APPROMATING CODIMENSION \( \geq 3 \) \( \sigma \)-COMPACTA WITH LOCALLY HOMOTOPICALLY UNKNOTTED EMBEDDINGS*

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The proof of Štan’ko’s embedding approximation theorem is simplified and extended to a relative version for non-locally-compact \( \sigma \)-compact subsets of arbitrary topological manifolds.

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locally homotopically unknotted dimension embedding pseudoisotopy majorant p-LCC

1. Introduction

Recently Štan’ko [34] proved an approximation theorem that seems likely to have fundamental importance in geometric topology: Any embedding of a compactum \( X \) into \( R^q \), \( \dim X \leq q - 3 \), can be approximated arbitrarily closely by an embedding \( g: X \to R^q \) which is locally homotopically 1 co-connected (1-LCC), that is, \( R^q - g(X) \) is locally homotopically 1-connected (1-LC) at \( g(X) \) (definitions below). Unfortunately, his proof has some difficult spots, primarily because of the uncommon operations of resolution and reconstitution. In this paper we present his proof in the language of immersions, and extend it to a relative version for \( \sigma \)-compact subsets of arbitrary (topological metric) manifolds. Here \( \sigma \)-compact means only a countable union of compact subsets; it does not imply in addition locally compact as in [17]. Such subsets arise naturally in topological self-general-position theory (see [36] or [18, § 4]) and taming theory (e.g., any embedding \( f: X \to R^q \) of a compactum \( X \), \( \dim X \leq q - 3 \), is 1-LCC mod some 0-dimensional \( \sigma \)-compactum \( F \subset X \), that is, \( R^q - f(X - F) \) is 1-LC at \( f(X) \); see [14, Theorem 6.1]).

Approximation Theorem (generalizing Štan’ko [34]). Suppose \( Q^q \) is a topological manifold and \( X \) is an arbitrary \( \sigma \)-compact subset with \( \dim X \leq q - 3 \). Given any

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majorant map \( \varepsilon : X \to (0, \infty) \), there exists an embedding \( g : X \to Q \) within \( \varepsilon \) of the inclusion such that \( g(X) \) is 1-LCC in \( Q \), that is, \( Q - g(X) \) is 1-LC at \( g(X) \).

If \( \partial Q \neq \emptyset \) and \( \dim(X \cap \partial Q) \leq q - 4 \), then \( g \) may be chosen so that in addition \( g(X \cap \partial Q) = g(X) \cap \partial Q \) and \( g(X \cap \partial Q) \) is 1-LCC in \( \partial Q \).

If \( Y \) is a closed subset of \( X \) such that \( Y \) is 1-LCC in \( Q \) and \( Y \cap \partial Q \) is 1-LCC in \( \partial Q \), then \( g \) may be taken to be the identity on \( Y \).

Addendum. In addition to the embedding \( g \) one can choose an arbitrarily small ambient pseudoisotopy of \( Q \) which, when applied to \( g \), provides an isotopy of \( g \) to the inclusion map \( M \hookrightarrow Q \). More precisely, given any majorant map \( \eta : Q \to [0, \infty) \) such that \( X - Y \subset \eta^{-1}(0, \infty) \), there exists an \( \eta \)-pseudoisotopy \( h_t : Q \to Q \), \( t \in [0, 1] \) (defined below), such that \( h_1 g = \text{id}(X) \). Furthermore, \( h_t \) may be chosen so that its only nondegenerate point inverses lie in \( h_t^{-1}(\text{cl}_U(X - Y)) \), where \( \text{cl}_U(X - Y) \) is the closure of \( X - Y \) in some arbitrarily small open neighborhood \( U \) of \( X - Y \) in \( Q \).

We use \( \text{id}(X) \) to denote both the identity map and the inclusion map \( X \hookrightarrow Q \). A homotopy \( h_t : Q \to Q \), \( t \in [0, 1] \), is an \( \eta \)-pseudoisotopy if \( h_0 = \text{id}(Q) \); each \( h_t \), \( t \in [0, 1] \), is a homeomorphism; \( h_1 \) is a proper surjection; and for each \( w \in Q \), the path \( \{h_t(w)\} | t \in [0, 1] \) has diameter \( \leq \eta(w) \). Recall that such an \( h_1 \) is necessarily cellular [20]. In the Addendum it may be necessary that \( h_t^{-1}(\partial Q) \cap \text{int} Q \neq \emptyset \) (see Section 7), and also that \( h_t^{-1}(\text{cl}_U(X - Y) - (X - Y)) \) contains nondegenerate point inverses (see Section 6). Note that for \( X \) closed in \( Q \) one may choose \( \text{cl}_U(X - Y) = X - Y \). The idea of realizing wild embeddings by pseudoisotopy of tame embeddings is due to Keldyš [23].

The following Corollary is discussed more fully in [18, Section 3].

Corollary (Same hypotheses). In the space of all embeddings of \( X \) into \( Q \) with the majorant topology, the subspace of all 1-LCC embeddings is a dense-by-ambient-pseudoisotopy \( G_6 \) subset.

Corollary (cf. Štank'ko [35]; discussed in [18]). The \( k \)-dimensional Menger compactum \( M^k_q \) in \( R^q \) and the \( k \)-dimensional Nöbeling space \( N^k_q \) in \( R^q \) are each universal for all \( k \)-dimensional \( \sigma \)-compacta \( \{X\} \) in \( R^q \), that is, any such \( X \) embeds in both \( M^k_q \) and \( N^k_q \) (no dimension restrictions).

The Approximation Theorem can be regarded as an extension to non-trivial dimensions of the classical Menger–Nöbeling embedding theorem [22, Theorem V-2], which works when \( 2 \dim X + 1 \leq q \) and produces a 1-LCC embedding when in addition \( \dim X \leq q - 3 \). Previous to Štank'ko the Approximation Theorem above was also known for \( X \) a topological manifold, in which case \( g \) can be chosen a locally flat embedding [28]. (Bryant extended this to polyhedra [8].) Interestingly, Štank'ko's Theorem offers an alternative route to Miller's theorem [31].
In an earlier paper [33], Štan’ko illustrated the significance of the 1-LCC property by introducing the notion of demension (= dimension of embedding). It turns out that arbitrary compacta in topological manifolds can often be handled analogously to polyhedra in PL manifolds, if one works with demension instead of dimension. An introduction to demension theory is given in [18].

The following remarks are increasingly technical.

Remark 1. If \( X \) is a \( \sigma \)-compact 1-LCC subset of a topological manifold \( Q^n \), then \( X \) is \( k \)-LCC in \( Q \) for all \( 0 \leq k \leq q - \dim X - 2 \) (generalizing [29, Theorem 4.1] to \( \sigma \)-compacta by a limit argument). This paper does not use this remark, nor its converse that if a \( \sigma \)-compactum \( X \) in \( Q^n \) has no interior and is \( k \)-LCC for \( 0 \leq k \leq q - n - 2 \), then \( \dim X \leq n \), but these facts have geometric significance in dimension theory.

Remark 2. The Approximation Theorem is also true if \( X \) is a codimension 1 submanifold of \( \text{int} \ Q (q \geq 5) \) for which there is a 0-dimensional \( \sigma \)-compact subset \( F \subset X \) such that \( Q - (X - F) \) is 1-LC in \( Q \). As a consequence such an \( X \) is locally flatly approximable ([16], also [13]). Details are in [9]. Still open are the questions of whether all codimension 1 submanifolds have such subsets \( F \) (c.f. [15, Theorem 2]) or whether all codimension 1 submanifolds have 1-LCC approximations.

Remark 3. If one merely asks that \( g(X \cap \partial Q) \) be 1-LCC in \( \partial Q \), then it suffices to assume \( X \cap \partial Q \) is \( \sigma \)-compact and \( \dim(X \cap \partial Q) \leq q - 4 \), with no assumptions at all on \( X - \partial Q \). For example \( X - \partial Q \) could be all of \( Q - \partial Q \). This version, like the \( \partial Q \neq \emptyset \) case of the Theorem, is a corollary of the Addendum (applied to \( \partial Q \)), and is explained more fully in Section 7. In general, if \( M^m \) is a locally flat submanifold of \( Q \) (say \( \partial M = \emptyset = \partial Q \) for simplicity) and \( X \) is an arbitrary subset of \( Q \) such that \( X \cap M \) is a \( \sigma \)-compact and \( \dim(X \cap M) \leq m - 3 \), then there exists an embedding \( g : X \to Q \), arbitrarily close to the inclusion and realizable by appropriate pseudo-isotopy of \( (Q, M) \), such that \( g^{-1}(M) = X \cap M \) and \( g(X \cap M) \) is 1-LCC in \( M \).

Remark 4. The proof of the Theorem actually produces an embedding of \( X \) into a Nöbeling space (if \( Q = R^4 \)). If \( q = 4 \) (where the Theorem is classical), this conceivably may be a stronger conclusion than being 1-LCC embedded.

A subset \( X \subset Q \) is locally homotopically \( p \)-co-connected \((p \geq 0; \text{abbreviated } p-\text{LCC})\) if \( Q - X \) is \( p \)-LC at \( X \), that is, for each \( x \in X \) and each neighborhood \( U \) of \( x \) in \( Q \), there is a neighborhood \( V \) of \( x \) in \( Q \) such that any map \( a : S^p \to V - X \) is nullhomotopic in \( U - X \). If \( Q \) is metric and \( X \) is compact, this property is uniform for \( x \in X \) (hence \( p-\text{ULCC} \), compared to \( p-\text{ULC} \)), i.e., given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( x \in X \), if \( U \) is the open \( \varepsilon \)-neighborhood \( N(x, \varepsilon) \), then \( V \) may be taken to be \( N(x, \delta) \). \( \text{LCC}^k \) means \( p-\text{LCC} \) for all \( 0 \leq p \leq k \); similarly for \( \text{ULCC}^k \). A map
\( f : X \to Q \), (especially for us an embedding) is \( p \)-LCC if \( f(X) \) is \( p \)-LCC in \( Q \). We recall that if \( Q \) is metric, then the \( LCC^k \) property is hereditary for arbitrary subsets of \( Q \) without interior (that is, if \( X \hookrightarrow Q \) is \( LCC^k \) and \( X \) has no interior, then \( Y \hookrightarrow Q \) is \( LCC^k \) for any \( Y \subset X \); [19]). Also if \( Q \) is complete metric, then the \( LCC^k \) property is preserved under countable union of \( \sigma \)-compact subsets of \( Q \) without interior.

**Note on terminology**

Originally we used the expression 1-LC *embedding* instead of 1-LCC *embedding*, but there were some justified objections that this didn’t quite mesh with the accepted terminology that a \( k \)-LC map is one whose point inverses are locally \( k \)-connected (compare \( k \)-UV), and the fact that it is the local connectivity of the complement that one is measuring (not the embedding or its mapping cylinder; for example, the mapping cylinder of any map of a compact manifold to a manifold is locally contractible). We choose not to use such previous expressions as \( Q - f(X) \) is \( k \)-LC *at* \( f(X) \) and locally nice as being either too cumbersome or imprecise. The terminology \( k \)-LCC was suggested by Chris Lacher.

2. Preliminaries and statement of the fundamental lemma

Let \( B^k = [-1, 1]^k \subset R^k = R^k \times 0 \subset R^q \). If \( (Q, d_Q) \) is a metric space, let \( dist_Q(A, B) \) denote the distance between subsets \( A \) and \( B \) of \( Q \). Given a function \( \varepsilon : Q \to (0, \infty) \), let \( N(x, \varepsilon) = \{w \in Q \mid d_Q(x, w) < \varepsilon(x)\} \) and \( N(A, \varepsilon) = \bigcup_{x \in A} N(x, \varepsilon) \). The closure and frontier of a subset \( A \) of a topological space are denoted \( \text{cl} \ A \) and \( \text{fr} \ A \), and topological homeomorphism is denoted \( \approx \). We will make use of the following elementary facts.

**Proposition 1.** Suppose \( g : X \to Y \) is a map of topological spaces, where \( X \) is \( T_0 \). Then \( g \) is an embedding if and only if for each closed subset \( B \subset X \) and each \( x \in X - B \), \( g(x) \notin \text{cl} \ g(B) \).

**Proposition 2** (Convergence Criterion; cf. [4, Theorem 7; 1, Lemma 2.1]). Suppose \( X \) is a metric space and \( Q \) is a complete metric space. Suppose one constructs a sequence \( \{\{\varepsilon_i, g_i\} \mid i = 1, 2, \ldots\} \) of majorant maps \( \{\varepsilon : X \to (0, \infty)\} \) and embeddings \( \{g_i : X \to Q\} \) in such a manner that each \( \varepsilon_i \) can depend on \( \varepsilon_{i-1} \) and \( g_{i-1} \), and each \( g_i \) is an arbitrary embedding that is \( \varepsilon_i \)-close to \( g_{i-1} \). Then the sequence \( \{\varepsilon_i\} \) may be chosen so small that \( g = \lim g_i \) exists and is an embedding.

**Proof.** Given \( \varepsilon_{i-1} \) and \( g_{i-1} \), it suffices to choose \( \varepsilon_i(x) < \frac{1}{i} \min\{\varepsilon_{i-1}(x), \text{dist}_Q(g_{i-1}(x), g_{i-1}(X - N(x, 1/i)))\} \) for each \( x \in X \). Then \( g = \lim g_i \) is an embedding by Proposition 1. \( \Box \)
Proposition 3. Suppose $X$ is $\sigma$-compactum in $\mathbb{R}^q$, $\dim X \leq q - 2$ and $(\dim X, q) \neq (1, 3)$. Suppose $P$ is a countable union of compact, 1-dimensional subpolyhedra in $\mathbb{R}^q$, and suppose $\eta : \mathbb{R}^q \to [0, \infty)$ is a map such that $X \cap P \subset \eta^{-1}(0, \infty)$. Then there is an ambient $\eta$-isotopy of $\mathbb{R}^q$ which moves $X$ off of $P$.

Note. In all of our applications $P$ will be closed in some open subset of $\mathbb{R}^q$.

Proof. If $X$ is compact and $P$ is a polyhedron, this is well known (using [22, Theorem IV 41]; see [26, Lemma 3] for the $(0, 3)$ case; the examples in [27], [5] are counterexamples to the $(1, 3)$ case). The general case is proved by a countable number of applications of the compact case, making use of Proposition 2. ☐

Proposition 4 (For the $\sigma$-compact case). Suppose $X$ is a $\sigma$-compact subset of the metric space $Q$ and $\varepsilon : X \to (0, \infty)$ is a map. Then there exists a map $\eta : Q \to [0, \infty)$ such that $\eta^{-1}(0, \infty)$ is a neighborhood of $X$ and $\eta|X < \varepsilon$.

The heart of the paper is the following Lemma, whose proof occupies the next three sections. The case for $X$ compact is treated by itself in Sections 3, 4 and the $\sigma$-compact case is discussed in Section 5. We postpone the cases $Y \neq \emptyset \neq \partial Q$ and all discussion of the Addendum to the Theorem (concerning the pseudoisotopy) until Section 6.

Fundamental Lemma. Suppose $f : X \to \mathbb{R}^q$ is an embedding of a $\sigma$-compactum $X$, $\dim X \leq q - 3$, such that $f(X) \cap (\partial B^2 \times B^{q-2}) = \emptyset$. Given any majorant map $\varepsilon : X \to (0, \infty)$, there exists an embedding $g : X \to \mathbb{R}^q$ such that $g$ is $\varepsilon$-close to $f$, $g = f$ on $f^{-1}(\mathbb{R}^q - \text{int } B^q)$, $g(f^{-1}(\text{int } B^q)) \subset \text{int } B^q$ and $g(X) \cap B^2 = \emptyset$.

3. Fundamental Construction, compact case

The Lemma is based on the Fundamental Construction, whose statement requires the following ad hoc definition. Call a map $g : X \to \mathbb{R}^q$ of a space $X$ a nice immersion if there exists a finite collection $D_1^q, \ldots, D_j^q$ of closed disjoint $q$-cells in $\mathbb{R}^q$ (called the singularity cells of $g$), each factoring as $D_j^q = D_j \times D_j^{q-2}$, such that $g$ is the union $g = g_1 \cup g_2$ of two embeddings $g_i : U_i \to \mathbb{R}^q$, $i = 1, 2$, where $X = U_1 \cup U_2$, such that $U_1 = X - g_2^{-1}(\bigcup_{j=1}^r D_j^{q-2})$ and $U_2 = X - g_1^{-1}(\bigcup_{j=1}^r D_j^{q-2})$ are open, $g_1(U_1) \cap \bigcup_{j=1}^r (\partial D_j^q \times D_j^{q-2}) = \emptyset$ and $g_2(U_2) \cap \bigcup_{j=1}^r (\partial D_j^q \times D_j^{q-2}) = \emptyset$. Thus, in particular, $g$ is an immersion with the image of the singularity set lying in $\bigcup_{j=1}^r \text{int } D_j^q$ (see Fig. 1). If in addition $\varepsilon > 0$ is such that $\text{diam } D_j^q < \varepsilon$ and $\text{diam } g^{-1}(D_j^q) < \varepsilon$ for each $j$, then $g$ is a nice $\varepsilon$-immersion.
Fundamental Construction (compact case). Suppose $X$ is compact and $\varepsilon > 0$. Given an embedding $f$ as in the statement of the Fundamental Lemma, there exists a map $g$ as in the conclusion, except that $g$ is a nice $\varepsilon$-immersion (of $X$ into $\mathbb{R}^q - B^2$) whose singularity $q$-cells lie in $\text{int} B^q - B^2$.

Proof. First we observe that the $\varepsilon = \infty$ case of the Fundamental Construction implies the general $\varepsilon > 0$ case in straightforward fashion: Let $L$ be a fine 1-dimensional grid in $B^q$ which partitions $B^q$ into finitely many squares $\{B^q_j\}$ such that for each $j$, $\text{diam } B^q_j < \varepsilon$ and $\text{diam } f^{-1}(B^q_j) < \varepsilon$. By Proposition 3 we can assume that $f(X)$ misses $L = \bigcup \{\partial B^q_j\}$. Let $\lambda > 0$ be so small that for each $j$, $\text{diam } (B^q_j \times \lambda B^{q-2}) < \varepsilon$, $\text{diam } f^{-1}(B^q_j \times \lambda B^{q-2}) < \varepsilon$ and $f(X) \cap \partial B^q_j \times \lambda B^{q-2} = \emptyset$. Now apply the $\varepsilon = \infty$ case separately to each $q$-cell $B^q_j \times \lambda B^{q-2}$.

Hence we assume $\varepsilon = \infty$ in the following proof. If $q \leq 5$ the Construction is known ([22, p. 641; our $g$ may be taken to be an embedding). Hence assume $q \geq 6$. The basic observation is that there is a locally flat, connected orientable 2-manifold $M^2$ in $B^q$ such that $M^2 \cap \partial B^q = \partial M^2 = \partial B^2$. This follows from the fact that $\partial B^2$ is null-homologous in $B^q - f(X)$ (Alexander duality); or one can construct $M^2$ by appealing directly to dimension theory, as follows. Let $p : B^q \to B^{q-2}$ be the projection map. Then $p$ is homotopic rel $\partial B^q$ to a map $\pi : B^q \to B^{q-2}$ such that $0 \notin \pi(f(X) \cap B^3)$ ([22, Theorem VI 1]). Without loss $\pi$ is differentiable (or piecewise linear) and 0 is a regular value. Then the component of $\pi^{-1}(0)$ containing $\partial B^2$ is the desired $M^2$.

Now $M^2$ is a 2-cell with handles. Since $q \geq 6$, $M^2$ can be ambient isotoped onto a nice homeomorphic copy in $B^2 \times [0, 1] \subset B^3$; thus we can regard $M^2$ as being obtained from $B^2$ by attaching a finite number of unknotted, pairwise unlinked 1-handles $\{H^j = B^j_1 \times B^2\}_{1 \leq j \leq r}$ lying in $B^2 \times [0, 1]$, that is, $M^2 = B^2 - \bigcup_{j=1}^{r} \partial B^j_1 \times B^2 + \bigcup_{j=1}^{r} B^j_1 \times \partial B^j_1$ (see Figs. 2(b) and 3).

Our goal at this point is, for each elongated handle $H^j_3$, to define a reembedding of it $\varphi_j : H^j_3 \times [-1, 0] \to B^2 \times [-1, 0]$ beneath $B^2 \times 0 = B^2$ in $B^3$, as suggested by Fig. 3, and then to extend this to an embedding $\Phi_j : H^j_3 \times [-\mu, \mu]^{q-3} \to B^q - B^2$ of thickened $H^j_3$, as suggested by Fig. 4. If the reader can perceive the reembedding $\Phi = + \Phi_j$
with the help of Figs. 2, 3 and 4 and perhaps the paragraph after next, he need not read the following technical description.

We begin by defining some precise structure for the handles of $M^2$. For each $j$, let $C_j^2$ be a 2-cell in $\text{int} B^1 \times 0 \times \text{int} B^3$ such that $C_j^2 = \text{int} D_j^2 \cup G_j^2 \cup H_j^2$ is the union of five 2-cells as shown in Fig. 2(a), such that $C_{jn} \cap \text{int} = (0, u H_{j2}) \cap (A_j'u F_j' u G_j')$. Let $B'^+ = [-2, 2] \times \text{int} B'$ and suppose that $(H_{j1}' \cup F_{j1}' \cup G_{j1}') = (B'^+, B'^-) \times [-1, 1]$ in such a manner that $(H_{j1}' \cup F_{j1}' \cup G_{j1}') \cap B^1 \times 0 \times 0$ corresponds to $\partial B_j^1 \times [-1, 1]$ and $H_j^2 \cap D_j^2$ corresponds to $B_j^1 \times -1$ and $(F_j^2 \cup G_j^2) \cap A_j^2$ corresponds to $(B_j^1+ - \text{int} B_j^1) \times -1$. Elongate $H_j^2$ by defining $H_j^2+b = H_j^2 \cup F_j^2 \cup G_j^2$ and let $\xi H_j^2+b \subset H_j^2+b$ be the subset corresponding to $\partial B_j^1+b \times [-1, 1]$. We can suppose that $H_j^2$ (defined above) = $H_j^2 \times [-\lambda, \lambda]$ for some $\lambda > 0$ ($[-\lambda, \lambda]$ is really the second
Before: the embedding $f$, whose image intersects $B^2 \times 0$.

After: the immersion $g$, whose image misses $B^2 \times 0$.

(The singularities of $g$ do not show since they lie outside of $B^3$ in the transverse direction.)

Fig. 3. The fundamental push (shown here in $R^3$).

factor $0 \times [-\lambda, \lambda] \times 0'$ in $B^1 \times [-\lambda, \lambda] \times B^1 \approx B^2$, with $B^2_j$ in $H^3_j \approx B^1_j \times B^1_j$ corresponding to $[-\lambda, \lambda] \times [-1, 1]$ and that $f(X) \cap C^2_j \times [-\lambda, \lambda] \subseteq E^2_j \times (-\lambda, \lambda) \cup \text{int} \ D^2_j \times [-\lambda, \lambda]$, where $E^2_j$ is the open core of $H^2_{i_1^+}$ corresponding to $B^1_{i_1^+} \times (-1, 1)$. Define $H^3_{i_1^+} = H^2_{i_1^+} \times [-\lambda, \lambda]$. Let $\mu > 0$ be so small that $f(X) \cap C^3_j \times [-\lambda, \lambda] \times [-\mu, \mu]^{q-3} \subseteq (E^2_j \times (-\lambda, \lambda) \cup \text{int} \ D^2_j \times [-\lambda, \lambda]) \times [-\mu, \mu]^{q-3}$ and define $D^q_j = D^2_j \times [-\lambda, \lambda] \times [-\mu, \mu]^{q-3}$ and let $D^q_j \approx \emptyset$ correspond to the factor $[-\lambda, \lambda] \times [-\mu, \mu]^{q-3}$ in $D^q_j$. We assume the cells $\{C^3_j \times [-\lambda, \lambda] \times [-\mu, \mu]^{q-3}\}$ are disjoint.
The immersion $g$ is obtained from $f$ simply by pushing the handles $\{H_j^{3,+}\}$ underneath $B^2 \times 0$ in $B^3$ and damping this push in the $[-\mu, \mu]^a$ direction (see Figs. 3 and 4). Specifically, let $\psi_{j,s}, 0 \leq s \leq 1$, be an isotopy (= homotopy through embeddings) of $H_j^{3,+}$ in $C_j^3$, fixed on $\partial H_j^{3,+}$, such that $\psi_{j,0} = \text{identity}$ and $\psi_{j,1}(H_j^{3,+}) \subset A_j^3 \cup F_j^3 \cup G_j^3 - B^1 \times 0 \times 0$. Let $\varphi_{j,s} = \psi_{j,s} \times \text{id}([-\lambda, \lambda]) : H_j^{3,+} \times \lambda \rightarrow C_j^3 \times [-\lambda, \lambda]$ and let $\varphi_s = \oplus_{j=1}^q \varphi_{j,s}$ be the disjoint union of these isotopies. Let the embedding $\Phi : \bigcup_{j=1}^q H_j^{3,+} \times [-\mu, \mu] \rightarrow B^3$ be the natural extension of $\varphi : \bigcup_{j=1}^q H_j^{3,+} \times 0 \rightarrow B^3$ given by $\Phi \big| \bigcup_{j=1}^q H_j^{3,+} \times \{w\} = \varphi \big| \|w\|/\mu : \bigcup_{j=1}^q H_j^{3,+} \times \{w\} \rightarrow B^3 \times \{w\}$ for each $w \in [-\mu, \mu]^{a-3}$, where $\|w\|$ is the maximum-of-the-coordinates norm.
Let \( W = f^{-1}(\bigcup_{j=1}^{r} H_j^{3+} \times [-\mu, \mu]^{-3}) \) and define \( g : X \to R^q \) by \( g|W = \Phi f|W \) and \( g|X-W = f|X-W \). Then \( g \) is the desired nice immersion, being the union of two embeddings \( g|U_1 \) and \( g|U_2 \), where \( U_1 = X - (\Phi f|W)^{-1}(\bigcup_{j=1}^{r} D_j^q) \) and \( U_2 = X - f^{-1}(\bigcup_{j=1}^{r} D_j^q) \). The singularity cells \( \{ D_j^q \} \) may be shrunk slightly to miss \( B^2 \).

4. Proofs of Lemma and Theorem, Compact Case

Proof of Lemma (\( X \) compact). The proof of the Lemma involves repeated application of the Fundamental Construction. Let \( \varepsilon_0, \varepsilon_1, \ldots \) be a sequence of positive numbers such that \( \sum_{i=0}^{\infty} \varepsilon_i < \varepsilon \). The idea is to construct a sequence \( g_1, g_2, \ldots \) of maps of \( X \) into \( R^q - B^2 \) such that each \( g_i \) is a nice \( \varepsilon_i \)-immersion whose singularity \( q \)-cells lie in \( \text{int} B^q - B^2 \), and such that \( g_i \) agrees with \( f \) on \( f^{-1}(R^q - \text{int} B^q) \) and \( g_i(f^{-1}(\text{int} B^q)) \subset \text{int} B^q \) and \( g \) is \( \varepsilon_{i-1} \)-close to \( g_{i-1} \). The immersion \( g_{i+1} \) will be obtained from \( g_i \) by using the Fundamental Construction to remove the singularities of \( g_i \). This will introduce new singularities belonging to \( g_{i+1} \), but they will be of smaller size. The map \( g = \lim_{i \to \infty} g_i \) will be the desired embedding.

Figure 5 suggests all the 2-cells-with-handles \( \{ M^2 \} \) constructed during the countable number of applications of the Fundamental Construction in the proof of the Fundamental Lemma. The \( g_i \)'s reembed the solid handles of this collection of \( M^2 \)'s by pulling them through the spanning membranes. Some readers may prefer to regard the entire collection of \( M^2 \)'s as being constructed first, and then the reembeddings \( \{ g_i \} \) performed all at once. The advantage of the following inductive proof is its succinctness.

To start, let \( g_0 = f \) and let \( g_1 \) be a nice \( \varepsilon_1 \)-immersion within \( \varepsilon_0 \) of \( g_0 \) obtained by applying the Fundamental Construction to \( g_0 \). In general, given \( g_i \), let \( D_i = \{ D_j^q = D_j^q \times D_j^{q-2} | j \in J_i \} \) be the collection of associated singularity \( q \)-cells. Then \( g_{i+1} \) is obtained by altering \( g_i \) on \( g_i^{-1}(\bigcup D_i) \) as follows. Suppose \( g_{i+1} : U_1 \to R^q \) and

![Fig. 5. The model collection of \( M^2 \)'s.](image-url)
$g_{i,2}: U_2 \to R^q$ are the embeddings such that $g_i = g_{i,1} \cup g_{i,2}$. The Fundamental Construction (with $\varepsilon = \varepsilon_{i+1}$) may be applied separately to each of the embeddings $g_{i,j}|_D g_{i,j}(D^q_j)$, $j \in J_i$, to move them off $\bigcup \{ D^q_j \times 0 | j \in J_i \}$, and the results amalgamated to produce a nice $\varepsilon_{i+1}$-immersion $h_{i,1}: U_i R^q$. Thus $h_{i,1}(U_i) \cap D^q_j \times 0 = \emptyset$ for each $j \in J_i$, and the singularity $q$-cells $\mathcal{D}_{i+1} = \{ D^q_f | j \in J_i \}$ which arise in the construction of $h_{i,1}$ lie in $\bigcup \{ \text{int } D^q_f - D^q_j \times 0 | j \in J_i \}$. Let $h_{i,2}: U_2 \to R^q$ be an embedding constructed from $g_{i,2}$ by squeezing each image set $g_{i,2}(U_2) \cap D^q_j(j \in J_i)$ toward $D^q_j \times 0$ by sliding along the $D^q_f$ factor keeping $\partial D^q_f$ fixed, so that $h_{i,2}(U_2 - U_j) \cap (h_{i,1}(U_j) \cup \mathcal{D}_{i+1}) = \emptyset$. Then $g_{i+1} = h_{i,1} \cup h_{i,2}$ is the next nice immersion of the sequence. Its associated singularity $q$-cells are the members of $\mathcal{D}_{i+1}$, which are subsets of the interiors of members of $\mathcal{D}_i$.

To show that $g = \lim g_i$ is an embedding, we first make a general observation: Suppose $B \subset X$ and $i > 0$. Let $D(B, i) = \bigcup \{ \text{int } D^q_f | D^q_f$ is a singularity $q$-cell of $g_i$ such that $B \cap g_i^{-1}(D^q_f) \neq \emptyset \}$. Then $g(B) \cap g_i(B) \cup D(B, i)$. Now to show $g$ is an embedding, suppose $B$ is a closed subset of $X$ and $x \in X - B$ (cf. Proposition 1). Let $i$ be so large that $2\varepsilon_i < \text{dist}(x, B)$, hence $(\{g_i(x)\} \cup D\{x, i\}) \cap (\text{cl } g_i(B) \cup D(B, i)) = \emptyset$. By the above observation, $g(x) \cap \text{cl } g(B) = \emptyset$. (The cl's here are for the upcoming $\sigma$-compact case.) This completes the Lemma (compact case).

Proof of Theorem $(X$ compact, $\partial Q = \emptyset = Y$). The following proof is one variation of several (compare [30, Theorem 3]). Let $N^k$ be the set of all points in $R^q_k$ with $k$ or fewer coordinates rational (Nöbeling's $k$-dimensional universal set, cf. [22, p. 64]). For any subset $Z \subset N^q$, the inclusion $Z \to R^q$ has property ULCC$^{q-k-2}$ (cf. [6, Theorem 2] for the model case). Let $\{ D^2_i | i = 1, 2, \ldots \}$ be the countable collection of 2-cells whose union is $B^q - N^q$. Let $\{ \varphi_j : R \to Q | j = 1, 2, \ldots \}$ be a collection of coordinate neighborhoods of $Q$ such that $Q = \bigcup_{j=1}^\infty \varphi_j(\text{int } B^q)$. Let the set of all positive pairs $\{(i, j)\}$ be sequentially ordered. Using the Fundamental Lemma and Propositions 2 and 3, it is an easy matter to construct a sequence of embeddings $\{g_{i,j} : X \to Q\}$ such that

1. $g_{i,j}(X) \cap \varphi_j(D^2_i) = \emptyset$ and for each $(i', j') > (i, j)$ and each $x \in X$, $\text{dist}(g_{i', j'}(x), \varphi_j(D^2_i)) > \frac{1}{2} \text{dist}(g_{i,j}(x), \varphi_j(D^2_i))$, and
2. the sequence $\{g_{i,j}\}$ converges to an embedding $g : X \to Q$ such that $g$ is $\varepsilon$-close to $X \to Q$. Since $g(X) \cap \varphi_j(D^2_i) \neq \emptyset$ for each $(i, j)$, $g$ is a $1$-LCC embedding.

5. The $\sigma$-compact case

At first glance one might guess that the $\sigma$-compact case follows from a countable number of applications of the compact case, but this is not so. If $X_0$ is a compact subset of the $\sigma$-compactum $X \subset Q$, the compact case provides a reembedding of $X_0$ which is $1$-LCC, but does not provide an extension of this reembedding over $X$.

We remark that the proof in Sections 3 and 4 fails in the $\sigma$-compact case because of the nonexistence of the spanning surface $M^2$. That is, tame $S^1$'s in the complement
of a codimension $\geq 3$ $\sigma$-compactum may not bound 2-manifolds, as they did in the compact case. For example, let $X_0 = \bigcup_{k=1}^{\infty} K_k$ be a noncompact countable union of Antoine's necklaces in $R^3$ as pictured in Fig. 6, and let $S$ be a circle which 'links' each one once. Then there is no spanning surface of finite genus for $S^1$ in $R^3 - X$ (via a minimum genus argument as in [21, p. 141]). Nevertheless, there is a 2-cell-with-countably-many-handles which spans $S^1$ in $R^3 - X$, and this suggests a proof.

![Fig. 6.](image)

As before, the proof of the Theorem ultimately rests on the Fundamental Construction, whose extension to the $\sigma$-compact case uses the following extension of the notion of nice immersion. Suppose $W \subset R^q$ is open and $g: X \to W$ is a map. Then $g$ is a decent immersion of $X$ into $W$ if there exists a countable, discrete (=disjoint and the union of any subcollection is closed in $W$) collection $\{D^j_1: j = 1, \ldots, \infty\}$ of closed $q$-cells in $W$ (called the singularity cells of $g$), $D^j_0 = D^j_2 \times D^j_2$, such that $g = g_1 \cup g_2$ is the union of two embeddings $g_i: U_i \to W$, $i = 1, 2$, where $X = U_1 \cup U_2$, $U_1 = X - g_2^{-1}(\bigcup_{j=1}^{\infty} D^j_2)$ and $U_2 = X - g_1^{-1}(\bigcup_{j=1}^{\infty} D^j_2)$ are open, $g_1(U_1) \cap \bigcup_{j=1}^{\infty} (\partial D^j_2 \times D^j_2) = \emptyset$ and $g_2(U_2) \cap \bigcup_{j=1}^{\infty} (D^j_2 \times \partial D^j_2) = \emptyset$. If in addition $\varepsilon: X \to (0, \infty)$ is a majorant map such that $\text{diam } D^j_2 < \inf \varepsilon(g^{-1}(D^j_2))$ and $\text{diam } g^{-1}(D^j_2) < \sup \varepsilon(X)$ for each $j$, then $g$ is a decent $\varepsilon$-immersion.

**Fundamental Construction ($\sigma$-compact case).** Given a majorant map $\varepsilon: X \to (0, \infty)$ and an embedding $f$ as in the statement of the Fundamental Lemma (in Section 2), there exists a map $g$ as in the conclusion, except that $g$ is a decent $\varepsilon$-immersion of $X$ into $R^q - B^2$ whose singularity $q$-cells lie in $\text{int } B^q - B^2$.

**Proof.** The argument consists of constructing a sequence $\{g_i: X \to R^q\}$ of nice immersions such that $g = \lim g_i$ is the desired decent immersion. We do only the $\varepsilon = \infty$ case, since the general case follows from this case much as before, e.g., let $U \supset f(X) \cap B^2$ be an open subset of $\text{int } B^2$ such that $U$ contains an infinite, locally finite 1-dimensional grid $L$ which partitions $U$ into a countable collection $\{B^2_j\}$ of squares with the property: For each $j$, there is a neighborhood $V_j$ of $B^2_j$ in $R^q$ such that $\text{diam } V_j < \varepsilon(f^{-1}(V_j))$ and $\text{diam } f^{-1}(V_j) < \sup \varepsilon(X)$. By Proposition 3, there is an arbitrarily small push rel($R^q - \text{int } B^2) \cup (B^2 - U)$ which moves $f(X)$ off of $L$. 
Let each $B_j^q$ be fattened in the perpendicular coordinates to give a $q$-cell $B_j^q \subset V_j$ such that $(B_j^q, B_j) = (B_j^q \times B_j^{q-2}, B_j)$. Push $f(X)$ off $\bigcup \{\partial B_j^q \times B_j^{q-2}\}$ by the inverse of the following specific map $\sigma$ (defined here for future reference): let $\sigma : R^q \rightarrow R^q$ be a map which is orthogonal projection to $\bigcup \{\partial B_j^q \times 0\}$ on $\bigcup \{\partial B_j^q \times B_j^{q-2}\}$ and is a homeomorphism elsewhere, and takes each $V_j$ onto itself and is the identity off $\bigcup \{V_j\}$. Replace $f$ by $\sigma^{-1}f$. Now the $\epsilon = \infty$ case can be applied separately to each $B_j^q$.

Hence we assume $\epsilon = \infty$ in the following proof. We consider only the $q \geq 6$ case, since the classical proof works when $q \leq 5$, as before. Suppose $X = \bigcup_{k=1}^{\infty} X_k$, where each $X_k$ is compact and the union is monotone, that is, $X_k \subset X_{k+1}$. Let $g_0 = f$. Given $k \geq 0$, suppose that a nice immersion $g_k : X \rightarrow R^q$ has been constructed with singularity $q$-cells $\{D_j^q | 1 \leq j \leq n_k\}$ in int $B^2 \setminus B^2$ such that $g_k(X_k) \cap B^2 = \emptyset$ (as opposed to $g_k(X) \cap B^2 = \emptyset$). Let $\eta \in (0, 1/2^k)$ be so small that $(g_k(X_k) \cup \{D_j^q | 1 \leq j \leq n_k\}) \cap B^2 \times \eta B^{q-2} = \emptyset$. We construct $g_{k+1}$ essentially by doing a modified Fundamental Construction (compact case) on the embedding $g_k | g_k^{-1}(B^2 \times \eta B^{q-2})$. Let $(M^2, \partial M^2)$ be a nice 2-manifold in $(B^2 \times [0, \eta], \partial B^2 \times 0)$ as before such that $M^2 \cap g_k(X_{k+1}) = \emptyset$. Let $A^2$ be a 2-cell in int $M^2 \cap B^2 \times 0$. Let $P^1$ be a 1-dimensional spine of $M^2 - \text{int } A^2$ such that $\partial M^2 \subset P^1$. By Proposition 3 we can push $g_k(X)$ off of $P^1$ by ambient isotopy of $R^q$ rel $R^q - \text{int}(B^2 \times \eta B^{q-2})$. Then by an isotopy of $R^q - (P^1 \cup \partial B^2 \times \eta B^{q-2})$ into itself rel $R^q - B^2 \times 0$ we can push $g_k(X)$ off of $N$, where $N$ is some neighborhood of $M^2 - \text{int } A^2$ in $B^2 = \eta B^{q-2}$. (For future reference let $\tau_k : R^q \rightarrow R^q$ denote the cellular map such that $\tau_k^{-1} | R^q - (P^1 \cup \partial B^2 \times \eta B^{q-2})$ is the end embedding of this isotopy.) Thus we can assume that $g_k(X_{k+1}) \cap M^2 = \emptyset$ and $g_k(X) \cap N = \emptyset$. Now the same handle moving operations as before can be done. They are performed in $\text{int}(B^2 = \eta B^{q-2})$, making sure the cells $\{C^j \times [-\lambda, \lambda] \times (-\mu, \mu)^{q-3}\}$ are chosen disjoint from $A^2$, where $j$ is numbered to run from $n_k + 1$ to $n_{k+1}$. This produces a nice immersion $g_{k+1}$ with singularity cells $\{D_j^q | 1 \leq j < n_{k+1}\}$.

Let $g = \lim_{k \to \infty} g_k$. Note that for each $k$ there is a neighborhood $W_k$ of $X_k$ in $X$ such that $g_k | W_k = g_k | W_k$ for all $j \geq k$. Hence $g : X \rightarrow R^q - B^2$ is a decent immersion with singularity cells $\{D_j^q | 1 \leq j < \infty\}$ lying in $\text{int } B^2 \setminus B^2$.

The proof of the $\sigma$-compact case of the Fundamental Lemma is the same as the compact case, except for the following technical changes. ‘Nice’ is replaced by ‘decent’ and the $\{\epsilon_i\}$ are understood to be majorant maps such that $\sum \epsilon_i = \infty$ and (without loss) $\sup \epsilon_i(X) \rightarrow 0$. In the construction of $h_{i+2}$ from $g_{i+2}$ we first push $h_{i+1}^{-1}(U_i) \cup \cup D_{i+1}$ a little in $\cup D_i$ so that for each $j \in J_i, (h_{i+1}(U_i) \cup \cup D_{i+1}) \cap D_j^q$ misses some neighborhood of $D_j^q \times 0 \cup \partial D_j^q \times \text{int } D_j^q$ in $D_j^q$. (For future reference let $\gamma_j : R^q \rightarrow R^q$, with support in $\cup D_i$, be a cellular map such that $\gamma_j^{-1} \cap \text{int } D_j^q$ is this push. That is, $\gamma_j^{-1}(D_j^q \times 0 \cup \partial D_j^q \times \text{int } D_j^q)$ is a neighborhood of $D_j^q \times 0 \cup \partial D_j^q \times \text{int } D_j^q$ in $D_j^q$.) The ranges of the $\gamma_i$'s, although unspecified here, are readily provided. In the discussion on convergence, the inequality $2 \epsilon_i < \text{dist}(x, B)$ should be interpreted as $2 \sup \epsilon_i(X) < \text{dist}(x, B)$; the rest of that sentence uses the fact that $g_i(x) \in g_i(\text{cl } B) \Rightarrow g_i(x) \in \text{cl } \cup D_i$.

The Fundamental Lemma also holds under the weaker hypothesis that $f(X) \cap \partial B^2 \times 0 = \emptyset$ instead of $f(X) \cap (\partial B^2 \times B^{q-2}) = \emptyset$. This can be deduced from the stated
version by an argument as in the $\varepsilon = \infty \Rightarrow \varepsilon > 0$ discussion in the Fundamental Construction, $\alpha$-compact case. Given this, the proof of the $\alpha$-compact case of the Theorem ($\partial Q = \emptyset = Y$) is word-for-word the same as the compact case. The relative $Y \neq \emptyset$ case of the Theorem is immediate from an application of the absolute case to $X - Y$, using the fact that if both $g(Y)$ and $g(X - Y)$ are 1-LCC subsets of $Q$, then so is $g(X)$. This completes the Theorem for $\partial Q = \emptyset$. □

6. Proof of the Addendum to the Theorem ($\partial Q = \emptyset$).

Always $\partial Q = \emptyset$ in this section; the $\partial Q \neq \emptyset$ cases are taken up in Section 7.

In the Addendum to the Theorem, the key to visualizing the pseudoisotopy taking $g$ to $\text{id}(X)$ is to think of running the construction of $g$ backwards, that is, of constructing $\text{id}(X)$ from $g$ by undoing all of the handle moves in the original proof. Figs. 3, 4 and 5 make it seem clear that one can do this without introducing singularities in $g(X)$. The details which follow are to confirm this.

To focus on the heart of the pseudoisotopy construction, we first examine the compact case. The Addendum to the Theorem is deduced from the following lemma.

**Fundamental Lemma with Pseudoisotopy** (compact version, $q \geq 6$). Suppose the data of the original Fundamental Lemma (Section 2), except that $X$ is compact and $\varepsilon > 0$ is constant. Then there exists in addition to the embedding $g: X \to \mathbb{R}^q - B^2$, an $\varepsilon$-pseudoisotopy $h: \mathbb{R}^q \to \mathbb{R}^q$, $t \in [0, 1]$, with support in $\text{int} B^q$, such that $h \cdot g = f$, and such that the only nondegenerate point inverses of $h$, lie in $h^{-1}_1(f(X) \cap \text{int} B^q)$.

**Proof.** We assume $\varepsilon = \infty$ throughout, since the previous reductions of the $\varepsilon > 0$ case to the $\varepsilon = \infty$ case also apply here. The proof of this Lemma entails looking at the original proof more closely. In the Fundamental Construction (Section 3), $g$ is actually obtained from $f$ by a regular homotopy $g': (= \text{homotopy through immersions})$ of a neighborhood of $f(X)$. To make this concrete let $\Phi: \bigcup_{j=1}^r H_{i,j}^2 \times [-\mu, \mu]^{q-3} \to B^q$, $t \in [0, 1]$, be the natural isotopy of the thickened elongated handles, joining $\Phi_0 = \text{identity}$ to $\Phi_1 = \Phi$, defined by $\Phi_t|\bigcup_{j=1}^r H_{i,j}^2 \times \{w\} = \varphi_t(1-|w|/\mu)$ for each $w \in [-\mu, \mu]^{q-3}$ and $t \in [0, 1]$ (notation from Section 3). Let the regular homotopy $g'$ be defined using $\Phi_t$ exactly as $g$ was defined using $\Phi$.

The proof of the Fundamental Lemma in Section 4 is accomplished by application of a countable number of these regular homotopies. Regarding for subscript convenience the $i$th step as the one which produces $g_i$, it commences with the nice immersion $g_{i-1}: X \to \mathbb{R}^q$ with singularity cells $\mathcal{D}_{i-1} = \{D_j^q \approx D_j^3 \times D_j^q - 2 | j \in J_{i-1}\}$ (regard $\mathcal{D}_0 = \{B^q = B^2 \times B^{q-2}\}$). This $g_{i-1}$ is transformed to $g_i: X \to \mathbb{R}^q$ by first moving the thickened elongated handles $\mathcal{H}_i = \{H_{i,j}^3 \times [-\mu, \mu]^{q-3} | j \in J_i\}$ (lying in $\bigcup \mathcal{D}_{i-1}$) off the 2-cells $\{D_j^3 \times 0 | j \in J_{i-1}\}$, and then squeezing the images of the previously moved handles (from the collection $\mathcal{H}_{i-1}$) toward the same 2-cells. Thus, the individual handles in $\mathcal{H}_i$, which are defined in step $i$, are moved during steps $i$ and
For the purposes of this section, we wish to regard this combined move a bit differently. Namely, instead of moving and then squeezing, we wish to move the handles down the squeezed tracks themselves, so that they do not pass through any of the yet-to-be-defined singularity cells in \( D_j, j > i \).

For each \( i \geq 1 \), let this isotopy down the squeezed tracks be denoted \( \Omega_{i,t} : \bigcup H_i \to \bigcup D_{i-1}, 0 \leq t \leq 1 \), with \( \Omega_{i,0} = \text{identity} \). It can be defined explicitly as \( \Omega_{i,t} = \omega_i \Phi_{i,t} \), where \( \Phi_{i,t} : \bigcup H_i \to \bigcup D_{i-1} \) is the analogue (i.e. conjugate) in \( \bigcup D_{i-1} \) of the natural isotopy \( \Phi_i \) of the Fundamental Construction defined above, and \( \omega : R^q \to R^q \) is the squeezing homeomorphism of step \( i + 1 \) mentioned above (unlabeled in Section 4), with support in \( \bigcup D_i \subseteq \bigcup D_{i-1} \), which transforms \( g_{i+1} \) to \( \omega g_{i+2} = h_{i+2} \). Note \( \Omega_i \) is an \( \epsilon_{i-1} \)-isotopy.

For notational convenience, define for \( i \geq 1 \),

\[
H_i = \bigcup H_i = \bigcup \{ H_j^{j+} \times [-\mu, \mu]^{q-3} | j \in J_i \},
\]

\[
H_i' = \Omega_{i,1}(H_i) = \bigcup \{ \Omega_{i,1}(H) | H \in H_i \},
\]

\[
\text{tr} H_i (\text{track of } H_i) = \bigcup \{ \text{tr} \Omega_{i,1}(H) | t \in [0, 1] \}, \quad \text{and} \quad D_i = \bigcup D_i.
\]

Recall from Sections 3 and 4 that, for each \( i \),

(a) \( D_i \cap H_i = \emptyset \); hence \( H_i \cap H_j = \emptyset \) for \( i \neq j \), and

(b) \( g[f^{-1}(H_i)] = \Omega_{i,1} f \), and \( g[f^{-1}(R^q - \bigcup_{j<i} H_j)] = f \).

The isotopy \( \Omega_i \) described above has the property that, for each \( i \geq 1 \),

(c) \( \text{tr} H_i \cap (H_{i+1} \cup D_{i+1}) = \emptyset \)

(as opposed to just \( H_i \cap (H_{i+1} \cup D_{i+1}) = \emptyset \), as the original description would have given).

Let \( \Omega_{i,t} : H_i \to D_{i-1} \) be the reflection-in-time of \( \Omega_{i,t} : H_i \to D_{i-1} \), that is, \( \Omega_{i,t} = \Omega_{i,1-\epsilon_1} \Omega_{i,1}^{-1} \), \( 0 \leq t \leq 1 \) (so that \( \Omega_{i,0} = \text{id}(H_i) \)).

To simplify notation, let \([0, \infty)\) serve as the time interval for upcoming homotopies and isotopies, instead of \([0, 1]\).

Let \( g_i : X \to R^q, t \in [0, \infty) \), be the homotopy defined by

\[
g_i = \Omega_{i-1,1} \cdot \Omega_{i-1,1} \cdot \cdots \Omega_{1,1} \Omega_{1,i} g \quad \text{for } t \in [i-1, i], \quad \text{and} \quad g_{\infty} = f_i,
\]

where it is understood that each \( \Omega_{i,t} \) is extended via the identity wherever needed in the definition. Then \( g_i \) is a well-defined isotopy, as can be seen by using (a), (b) and (c) above and induction on increasing \( i \) to establish:

\[
(*) \quad \text{for } t \in [i-1, i], \quad g_{i,t} : X \to R^q \quad \text{is an embedding with image } g_i(X) \subseteq (f(X) - \bigcup_{j<i} \text{tr } H_j) \cup \bigcup_{j<i} H_j \cup \bigcup_{j=i} \Omega_{i+1,j} (H'_j) \cup \bigcup_{j=i} H'_j, \quad \text{and furthermore } \{ x \in X | g_i(x) \neq f(x) \} \subseteq f^{-1}(\bigcup_{j=i} H_j).
\]

Concerning (\(\ast\)), it helps to observe that for the interval \( t \in [i-1, i] \), the motion of \( g_i \) lies in \( \text{tr } H_i \), while from (a) and (c) it follows that \( \text{tr } H_i \cap \bigcup_{j<i} H_j \cup \bigcup_{j>i} H'_j = \emptyset \). (Note: the embedding \( g_{i-1} \) here is not to be confused with the nice immersion \( g_i \) above.)
To extend this isotopy \( g_t \) to a pseudoisotopy of \( R^q \), just observe that from the construction and its properties, each \( \Omega_{i,t} \) hence each \( \Omega_{i_1} \) can readily be extended to an ambient \( \epsilon_{i_1-1} \)-isotopy of \( R^q \), with support in \( \text{nbhd}(\tau H_i) - (H'_{i+1} \cup D_{i+1}) \subseteq D_{i-1} \), so that the definition of \( g_t \) is unaffected. Define \( h_t : R^q \to R^q, t \in [0, \infty) \), using the same equation which defines \( g_t \) above. Then \( h_t = \lim_{t \to \infty} h_t \) is a proper surjection of \( R^q \), whose only nondegenerate point inverses lie in \( h_t^{-1}(f(X)) \) (assuming, without loss, that each component handle of each \( \mathcal{H}_i \) intersects \( f(X) \)). Thus \( h_t : R^q \to R^q, t \in [0, \infty) \), is the desired pseudoisotopy of the compact version.

We turn now to the general \( \sigma \)-compact case. The principal goal of this section is to prove the following strengthened version of the Fundamental Lemma (Section 2).

**Fundamental Lemma with Pseudoisotopy** (full \( \sigma \)-compact version). Suppose \( f : X \to R^q \) is an embedding of a \( \sigma \)-compactum \( X \), \( \dim X \leq q - 3 \), and suppose \( \eta : R^q \to [0, \infty) \) is any majorant map. Then there exists an embedding \( g : X \to R^q - (R^2 \cap \eta^{-1}(0, \infty)) \) and an \( \eta \)-pseudoisotopy \( h_t : R^q \to R^q, t \in [0, 1] \), such that \( h_t g = f \). Furthermore, letting \( U = \eta^{-1}(0, \infty) \), then \( h_t \) may be chosen so that the only nondegenerate point inverses of \( h_t \) lie in \( h_t^{-1}(\overline{U}(f(X) \cap U)) \).

**Note.** The composition \( h_t g : X \to R^q, t \in [0, 1] \), is automatically an isotopy from \( g \) to \( f \). That is, \( \{ h_t g | t \in [0, 1] \} : X \times I \to R^q \times I \) is a level-preserving embedding.

**Example.** This example shows that in the above Lemma (and similarly in the Theorem), if \( X \) is \( \sigma \)-compact, then one cannot hope to have the nondegenerate point inverses of \( h_t \) lie only in \( h_t^{-1}(f(X)) \). Let \( f : X \to R^q \) be any embedding of a \( \sigma \)-compactum \( X \) such that \( f(X) \) is dense in \( R^q \). The following fact shows that any \( h_t \) provided by the Fundamental Lemma, with nondegenerate point inverses lying only in \( h_t^{-1}(f(X)) \), would have to be a homeomorphism.

**Fact.** Suppose \( h : R^q \to R^q \) is a proper surjection and \( f(X) = X \subseteq R^q \) is a dense subset such that

(i) \( \deg h \neq 0 \),

(ii) \( h \) is \( 1 \)-\( 1 \) on \( h^{-1}(R^q - X) \), and

(iii) there exists an embedding \( g : X \to R^q \) such that \( hg = \text{id}(X) \).

Then \( h \) is a homeomorphism.

**Proof of Fact.** We wish to show that \( g(X) = h^{-1}(X) \), not just \( g(X) \subseteq h^{-1}(X) \). Define a function \( \varphi : R^q \to R^q \) by \( \varphi h^{-1}(X) = gh \) and \( \varphi h^{-1}(R^q - X) = \text{identity} \). Then \( \varphi \) is continuous: this is clear at points of \( h^{-1}(R^q - X) \) (remembering that \( h \) is closed, hence point inverses \( \{ h^{-1}(y) | y \in R^q \} \) have arbitrarily small saturated neighborhoods), while at points of \( h^{-1}(X) \) this uses that \( h^{-1}(X) \) is dense in \( R^q \) (from invariance of domain). Furthermore \( \varphi(R^q) \subseteq \overline{g(X)} \). Now \( h \varphi - h \) hence \( h \) is proper and \( \deg \varphi \neq 0 \), hence \( \overline{g(X)} = R^q \). But then \( \varphi = \text{identity} \) and therefore \( g(X) = h^{-1}(X) \) as sought.  \( \square \)
The proof of the above Fundamental Lemma with Pseudoisotopy is somewhat intricate and splintered, so we first prove the following:

Fundamental Lemma with Pseudoisotopy $\Rightarrow$ Addendum to Theorem ($\partial Q = \emptyset$).

**Proof.** Suppose $X \subset Q$ and $\eta: Q \to [0, \infty)$ are given by the Theorem. Note $Y$ is irrelevant, since we may as well assume $Y \cap \eta^{-1}(0, \infty) = \emptyset$. Following the Proof of Theorem in Section 4, but using index $k$ in place of $(i, j)$, let $\{D^2_k\} | 1 \leq k < \infty$ be the countable number of 2-cells in $Q$ off of which $X$ is to be moved. Let $g_0 = \text{id}(X): X \to Q$ and in general given an embedding $g_{k-1}: X \to Q - \bigcup \{D^2_l\} | 1 \leq l \leq k-1$, let $g_k: X \to Q - \bigcup \{D^2_l\} | 1 \leq l \leq k$ be a nearby embedding constructed as in the original proof, this time along with a small pseudoisotopy $h_k: Q \to Q$, $t \in [0, 1]$, such that $h_{k,0} = \text{id}$ and $h_{k,1}g_k = g_{k-1}$. Before discussing epsilonics, we exhibit the method of composing these pseudoisotopies to get the pseudoisotopy of the Theorem. For each $k$, let $h_{k,t}: Q \to Q$, $t \in [0, 1]$, be $h_{k,t}$ with new time parametrization, so that all its action takes place between time $t = (k-1)/k$ and $t = 1$. That is, $h_{k,t} = \text{id}$ for $0 \leq t \leq (k-1)/k$, and $h_{k,t} = h_{k-1-k+kt}$ for $(k-1)/k \leq t \leq 1$. The pseudoisotopy $h_t: Q \to Q$, $t \in [0, 1]$, of the Theorem is defined by $h_t = \lim_{k \to \infty} h_{k,1}h_{k,2} \cdots h_{k,t}$. (This method of composition, instead of $h_t = h_{k+1,t}h_{k,t} \cdots h_{1,t}$, makes it easier to see that the limit is a pseudoisotopy.) As before let $g = \lim_{k \to \infty} g_k$. Observe that one can choose the sequence $h_{k,t}$, $k = 1, 2, \ldots$, sufficiently rapidly convergent so that $h_t$ is an $\eta$-pseudoisotopy, in addition to $g$ having the properties of the previous proof in Section 4. Clearly then $h_tg = g_0 = \text{id}(X)$. Concerning the location of nondegenerate point inverses of $h_t$, first note that one may as well regard $Q$ as being $U$, in which case we need only arrange that $h_t$ be $1-1$ off of $h_t^{-1}(\text{cl}O\text{cl}(X))$. Now the above Fundamental Lemma with Pseudoisotopy says that any individual $h_{k,1}$ can be assumed to be $1-1$ off of $h_{k,1}^{-1}(\text{cl}O\text{cl}(X))$, hence any finite composition $h_{1,1} \cdots h_{k,1}$ can be assumed to be $1-1$ off of $h_{k,1}^{-1} \cdots h_{1,1}^{-1}(\text{cl}O\text{cl}(X))$. So if one chooses $h_{i,t} = \text{id}$ on $h_{k,1}^{-1} \cdots h_{1,1}^{-1}(Q - \text{N}(X,1/k))$ for each $i > k$, it will follow that $h_t$ is $1-1$ off of $h_t^{-1}(\text{cl}O\text{cl}(X))$. This completes the proof at hand. \(\square\)

For subsequent discussion it is useful to have available the notion of $\text{dem} X \leq q-2$ for a $\sigma$-compact subset $X$ of $R^q$. This means simply that $X$ satisfies the conclusion of Proposition 3 in Section 2; when $q \neq 3$ this notion coincides with $\text{dim} X \leq q-2$, but when $q = 3$ it is strictly between $\text{dim} X \leq q-3$ and $\text{dim} X \leq q-2$. (The general notion of dem(=dimension of embedding), introduced for compacta in [33], is defined for $\sigma$-compacta in [18].)

**Modified Proposition 3.** Suppose the hypotheses on $X$ of Proposition 3 in Section 2. Then there exists an embedding $g: X \to R^q$ and an $\eta$-pseudoisotopy $h_t: R^q \to R^q$, $t \in [0, 1]$, such that $h_tg = \text{id}(X)$ and $\text{dem} \{g(X) \cap \eta^{-1}(0, \infty)\} \leq q-2$. Furthermore $h_t$ can be chosen so that its only nondegenerate point inverses lie in $h_t^{-1}[\text{cl}(X \cap \eta^{-1}(0, \infty))]$. 
Proof. For simplicity assume $\eta = \infty$. Write $N_q^{q-2} = R^q - \bigcup_{i=1}^{\infty} R_i^1$ where $N_q^{q-2}$ is Nöbeling’s universal $(q-2)$-dimensional set (see Proof of Theorem, Section 4) and each $R_i^1$ is a 1-dimensional hyperplane in $R^q$. The embedding $g$ will be such that for each $i$, $\text{cl}(g(X) \cap R_i^1) = \emptyset$, hence $\text{dem cl} \ g(X) \leq q-2$. To construct $g$, we move $\text{cl} X$ off one $R_i^1$ at a time, by first moving $X$ off of $R_i^1$ by an ambient isotopy (cf. original Proposition 3), and then blowing up $\text{cl}(X \cap R_i^1)$ to $[\text{cl}(X \cap R_i^1)] \times D^{q-1}$ by the inverse of a pseudoisotopy. Do this for each $1 \leq i < \infty$ in the usual convergent fashion to construct $g$. The sequence of pseudoisotopies so obtained can be composed as in the preceding proof to get the final pseudoisotopy.

Turning to the proof of the general $\sigma$-compact case of the Fundamental Lemma with Pseudoisotopy, we first make two reductions of the problem. The first is done to obviate certain auxiliary cellular maps $\sigma$ and $\tau_k$ used in Section 5. This reduction is that, by applying the Modified Proposition 3 above to $f(X)$ at the start of the Fundamental Lemma with Pseudoisotopy, we can assume that $\text{dem cl} [f(X) \cap \eta^{-1}(0, \infty)] \leq q-2$, which we do from now on. The second reduction of the problem is that, arguing as in the $\varepsilon = \infty \Rightarrow \varepsilon > 0$ remarks in Section 5 (without need of $\sigma$ now, because of the first reduction), it suffices to prove the following.

Fundamental Lemma with Pseudoisotopy (reduced $\sigma$-compact version). Suppose $f: X \to R^q$ is an embedding of a $\sigma$-compactum $X$, $\dim X \leq q - 3$, such that $\text{cl} [f(X) \cap \partial B^2 \times B^{q-2}] = \emptyset$ and $\text{dem cl} [f(X) \cap \text{int} B^q] \leq q-2$. Then there exists an embedding $g: X \to R^q - B^2$ and a pseudoisotopy $h_t: R^q \to R^q$, $t \in [0, 1]$, with support in $\text{int} B^q$, such that $h_t g = f$. Furthermore, $h_t$ may be chosen so that its only nondegenerate point inverses lie in $h_t^{-1}[\text{cl} (f(X) \cap \text{int} B)]$.

Proof. This proof is virtually a direct adaption to the $\sigma$-compact case of the above compact case pseudoisotopy construction, accomplished by paralleling the manner in which Section 5 was adapted from Sections 3, 4. However the auxiliary cellular maps $\tau_k$ and $\gamma_k$ used in Section 5 require special attention. The problem with $\tau_k$, which occurs in the Fundamental Construction in Section 5, is that even though one can arrange each finite composition $\tau_1 \cdots \tau_k$ to be well-defined, the limit as $k \to \infty$ will not exist because it will necessarily blow up the bad point which is the limit in $B^2 \times 0$ of the cells $\{A^2\}$.

We remedy this as follows. First note that $\tau_k$ can now be chosen a homeomorphism, because of the assumption $\text{dem cl} [f(X) \cap \text{int} B^q] \leq q-2$. However, $\lim_{k \to \infty} \tau_1 \cdots \tau_k$ will still fail to exist. We can make this limit converge, however, even to a homeomorphism using Proposition 2, by modifying each stage of the Fundamental Construction using the following observation. If $A^2 \subset B^2$ is a 2-cell such that $\text{cl} [g_k(X) \cap B^2 \times 0] \subset \text{int} A^2 \times 0$, then $A^2$ can be subdivided into many small 2-cells $A^2 = \bigcup A_i^2$ and $\text{cl} g_k(X)$ can be ambient isotoped off $\bigcup A_i^2 \times 0$ (since $\text{dem cl} g_k(X) \leq q-2$) to arrange that $\text{cl} [g_k(X) \cap B^2 \times 0] \subset \bigcup_i \text{int} A_i^2 \times 0$. Then in the Fundamental Construction further activity can be confined to small disjoint neighborhoods of the $\text{int} A_i^2 \times 0$'s. This way we can arrange $\lim_{k \to \infty} \tau_1 \cdots \tau_k$ to be a homeomorphism, and so we
can disregard it (by absorbing it into \( h \) in the usual manner). Formerly the limit of the \( A^2 \)'s could be a point, whereas now the limit of the \( A_5^2 \)'s may necessarily be as large as a Cantor set, say \( Z \). In short, then, we will assume that in the Fundamental Construction (\( \sigma \)-compact case), \( g \) is gotten from \( f \) by a regular homotopy \( \Phi_t \) of a countable collection of handles which lie in \( \text{int } B^q - B^2 \) and comprise a discrete collection in \( \text{int } B^q - Z \).

Now to prove the Fundamental Lemma with Pseudoisotopy (reduced \( \sigma \)-compact version) one argues as in the earlier compact version, constructing \( \Omega_{t_0}, \Omega'_t, \) etc. and pushing along the squeezed tracks. However one must take special account of the auxiliary cellular map \( \gamma_i \) used in the original proof of the Fundamental Lemma (\( \sigma \)-compact case, Section 5). Let \( \gamma_i : R^q \to R^q \) be a pseudoisotopy of \( \gamma_{i,0} = \text{identity} \), say \( \gamma_i \). The natural definition of the isotopy \( g_i : X \to R^q, t \in [0, \infty), \) in this case, corresponding to the definition of \( g : X \to R^q \) in the compact case above, is

\[
g_i = \Omega_{t_0}^{l,(i+1-1)} \gamma_{i-1,i} \Omega_{t_0}^{l,1} \cdots \gamma_{i,1} \Omega_{t_0}^{l,1} g \quad \text{for } t \in [i-1, i-\frac{1}{2}],
\]
\[
g_i = \Omega_{t_0}^{l,(i+1/2-1)} \gamma_{i,0} \Omega_{t_0}^{l,1} \cdots \gamma_{i,1} \Omega_{t_0}^{l,1} g \quad \text{for } t \in [i-\frac{1}{2}, i],
\]

and

\[
g_{\infty} = f.
\]

This isotopy naturally extends to a cellular homotopy \( h_i : R^q \to R^q \) by extending the \( \Omega_t \) as in the compact case. However we desire \( h_t \) to be a pseudoisotopy. But \( h_i \) can be adjusted to be a pseudoisotopy \( h_i : R^q \to R^q, t \in [0, \infty), \) by slowing down the \( \gamma_i \) so that they all are homeomorphisms until the last instant \( t = \infty \). That is, in the above-definition of \( g, \) every \( \gamma_i \) and \( \gamma_{i,1} \) should be replaced for example by \( \gamma_i^{(i-f+1/2)}(i-f+3/2) \). This new \( h_i \) no longer covers \( g \) as \( h_i \) did, but that is not necessary. The desired property \( h_{\infty} g = \text{id}(X) \) does hold. As in the compact case, the nondegenerate point inverses of \( h_{\infty} \) can be assumed to occur only where they should.

It remains to consider the cases \( q = 3, 4 \) or 5 (still \( \partial Q = \emptyset \)). The classical proof which we invoked in Sections 3 and 5 for these cases only provides an approximating embedding, not a pseudoisotopy.

In the original construction in Section 3, the dimension restriction \( q \geq 6 \) was used only in the construction of the ambient isotopy of \( B^q \) rel \( \partial B^q \) taking the 2-manifold \( M^2 \subset \pi^{-1}(0) \) onto a standardly embedded copy in \( B^2 \times [0, 1] \subset B^q \). If \( q = 5 \), this can still be done, as any two faithful smooth (or PL) embeddings of \( (M^2, \partial M^2) \) into \( (B^q, \partial B^q) \) are equivalent for \( q = 5 \) (see [38, Theorem 24]; the idea is to construct a concordance of \( M^2 \times I \) in \( B^q \times I \) by using the Whitney trick to remove 0-dimensional singularities, and then use concordance \( \Rightarrow \) isotopy).

If \( q = 3 \), the handles of \( M^2 \subset \pi^{-1}(0) \) may be quite badly linked and knotted, but we can construct a nicely embedded 2-cell-with-handles \( N^2 \) which misses \( f(X) \) by tunneling as follows. Let \( P^3 \) be one of the closed complementary domains of \( M^2 \) in \( B^3 \) and let \( \delta P^3 = \text{fr} B^3, P^3 = M^2, \) initially). Let \( P^3 \) be represented as \( B^3 \times [1, 0] \) with 1-handles and 2-handles attached. Let \( \alpha_1, \ldots, \alpha_n \) be the complementary 1-dimensional cores of the 2-handles, perturbed slightly by ambient homeomorphism
to miss \( f(X) \) (Proposition 3). For each \( i \), remove from \( P^3 \) a small open tubular neighborhood of \( \alpha_i \), so that \( \delta P^3 \cap f(X) = \emptyset \) still holds. If we similarly remove 2-handles from the complement of \( P^3 \) by adding 1-handles to \( P^3 \), we can arrange that both \( P^3 \) and its complement are cubes with handles, and so can conclude that the handles of \( P^3 \) are unlinked and unknotted (cf. [32, p. 841 and [37]). Finally, let \( N^2 = \delta P^3 \) play the role of \( M^2 \) in the proof. \( \square \)

In either of the above cases, then, the pseudoisotopy construction above can be carried out, even when \( X \) is \( \sigma \)-compact.

If \( q = 4 \), we return to the classical proof, adapting it to provide the desired pseudoisotopy. Namely, we prove the \( q = 4 \) case of the Fundamental Lemma with Pseudoisotopy directly, using the classical approximation theorem and the following Proposition. (For the definition of \( \text{dem} \) for \( \sigma \)-compacts, see [18].)

**Proposition.** Suppose \( X \) is a \( \sigma \)-compact subset of \( R^q \) and \( g : X \to R^q \) is an embedding such that \( 2 \text{dem} \text{cl} g(X) + 2 \leq q \) and \( g^{-1} : g(X) \to X \to R^q \) extends to a map \( \gamma : \text{cl} g(X) \to R^q \). Then there exists an ambient isotopy \( h_t : R^q \to R^q, t \in [0, 1] \), such that \( h_t g \) is arbitrarily uniform-close to \( \text{id}(X) \). Furthermore, if \( \gamma \) is \( \eta \)-close to identity for some majorant map \( \eta : R^q \to [0, \infty) \), then \( h_t \) may be chosen an \( \eta \)-isotopy.

This proposition is a routine generalization of [6]; here \( \eta \)-close means \( <\eta \)-close except where \( \eta = 0 \). (The \( q = 2 \) case of the Proposition requires [11].) By passing to a limit, one can construct an \( \eta \)-pseudoisotopy \( h_t : R^q \to R^q, t \in [0, 1] \), such that \( h_t g = \text{id}(X) \). Given \( X \) as in the Approximation Theorem \( (q = 4) \), the classical embedding theorem [22, Theorem V3] can be adapted to provide \( g \) as in the Proposition. We omit details.

We remark that the above Proposition leads to a generalization of [6], to give a natural condition for two embeddings of a \( \sigma \)-compactum into \( R^q \) to be ambient isotopic, in the trivial dimension range. Details are in [18, § 3].

7. The \( \partial Q \neq \emptyset \) cases

This section starts with an example, to illustrate something which cannot be done at \( \partial Q \).

**Example.** This shows that in the Theorem with Addendum \( (\partial Q \neq \emptyset = Y; X \text{ compact say}) \), even if one only asks that \( g(X \cap \partial Q) \) be 1-LCC in \( \partial Q \) (but not \( g(X \cap \text{int } Q) \) be 1-LCC in \( \text{int } Q \)), nevertheless one cannot insist that \( h_t^{-1}(\partial Q) \cap \text{int } Q = \emptyset \). Rather, some nondegenerate point inverses in \( h_t^{-1}(X \cap \partial Q) \) may have to intersect \( \text{int } Q \). (Note however if \( X \subset \partial Q \), then one can arrange the above conclusion by extending appropriately the pseudoisotopy of \( \partial Q \) over a collar neighborhood of \( \partial Q \) in \( Q \).)

For the example let \( X_0 \subset \partial Q \) be any non-1-LCC compact subset of the boundary
of any manifold $Q^q$, $\dim X_0 \leq q - 4$, and let $X = X_0 \times [0, 1] \subset \partial Q \times [0, 1]$, the latter set regarded as a collar neighborhood for $\partial Q = \partial Q \times 0$ in $Q$. We show that if one could construct $g$ and $h$, as indicated above, then $X_0$ would be 1-LCC in $\partial Q$. Let $\alpha : S^1 \to \partial Q - X_0$ be a small loop near $X_0$. Suppose an embedding $g : X \to Q$ and pseudoisotopy $h : Q \to Q$, $t \in [0, 1]$ are as above, so that $g(X_0)$ is 1-LCC in $\partial Q$, $h_1g = \text{id}(X)$, $h_1(\text{int } Q) = \text{int } Q$ and support $h \subset \partial Q \times [0, 1] - \alpha(S^1)$. Now $\alpha(S^1)$ bounds a small singular disc $\alpha(D^2) \subset \partial Q - g(X_0)$. By a small homotopy push of this singular disc into int $Q$ rel $\alpha(S^1)$, we can arrange $\alpha(D^2) \subset \partial Q \times (0, 1) - g(X)$. But then $h_1\alpha(D^2) \subset \partial Q \times (0, 1) - X$ and so projection to $\partial Q$ gives a small null homotopy in $\partial Q - X_0$ of $h_1\alpha(S^1) = \alpha(S^1)$.

We proceed to the proofs. There are two independent ways of working at the boundary, each with its own virtues. We describe both. As usual $Y = \emptyset$; the $Y \neq \emptyset$ cases follow from the $Y = \emptyset$ cases as before. Let $\partial X$ denote $X \cap \partial Q$.

Both ways have the common feature that they first produce an embedding $g_* : X \to Q$, with $g_*^{-1}(\partial Q) = \partial X$, and a small pseudoisotopy $h_{*,t} : Q \to Q$, $t \in [0, 1]$, such that $h_{*,t}g_* = \text{id}(X)$ and $g_*(\partial X)$ is 1-LCC in $\partial Q$. Then the original without-boundary case can be applied to the restriction of this embedding $g_* : X \cap \text{int } Q \to \text{int } Q$ to produce an embedding $g' : X \cap \text{int } Q \to \text{int } Q$ and small pseudoisotopy $h' : \text{int } Q \to \text{int } Q$, $t \in [0, 1]$, such that $h'g' = g_*(X \cap \text{int } Q)$ in $\text{int } Q$. By choosing $h'$ sufficiently small we can assume that $h'$ extends via the identity over $\partial Q$ to a pseudoisotopy $h' : Q \to Q$, $t \in [0, 1]$. The desired embedding $g : X \to Q$ is defined by $g|_{\partial X} = g_*|_{\partial X}$ and $g|_{X \cap \text{int } Q} = g'$, and the final pseudoisotopy of the Theorem is the composition $h = h_{*,h'} : Q \to Q$, $t \in [0, 1]$. The first method of constructing $g_*$ and $h_{*,t}$ is to simply adapt all of the original without-boundary constructions of Sections 3–6 to the with-boundary case. To simplify the discussion, we concentrate only on adapting Sections 3, 4 and 6, noting that the extension of Section 5 follows by exact analogy. Thus, we assume $X$ is compact.

The primary goal is to prove:

**Addendum to the Fundamental Lemma** (for application at $\partial Q; X$ compact). The Fundamental Lemma (Section 2) also holds with $R^q$ replaced by $R^q_+ \equiv \{(x_1, \ldots, x_q) \in R^q | x_q \geq 0\}$, $B^q$ replaced by $B^q_+ = B^q \cap R^q_+$ and $B^{q-2}$ replaced by $B^{q-2}_+$ ($B^2$ remains unchanged), where $\text{int } B^q_+ = \text{int } B^q \cap R^q_+$. It is only assumed that $\dim(X \cap \partial R^q_+) \leq q - 4$, and it is further concluded that $g^{-1}(\partial R^q_+) = f^{-1}(\partial R^q_+)$. This Addendum is proved exactly as the original Fundamental Lemma was in Section 4, using in place of the original Fundamental Construction the Addendum below. It in turn requires the following modification of the definition of nice immersion. A map $f : X \to R^q_+$ is a nice immersion at the boundary if in the original definition of nice immersion we replace $D^{q-2}_+ \subset \partial D^{q-2}_+ \times [0, 1]$ by $D^{q-2}_+ \subset \partial D^{q-2}_+ \times [0, 1] \cup D^{q-3}_+ \times 1(= \text{frontier of } D^{q-2}_+ \text{ in } R^{q-2}_+; \text{ see Fig. 7.})$

We can now state
Addendum to the Fundamental Construction (for application at $\partial Q; X$ compact). Suppose $X$ is compact and $\varepsilon > 0$. Given an embedding $f: X \to \mathbb{R}^q_+$ as in the preceding Addendum to the Fundamental Lemma, there exists a map $g: X \to \mathbb{R}^q_+ - B^2$ as in the conclusion, except that $g$ is a nice $\varepsilon$-immersion at the boundary, whose singularity $q$-cells lie in $(\text{int } B^q \cap \mathbb{R}^q_+) - B^2$.

The original proof of the Fundamental Construction in Section 3 also serves for this Addendum, merely by interpreting the last factor in all $q$-cells to be $[0,1]$ instead of $[-1,1]$, and considering the domains of $p$ and $\pi$ to be $B^{q-1}(= B^q_+ \cap \mathbb{R}^q_1)$ instead of $B^q$ (thus $\pi$ is not extended over $B^q_+ \cap \mathbb{R}^q_1$).

The $q = 4,6$ cases of the preceding Addendum can be proved using the remarks from the discussion of the $q = 3,5$ cases at the end of Section 6. However the $q = 5$ case of the Addendum to the Fundamental Lemma rests on the classical embedding theorem, which can readily be adapted to the with-boundary context.

The pseudoisotopy $h_{q,i}$ is constructed for the $q \neq 5$ cases by mimicking the construction in Section 6. It is interesting to check than an $h_{q,i}$ so constructed may well have $h_{q,i}^{-1}(\partial Q) \cap \text{int } Q \neq \emptyset$. In the $q = 5$ case, the above-mentioned adaptation of the classical embedding theorem cannot obviously be done by pseudoisotopy (unless $\dim X \leq 1$), hence for this single case we rely on the subsequent alternative method of constructing $h_{q,i}$.

The second method of constructing $g_q$ and $h_{q,i}$ is to first apply the without-boundary case in the boundary itself, obtaining a 1-LCC embedding $g_\partial: \partial X \to \partial Q$ and pseudoisotopy $h_{\partial,t}: \partial Q \to \partial Q, t \in [0,1]$, and then to extend this embedding and pseudoisotopy into int $Q$ in appropriate fashion. Unfortunately this extension process requires more than just simple collar extension, so we give some details. The idea is one used by Cantrell in [12] (or even Bing in [3]), and subsequently generalized to similar problems by Bryant–Seebeck, Cantrell–Lacher and others.

The following description of the construction is completely without epsilonics and without certain tapering and support restrictions, but the interested reader will
be able to supply them. The problem is this (compact case): Given a compact subset \(X \subset Q\), \(\dim \partial X \leq q - 4\), and given a 1-LCC embedding \(g_\# : \partial X \to \partial Q\) and a pseudoisotopy \(h_{\#, t} : \partial Q \to \partial Q, t \in [0, 1]\), such that \(h_{\#, 0}g_\# = \text{id}(\partial X)\), to produce an extension embedding \(g_\# : X \to Q\) of \(g_\#\) and an extension pseudoisotopy \(h_{\#, t} : Q \to Q, t \in [0, 1]\), of \(h_{\#, t}\) such that \(h_{\#, t}g_\# = \text{id}(X)\). The construction goes as follows. Let \(\partial Q \times [0, 2] \subset Q\) be a collar neighborhood for \(\partial Q = \partial Q \times 0\) in \(Q\). Let \(\rho : Q \to Q\) be the extension-via-identity of the map \(\text{id}(\partial Q) \times \rho : \partial Q \times [0, 2] \to \partial Q \times [0, 2]\), where \(\rho : [0, 2] \to [0, 2]\) is defined by \(\rho([0, 1]) = 0, \rho(2) = 2\) and \(\rho'\) is linear elsewhere. Define compact subset \(X_\# = \rho^{-1}(X) = \rho^{-1}(X \cap \text{int} Q) \cup \partial X \times [0, 1] \subset Q\); define embedding \(g_\# : X_\# \to Q\) by \(g_\#|_{\partial X \times [0, 1]} = \text{id}(\partial X \times [0, 1])\) for each \(s \in [0, 1]\); and define pseudoisotopy \(h_{\#, t} : Q \to Q, t \in [0, 1]\), by \(h_{\#, t}|_{Q - \partial Q \times [0, 1]} = \text{id}(Q - \partial Q \times [0, 1])\) and \(h_{\#, t}|_{\partial Q \times s} = h_{\#, t-1}h_{\#, t}^{-1}\text{id}(s)\). Then \(h_{\#, t}g_\# = \text{id}(X_\#)\). It remains to construct \(g_\#\) from \(g_\#\) and \(h_{\#, t}\) from \(h_{\#, t}\). This uses the collapsing trick alluded to above. Suppose \(\rho_1 : Q \to Q\) is a map with the following properties: \(\rho_1|_{Q - \partial Q \times (0, 2)} = \text{id}(Q - \partial Q \times (0, 2)), \rho_1|_{X_\#} = \rho|_{X_\#}\), and \(\rho_1\) is a homeomorphism elsewhere; hence, \(\rho_1\) takes \(Q - \partial X \times [0, 1]\) homeomorphically onto \(Q - \partial Q \times 0\). Such a \(\rho_1\) can be defined explicitly by pushing only vertically in the collar structure, by amounts determined by Urysohn functions. Let \(\rho_1 : Q \to Q, t \in [0, 1]\), be the natural linear pseudoisotopy which realizes \(\rho_1\) from \(\rho_0 = \text{id}(Q)\).

Now suppose we had an analogous map \(\omega_1 : Q \to Q\) and pseudoisotopy \(\omega_1 : Q \to Q, t \in [0, 1]\), which collapsed the subset \(g_\#(\partial X \times [0, 1])\) to \(g_\#(\partial X \times 0)\), keeping \(\partial Q\) fixed and taking \(Q - g_\#(\partial X \times [0, 1])\) homeomorphically onto \(Q - g_\#(\partial X \times 0)\). Then \(g_\#\) and \(h_{\#, t}\) can be defined by: \(g_\#|_{X \cap \text{int} Q} = \omega_1|_{X \cap \text{int} Q}\) and \(g_\#|_{\partial X \times 0} = -g_\#\); \(h_{\#, t} = h_{\#, t-1}\omega_1^{-1}\), which is well-defined when \(t = 1\) because the non-degenerate point inverses of \(\omega_1\) are mapped to points by \(\rho_1\). Note that if the only nondegenerate point inverses of \(h_{\#, 1}\) lie in \(h_{\#, 1}(\partial X)\), then the same is true for \(h_{\#, 1}\).

Concerning the construction of \(\omega_1\), it seems at first glance that one ought to be able to accomplish it by a clever arrangement of available maps, but the following codimension 2 example suggests the impossibility of doing this. Let \(\Sigma^{q - 3}\) be a locally flat, knotted codimension 2 sphere lying in a coordinate chart \(U\) of \(\partial Q\), such that \((U, \Sigma^{q - 3}) = [(R^{q - 1}, S^{q - 3}) - (D^{q - 1}, D^{q - 3})] \cup (D^{q - 1}, D^{q - 3})\), where \((D^{q - 1}, D^{q - 3})\) is homeomorphic to the standard ball pair, with \(D^{q - 1} \cap S^{q - 3} = D^{q - 3}\), and \((D^{q - 1}, D^{q - 3})\) is a locally flat knotted ball pair. Let \(h_{\#, t}\), be the pseudoisotopy of \(\partial Q\) which Alexander pseudoisotopes the knot to a point, at the same time giving an isotopy of \(\Sigma^{q - 3}\) to \(S^{q - 3}\). Then there exists no pseudoisotopy \(\omega_1\) for this particular \(h_{\#, t}\).

Nevertheless, in our codimension 3 situation above \(\omega_1\) can be defined by radial engulfing, in the manner first used by Bryant–Seebeck [10]. That is, \(g_\#(\partial X \times [0, 1])\) can be ambient isotopically shrunk down as close as desired to \(g_\#(\partial X \times 0)\) in a well-controlled manner, using radial engulfing, showing that the Bing Shrinking Criterion with Isotopy (elegantly presented in [25]) is satisfied. Then this criterion can be applied to construct the shrinking pseudoisotopy \(\omega_1\). There is no dimension restriction on \(q\) here; since \(\text{dem} g_\#(\partial X \times [0, 1]) \leq q - 3\), the engulfing works even when \(q = 4\).
Before discussing some of the details of the \( \sigma \)-compact case, we point out that in the above construction of \( g_{\sigma} \) and \( h_{\sigma,1} \), from \( g_{\partial} \) and \( h_{\partial,1} \), if \( X \) is \( \sigma \)-compact, then one must expect \( h_{\sigma,1} \) to have nondegenerate point inverses in \( h_{\sigma,1}^{-1}(cl \ X \cap \partial Q) \) rather than just in \( h_{\sigma,1}^{-1}(cl \ \partial X) \), even if \( \partial X \) is compact. To see this consider the case where \( X \cap int \ Q \) is dense in int \( Q \); necessarily here \( g_{\sigma} \) and \( h_{\sigma,1} \) must be such that \( cl \ g_{\sigma}(X \cap int \ Q) \cap h_{\sigma,1}^{-1}(\partial X) \subset g_{\sigma}(\partial X) \), so \( h_{\sigma,1} \) must have nondegenerate point inverses outside of \( h_{\sigma,1}^{-1}(\partial X) \).

In the \( \sigma \)-compact case, the above construction works after suitable modifications. Given \( X \), first arrange that \( cl \ \partial X = cl \ X \cap \partial Q \) (instead of just \( < \)) by pushing \( X \) away from \( \partial Q - cl \ \partial X \) keeping \( cl \ \partial X \) fixed, by the inverse of a pseudoisotopy. As usual we henceforth disregard this pseudoisotopy by tacitly absorbing it into the subsequent pseudoisotopy. Thus we assume \( cl \ \partial X = cl \ X \cap \partial Q \). Now given \( g_{\sigma} \) and \( h_{\partial,1} \) as above, with the only nondegenerate point inverses of \( h_{\partial,1} \) lying in \( h_{\partial,1}^{-1}(cl \ \partial X) \), we will construct \( h_{\sigma,1} \) to have its only nondegenerate point inverses in \( h_{\sigma,1}^{-1}(cl \ \partial X) \).

As before we seek shrinking homotopies \( \rho_t \) and \( \omega_t \) now with time parametrization \([0, \infty) \) instead of \([0, 1] \). Let \( \rho_{\infty}: Q \to Q \) be defined as \( \rho_1 \) was above, with \( \partial X \) replaced by \( \partial X \). The new \( \omega_{\infty} \) requires a special definition, because for \( X \) \( \sigma \)-compact there may not exist any map like the previous \( \omega_t \), shrinking \( g_{\sigma}(\partial X \times [0, 1]) \) to \( g_{\sigma}(\partial X \times 0) \). (The reader may wish to verify this point for himself with an example, because the following paragraphs are designed expressly to circumvent this problem.) We first define a cellular homotopy \( \omega_t \) on \([0, \infty) \) a segment \([k - 1, k]\) at a time. Write \( X = \bigcup_{k=1}^{\infty} X_k \) as a union of increasing compacta. To start, let \( \omega_{t,1} \), \( t \in [0, 1] \), be a pseudoisotopy which shrinks \( g_{\sigma}(\partial X_1 \times [0, 1]) \) to \( g_{\sigma}(\partial X_1 \times 0) \), just as in the compact case. Extend \( \omega_t \) over the interval \([1, 2]\) so that during that time it squeezes \( g_{\sigma}(((\partial X_2 - \partial X_1) \times [0, 1])) \) to \( g_{\sigma}(((\partial X_2 - \partial X_1) \times 0)) \), always keeping \( \partial Q \) fixed. Continue this to \( \infty \), so that for \( t \in [k - 1, k] \), the only nondegenerate point inverses of \( \omega_t \) are precisely \( \{\omega^{-1}_t(g_{\sigma}(x \times 0)) = g_{\sigma}(x \times [0, 1]) \mid x \in \partial X_{k-1}\} \). Now \( \omega_{\infty} = \lim_{t \to \infty} \omega_t \) is not in general defined, because it can go amuck on \((\partial X - \partial \partial X) \times 1\). But it is not necessary that \( \omega_{\infty} \) be well-defined on \( Q \), as we will show. We wish to use \( \omega_{\infty} \) to define the desired maps \( g_{\sigma} \) and \( h_{\sigma,1} \) by letting \( g_{\sigma} = \omega_{\infty} g_{\sigma} \rho_{\infty}^{-1} \) and \( h_{\sigma,1} = \rho_{\infty} h_{\sigma,1} \omega_{\infty}^{-1} \). For these definitions to make sense, it is only necessary that the following conditions hold:

1. letting \( Y = (Q - (\rho_{\infty} h_{\sigma,1})^{-1}(cl \ \partial X)) \cup \partial X \times 1 \), then \( \omega_{\infty}|Y = \lim_{t \to \infty} \omega_t|Y \) is a well-defined embedding, and
2. letting \( \mathcal{G} \) be the uppersemicontinuous decomposition of \( Q \) defined by the point inverses of \( \rho_{\infty} h_{\sigma,1} \), that is, \( \mathcal{G} = \{(\rho_{\infty} h_{\sigma,1})^{-1}(w) \mid w \in Q\} \) (note the only nondegenerate elements of \( \mathcal{G} \) are in \((\rho_{\infty} h_{\sigma,1})^{-1}(cl \ \partial X \times 0)\) ), then \( \omega_{\infty}^{-1} = \lim_{t \to \infty} \omega_t^{-1} \) induces a well-defined map from \( Q \) to \( Q/\mathcal{G} \).

These two conditions are arranged by choosing each partial homotopy \( \omega_t \), \( t \in [k - 1, k] \), to move only points arbitrarily near \( \omega_{k-1} g_{\sigma}((\partial X_k - \partial X_{k-1}) \times [0, 1]) \), and to move each point \( w \in Q \) only in a small neighborhood of an arc \( \omega_{k-1} g_{\sigma}(x(w) \times [0, 1]) \). That this suffices to ensure condition (1) is clear, and for (2) one can argue as follows: for each \( G \in \mathcal{G} \), define \( \omega_{\infty}(G) = \bigcap_{s > 0} cl \ \bigcup \{\omega_t(G) \mid t \geq s\} \). By choosing the \( \omega_t \)'s carefully as described above, it can be arranged that the sets \( \{\omega_{\infty}(G) \mid G \in \mathcal{G}\} \)
are disjoint compacta which comprise an uppersemicontinuous decomposition of $Q$, denoted $\omega_\infty(\mathcal{G})$. Now $\omega_\infty^{-1}$ defines a map from $Q$ to $Q/\mathcal{G}$, by sending each $\omega_\infty(G)$ to the point $G$ in $Q/\mathcal{G}$. For given any neighborhood $U$ of any $G \in \mathcal{G}$, there exists an $s$ such that for all $t \geq s$, $\omega_t^{-1}(\omega_\infty(G)) \subset U$.

So now $g_\varphi$ and $h_{\varphi,1}$ can be defined as above. In the usual fashion $h_{\varphi,t}$ can be defined by slowing down the shrinking action of $\omega_t$ (by slowing down the time parametrizations of each of its component pseudoisotopies on each $[k-1, k]$) so that $\omega_t$ has no nondegenerate point inverses until $t = \infty$.

8. Comments

For arbitrary codimension $\geq 3$ subsets of $R^q$ the proof of the Approximation Theorem given in this paper breaks down badly. For consider the following example, which makes use of Bing's example of a hereditarily indecomposable continuum $S$ in $R^q$ which separates two given points $x$ and $y$ [2, Theorem 3]. Let $G$ be a countable union of such $S$'s so that $X = R^q - G$ is 0-dimensional. Then $R^q - X$ contains no path, so that one cannot even get started in the proof of the Approximation Theorem. Of course, there exist 1-LCC approximations to this $X$ because it is 0-dimensional. But in the general situation little is known; for example: Is it true that given an arbitrary $X \subset R^q$, with $\dim X \leq q - 2$, there exists a nearby embedding $g : X \to R^q$ such that $R^q - g(X)$ is 0-LC at $g(X)$?

Appendix

These are a supplementary sequence of figures to illustrate the idea of the Fundamental Construction - Fundamental Lemma (Sections 3, 4). Here the embedding $f$ is $\text{id}(X) : X \hookrightarrow R^q$.

See Fig. A.1. The compactum $X$ associated with Fig. 4 might look like this (dimensions exaggerated). Here $X \approx I \times (1 + *)$ where $I = \text{interval}$ and $+ = \text{disjoint union}$. The intersection of $X$ with $R^3$ is a parabolic shaped interval union a point. The goal is to reembed $X$ to miss $B^2 \subset R^3$.

![Fig. A.1.](image-url)
See Fig. A.2. This depicts \( g(X) = g_1(X) \subset R^4 - B^2 \), which is the image of \( X \) under the first (nice) immersion. This immersion \( g_1 \) is obtained by isotoping the handle \( H_j \subset R^3 \) underneath \( B^2 \) in \( R^3 \) (\( H_j \) is not shown above; see Fig. 4). In the Fig. A.2, \( g_1(X) \) has two singular points, corresponding to four points in \( X \).

See Fig. A.3. Let \( X_1 = I \times * \) be the component of \( X \) not moved by \( g_1 \), and let \( D_j^2 \) be the spanning 2-disc in \( R^3 \) for the handle \( H_j \subset R^3 \).

If it were the case that \( X_1 \cap D_j^2 = \emptyset \), then the singularities of \( g_1(X) \) could be removed by squeezing \( g_1(X - X_1) \) toward \( D_j^2 \), so that this image passed through the gap indicated in Fig. A.4. The key step in Štan’ko’s argument is observing that the problem of making \( X_1 \cap D_j^2 = \emptyset \) is like the original problem of making \( X \cap B^2 = \emptyset \). So mimicking the construction of \( g_1 \), there exists an immersion \( g'_1 : X_1 \to R^4 - D_j^2 \),
arbitrarily close to $g_1|X_1 = \text{id}(X_1)$, which creates a gap. Then the singularities of $g_1$ can be removed by squeezing toward the gap as indicated above, to produce an immersion $g_2: X \to R^q - B^2$ with singularities much smaller than those of $g_1$. Carried to the limit, this process produces the desired reembedding of $X$ into $R^q - B^2$.

References


