Note

Engel's Inequality for Bell Numbers*

E. RODNEY CANFIELD

Department of Computer Science, University of Georgia, Athens, Georgia 30602

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K. Engel has conjectured that the average number of blocks in a partition of an $n$-set is a concave function of $n$. The average in question is a quotient of two Bell numbers less 1, and we prove Engel's conjecture for all $n$ sufficiently large by an extension of the Moser-Wyman asymptotic formula for the Bell numbers. We also give a general theorem which specializes to an inequality about Bell numbers less complex than Engel's, in that fewer terms of the asymptotic expansion are needed to verify it for all sufficiently large $n$. © 1995 Academic Press, Inc.

In this note $\tau_n$ denotes the average number of blocks in a partition of the generic $n$-set $[n] = \{1, 2, \ldots, n\}$. A partition of a set $S$ is a collection of non-empty subsets, called blocks, which are pairwise disjoint and whose union equals $S$. The number of partitions of $[n]$ having exactly $k$ blocks is the Stirling number of the second kind $S(n, k)$, and the total number of partitions of $[n]$ is the $n$-th Bell number $B_n$:

$$B_n = \sum_{k=1}^{n} S(n, k), \quad n \geq 1.$$ 

Since $\tau_n$ equals $\sum_k kS(n, k)/B_n$, it follows from the recursion

$$S(n + 1, k) = kS(n, k) + S(n, k - 1)$$

(1)

that

$$\tau_n = \frac{B_{n+1}}{B_n} - 1.$$ 

(2)

For an introduction to Bell and Stirling numbers, see [3, Chap. V].

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In studying the average rank within a filter of the partition lattice Engel [4] was led to conjecture that the sequence \( \tau_n \) is concave, that is
\[
\tau_n \geq \frac{1}{2} (\tau_{n-1} + \tau_{n+1}).
\]  
(3)
The latter inequality has been verified computationally for \( 2 \leq n \leq 1200 \) [5], but no general proof has been found. The first purpose of this note is to prove the following theorem.

**THEOREM 1.** There is an integer \( n_0 \) such that inequality (3) holds for \( n \geq n_0 \).

**Proof.** Moser and Wyman [7] give an asymptotic expansion for \( B_n \) which involves the quantity \( r = r(n) \) defined by
\[
re^r = n.
\]
If we use the Moser-Wyman formula directly in (2) it is necessary to relate the quantities \( r(n+h) \) for \( h \in \{-1, 0, 1, 2\} \). As pointed out in [2] one may follow the Moser-Wyman proof with minor modification to develop an estimate for \( B_{n+h} \) which holds uniformly for \( h = O(\log n) \) and which involves the single parameter \( r \). To settle Engel's conjecture it is necessary to carry out the expansion to a third term as follows: (uniformly for \( h = O(\log n) \) as \( n \to \infty \))
\[
B_{n+h} = \frac{(n+h)! \exp(e^r - 1)}{r^{n+h} (2\pi B)^{1/2}} \times \left( 1 + \frac{P_0 + hP_1 + h^2P_2}{e^r} + \frac{Q_0 + hQ_1 + h^2Q_2 + h^3Q_3 + h^4Q_4}{e^{2r}} \right) + O(e^{-3r}).
\]  
(4)
In the latter, \( B = (r^2 + r) e^r \), and \( P_i, Q_i \) are explicitly known rational functions of \( r \). Using (2) and (4) we compute
\[
\tau_n - (\tau_{n-1} + \tau_{n+1})/2 = \frac{-3Q_3 - 6Q_4 + P_2^2 + 3P_1P_2 - 2P_2/r}{e^r} + O(e^{-2r})
\]
\[
= \frac{1/2 + O(r^{-1})}{r^3 e^r},
\]
thus proving the theorem. The last equality requires the equations
\( P_1 = -1/2r + O(r^{-2}) \), \( P_2 = -1/2r^2 + O(r^{-3}) \), \( Q_3 = 5/12r^3 + O(r^{-4}) \), and \( Q_4 = O(r^{-4}) \). We do not bother to list exact values for \( P_i \) and \( Q_i \).
The possibility of completing the proof of Engel's conjecture by finding an explicit value for \( n_0 \) and performing computer verification for \( n < n_0 \) is open. Initial consideration suggests that both parts of such an attack would be challenging.

When expressed in terms of Bell numbers, and cleared of denominators, the desired inequality (3) involves products of size three. Being unable to prove this inequality, and thinking that the complexity may be due to the triple products and be reflected by the necessity of using three terms in the Moser-Wyman expansion, it was decided to search for a new inequality involving double products and requiring only two terms of the expansion to verify asymptotically.

It is known [3, exercise 1, p. 291] that the Bell numbers are convex:

\[
B_n \leq \frac{1}{2} (B_{n-1} + B_{n+1}).
\]

The convexity of a positive sequence is implied by log-convexity:

\[
B_n^2 \leq B_{n-1} B_{n+1},
\]

and Engel [4] proves the latter. For comparison with the interpretation given in the final sentence of this note we offer an alternative proof of (5). By (2), log-convexity of the Bell numbers is equivalent to the statement that the average number of blocks \( \tau_n \) increases with \( n \). The latter is an intuitively appealing statement, and follows from the classical recursion (1) on noting that the average of any multiset \( M \) of positive integers must increase if each occurrence of an integer \( k \in M \) is replaced by \( k \) copies of itself and one copy of \( k + 1 \) to form a new multiset \( M' \).

Let us consider finally a question not previously addressed in the literature: the log-concavity of \( B_n/n! \). This inequality involves products of Bell numbers of length two, and using (4) out to two terms only we have

\[
(B_n/n!)^2 \geq (B_{n-1}/(n-1)!(B_{n+1}/(n+1)!) = \exp(2e^r - 2) \left( \frac{-2P_2}{e^r} + O(e^{-2r}) \right) = \frac{1}{2\pi r^{2n+4}} \exp(2e^r - 2r - 2 + O(r^{-1})).
\]

Thus the sequence \( B_n/n! \) is log concave for sufficiently large \( n \). Recall the exponential generating function [8]:

\[
\sum_{n=0}^{\infty} \frac{B_n}{n!} u^n = \exp(e^u - 1).
\]
The case $X_j = 1/(j-1)!$ of the following theorem shows that the sequence $B_n/n!$ is log-concave for $n \geq 1$.

**Theorem 2.** If the sequence $1, X_1, X_2, \ldots$ is nonnegative and log-concave, then so is the sequence $1, a_1, a_2, \ldots$ defined by the generating function equation

$$
\sum_{n=0}^{\infty} a_n u^n = \exp \left( \sum_{j=1}^{\infty} X_j u^{j/j} \right).
$$

The proof of Theorem 2, and further discussion, will appear [1]. The log-concavity of $B_n/n!$ is equivalent to the assertion that $nB_{n-1}/B_n$ is an increasing sequence. Thus an appealing interpretation of Theorem 2 in the case $X_j = 1/(j-1)!$ is that the average number of singleton blocks in a partition of an $n$-set is an increasing function of $n$.

**References**

1. E. A. Bender and E. R. Canfield, Log concavity and a related property of the cycle index polynomials, preprint.
5. J. R. Griggs, personal communication.