(t) = \frac{x_1(t)}{10 + (2 + \sin t)x_1(t)}(10 - 20x_1(t) - 5x_1(t - \tau)) + \frac{x_2(t) - x_1(t)}{20},$$
$$\dot{x}_2(t) = \frac{x_2(t)}{10 + 2x_2(t)}(10 + \sin t - 35x_2(t) - 2x_2(t - \tau)) + \frac{x_1(t) - x_2(t)}{40}.$$

We compute the value $h_1 = 2/3$, $h_2 = 9/35$, $h_3 = 1/2$, $h_4 = 8/37$, $h_5 = 1/3$. We choose $r = 2$, $n = 3/2$, $\xi = 1/3$. Then, we easily find those conditions of Theorems 9 and 10 hold. So we can conclude that each system has a unique positive and global attractivity 2π-period solution.

5. CONCLUSION

From this paper, we can find that diffusion rate has no effect on permanence and existence of a positive periodic solution.

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