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A Combinatorial Proof of the Multivariable Lagrange Inversion Formula

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Part I contains a combinatorial proof of a multivariable Lagrange inversion formula. Part II discusses the various multivariable Lagrange inversion formulas of Jacobi, Stieltjes, Good, Joni, and Abhyankar and shows how they can be derived from each other. © 1987 Academic Press, Inc.

1. *Introduction*

The Lagrange inversion formula is one of the most useful formulas in enumerative combinatorics. Many proofs of it are known, including combinatorial proofs of Raney [45] and Labelle [37, 38]. Multivariable generalizations of the Lagrange inversion formula have many applications [1, 4, 5, 11, 21–26, 31, 32]. In Part I we give a combinatorial proof of the following form of multivariable Lagrange inversion:

Let the formal power series $f_1,..., f_m$ in the variables $x_1,..., x_m$ be defined by

$$f_i = x_i g_i(f_1,..., f_m); \qquad i = 1,..., m,$$

for some formal power series $g_i(x_1,..., x_m)$. Then the coefficient of

$$\frac{x_1^{n_1}}{n_1!}\cdots\frac{x_m^{n_m}}{n_m!} \quad \text{in} \quad \frac{x_1^{k_1}}{k_1!}\cdots\frac{x_m^{k_m}}{k_m!}\, g_1^{n_1}(x_1,..., x_m)\cdots g_m^{n_m}(x_1,..., x_m)$$

is equal to the coefficient of

$$\frac{x_1^{n_1}}{n_1!}\cdots\frac{x_m^{n_m}}{n_m!} \quad \text{in} \quad \frac{f_1^{k_1}}{k_1!}\cdots\frac{f_m^{k_m}}{k_m!}\Big/\det(\delta_{ij} - x_i g_i^{(j)}(f_1,..., f_m)),$$

where $g_i^{(j)}(x_1,..., x_m) = (\partial/\partial x_j)\, g_i(x_1,..., x_m)$.

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The combinatorics underlying the proof is quite simple.Let A be a subset of a finite set B. To any function from A to B we associate a digraph on B with an arc from each element of A to its image. The connected components of these digraphs are of two types.The first type is a rooted tree with all arcs directed towards the root. The root is in $B-A$ and every other vertex is in A. The second type consists only of vertices in A, and contains a single directed cycle, with all arcs not in the cycle directed towards the cycle.

The defining relations $f_i = x_i g_i(f_1,..., f_m)$ have the interpretation that f_i counts trees with vertices colored in m colors, where the root has color i. The vertices are weighted by "color-refined" degree, and the g_i are the generating functions for these weights.

The coefficient of

$$\frac{x_1^{n_1}}{n_1!} \cdots \frac{x_m^{n_m}}{n_m!} \quad \text{in} \quad \frac{x_1^{n_1}}{k_1!} \cdots \frac{x_m^{n_m}}{k_m!} g_1^{n_1} \cdots g_m^{n_m}$$

counts functions from A to B, where B contains n_i vertices of color i and $B-A$ contains k_i vertices of color i. The digraph associated to such a function will have k_i roots of color i. It is easy to show that $(f_1^{k_1}/k_1!) \cdots (f_m^{k_m}/k_m!)$ counts sets of trees with k_i roots of color i. A little more work shows that the reciprocal of the determinant counts sets of components of the second type.

Labelle [37, 38] has given a proof of a one-variable Lagrange inversion formula similar to the one given here. Labelle proved a different version of the formula which, while more convenient in the one-variable case, apparently does not generalize easily.

The use of exponential generating functions enables us to work with labeled trees. Raney's proof [45] of the one-variable formula used ordinary generating functions and (implicitly) plane (or ordered) trees. Chottin [6, 7] and Cori [8] generalized Raney's proof to a specialized two-variable formula. A combinatorial proof of the general formula using only ordinary generating functions may be possible, but is probably more difficult than the proof given here. Such a proof might lead to a multivariable generalization of the noncommutative Lagrange inversion formulas of [17].

Many different multivariable Lagrange inversion formulas have been found, such as those of Jacobi [33], Stieltjes [48], Good [21], Joni [35], and Abhyankar [1]. In Part II we show how these formulas and the formula proved in Part I can be transformed into each other by simple algebraic manipulation. We also give a new version of multivariable Lagrange inversion (Theorem 4) which is more convenient in some applications than previous formulas.

PART I

2. *Colored Labeled Structures*

In this section we develop some of the properties of exponential generating functions in several variables.

An *exponential generating function in m variables* is a formal power series of the form

$$\sum_{n_1,\dots,n_m} a_{n_1,n_2,\dots,n_m} \frac{x_1^{n_1}}{n_1!}\cdots\frac{x_m^{n_m}}{n_m!}. \tag{2.1}$$

It will be convenient to denote m-vectors by boldface letters; thus $\mathbf{n}$ denotes the vector $(n_1, n_2,\dots, n_m)$. We add, subtract, multiply, and divide vectors term by term; however, we interpret $\mathbf{x}^\mathbf{n}$ as $x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$ and $n!$ as $n_1!\,n_2!\cdots n_m!$ Then (2.1) may be written as $\sum_\mathbf{n} a_\mathbf{n}(\mathbf{x}^\mathbf{n}/\mathbf{n}!)$. Exponential generating functions in m variables arise in counting labeled objects in which the labels are colored in m colors. In this section we review some of the basic enumerative properties of these generating functions. For other approaches to exponential generating functions, see Beissinger [2], Bender and Goldman [3], Doubilet, Rota, and Stanley [10], Foata [12], Getu and Shapiro [19], Goulden and Jackson [26], Joni [36], Joyal [37], Labelle [40], Reilly [46], and Stanley [47].

Let $\mathbb{P}$ and $\mathbb{N}$ be the positive and nonnegative integers, and let $[n]=\{1,2,\dots,n\}$, with $[0]=\varnothing$. We will need m disjoint copies of $\mathbb{P}$, which we may construct by setting $\mathbb{P}_i=\mathbb{P}\times\{i\}$ for i in $[m]$. We think of $\mathbb{P}_i$ as $\mathbb{P}$ "painted" in color i. Let $\mathbb{P}_*=\mathbb{P}\times[m]=\mathbb{P}_1\cup\cdots\cup\mathbb{P}_m$. Let $[n]_i\subseteq\mathbb{P}_i$ be $[n]\times\{i\}$. For $\mathbf{n}=(n_1,\dots,n_m)$ in $\mathbb{N}^m$, let $[\mathbf{n}]=[n_1]_1\cup[n_2]_2\cup\cdots\cup[n_m]_m$. For any subset A of $\mathbb{P}_*$, let $\|A\|\in\mathbb{N}^m$ be $(|A\cap\mathbb{P}_1|,\dots,|A\cap\mathbb{P}_m|)$. Thus, for example, $\|[\mathbf{n}]\|=\mathbf{n}$. Let $\mathbf{e}_i=(0,\dots,1,\dots,0)$, with a 1 in the ith place and 0's elsewhere.

An *m-colored labeled structure L* with *weight function w* assigns to each $A\subseteq\mathbb{P}_*$ a set $L(A)$ such that $\sum_{\alpha\in L(A)} w(\alpha)$ depends only on $\|A\|$. We require in addition that if $A\neq B$, $L(A)$ and $L(B)$ must be disjoint. We define the *generating function* $\Gamma(L)$ for L by

$$\Gamma(L)=\sum_{\mathbf{n}\in\mathbb{N}^m}\frac{\mathbf{x}^\mathbf{n}}{\mathbf{n}!}\sum_{\alpha\in L([\mathbf{n}])} w(\alpha),$$

assuming that the sum exists as a formal power series. Two structures L and M are *isomorphic* (written $L\cong M$), if for every $A\subseteq\mathbb{P}_*$ there is a weight-preserving bijection from $L(A)$ to $M(A)$. It is clear that if $L\cong M$ then $\Gamma(L)=\Gamma(M)$. (In most applications the weaker equivalence $\Gamma(L)=\Gamma(M)$ would be sufficient. On the other hand, one might require the stronger condition that the bijection $L(A) \to M(A)$ be "natural." See Joyal [37] and Labelle [40].)

If L and M are structures such that all sets $L(A)$ and $M(B)$ are disjoint, then we define the sum $L+M$ by $(L+M)\ (A) = L(A) \cup M(A)$ with the obvious weight. It is clear that $\Gamma(L+M) = \Gamma(L) + \Gamma(M)$. We define infinite sums of structures the same way.

We define the product LM for any structures L and M by

$$(LM)(A) = \bigcup_{\substack{B \cup C = A \\ B \cap C = \varnothing}} L(B) \times M(C),$$

where the weight of (α, β) is $w(\alpha)\, w(\beta)$. (By abuse of notation we denote all weights by w, even though the same object may have different weights in different structures.) A straightforward argument shows that $\Gamma(LM) = \Gamma(L)\, \Gamma(M)$.

If $L(\varnothing) = \varnothing$ we define a structure $L^{(k)}$ for k in $\mathbb{N}$: $L^{(k)}(A)$ is the set of all $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_k\}$ such that for some partition $\{A_1, ..., A_k\}$ of A, $\alpha_i \in L(A_i)$. Here $w(\alpha) = w(\alpha_1) \cdots w(\alpha_k)$. Since each element of $L^{(k)}(A)$ corresponds to $k!$ elements of $L^k(A)$, we have $\Gamma(L^{(k)}) = \Gamma(L)^k/k!$, and consequently, $\Gamma(\sum_{k=0}^{\infty} L^{(k)}) = e^{\Gamma(L)}$.

3. *The Proof*

THEOREM 1. *For each i in $[m]$ and $\mathbf{j}$ in $\mathbb{N}^m$, let $g_{i,\mathbf{j}}$ be an indeterminate. Set $g_i(\mathbf{x}) = \sum_{\mathbf{j}} g_{i,\mathbf{j}}(\mathbf{x}^{\mathbf{j}}/\mathbf{j}!)$. Then there is a unique formal power series solution $f_1(\mathbf{x}), ..., f_m(\mathbf{x})$ to the system*

$$f_i = x_i g_i(\mathbf{f}); \qquad i = 1, ..., m. \tag{3.1}$$

Let $g_i^{(j)}(\mathbf{x}) = \partial g_i(\mathbf{x})/\partial x_j$. Then the coefficient of $\mathbf{x}^{\mathbf{n}}/\mathbf{n}!$ in $(\mathbf{x}^{\mathbf{k}}/\mathbf{k}!)\, \mathbf{g}^{\mathbf{n}}(\mathbf{x})$ *is equal to the coefficient of* $\mathbf{x}^{\mathbf{n}}/\mathbf{n}!$ in $(\mathbf{f}^{\mathbf{k}}/\mathbf{f}!)/\det(\delta_{ij} - x_i g_i^{(j)}(\mathbf{f}))$.

Proof. Equating coefficients of $\mathbf{x}^{\mathbf{n}}$ in (3.1) gives a system of recurrences for the coefficients of the f_i which is easily seen to have a unique solution. Therefore it is sufficient to find a solution $\mathbf{f}$ which satisfies the conditions of the theorem. We shall do this by counting digraphs in two ways in which every vertex has out-degree 0 or 1. For simplicity, we use the word *digraph* for a digraph of this type.

Let D be a digraph with vertex set in $\mathbb{P}_*$. Let v be a vertex of D of color i, and suppose that of the vertices u for which uv is an arc of D, j_l are of color l for each l in $[m]$. We say that v is of *type* $(i, \mathbf{j})$. We define the weight of v to be $g_{i,\mathbf{j}}$, and we define the weight of D to be the product of the weights of its vertices. A *root* of a digraph is a vertex of out-degree zero. We say that a digraph is of type $\mathbf{k}$ if it has k_i roots of color i for each i.

We now prove:

(i) The sum of the weights of the digraphs on $[\mathbf{n}]$ of type $\mathbf{k}$ is the coefficient of $\mathbf{x}^{\mathbf{n}}/\mathbf{n}!$ in $(\mathbf{x}^{\mathbf{k}}/\mathbf{k}!)\,\mathbf{g}^{\mathbf{n}}(\mathbf{x})$.

It is easy to see that $(\mathbf{x}^{\mathbf{k}}/\mathbf{k}!)\,\mathbf{g}^{\mathbf{n}}(\mathbf{x})$ is the generating function for the structure $L_{\mathbf{k},\mathbf{n}}$ defined as follows: $L_{\mathbf{k},\mathbf{n}}(A)$ is the set of functions $\psi\colon A \to \{0\} \cup [\mathbf{n}]$ such that $\|\psi^{-1}(0)\| = \mathbf{k}$, in which the weight of ψ is $\prod_{v \in [\mathbf{n}]} g_{c(v), \|\psi^{-1}(v)\|}$, where $c(v)$ is the color of v. If ψ is in $L_{\mathbf{k},\mathbf{n}}(\mathbf{n})$ we associate to it the digraph on $\mathbf{n}$ in which there is an arc from u to v if and only if $\psi(u) = v$. This correspondence is easily seen to be a weight-preserving bijection onto the digraphs on $[\mathbf{n}]$ of type $\mathbf{k}$, and thus (i) is proved.

By the *components* of a digraph, we mean the connected components of its underlying graph. It is not hard to show that the number of roots in a component is either zero or one, and that a component with no roots contains a unique directed cycle. We call a component with one root a *tree* and a component with no roots a *unicyclic digraph.*

Let $F_{i,\mathbf{j}}$ be the structure of trees with root of type $(i, \mathbf{j})$ and let $F_i = \sum_{\mathbf{j}} F_{i,\mathbf{j}}$ be the structure of trees with root of color i. Let $X_{i,\mathbf{j}}$ be the structure such that if $\|A\| = \mathbf{e}_i$ then $X_{i,\mathbf{j}}(A) = A$, with weight $g_{i,\mathbf{j}}$; otherwise, $X_{i,\mathbf{j}}(A) = \varnothing$. In other words, an $X_{i,\mathbf{j}}$-object is a point of color i weighted as though it were a vertex of type $(i, \mathbf{j})$ in a digraph.

A tree with root of type $(i, \mathbf{j})$ consists of a root of color i together with j_l trees of root color l for each l in $[m]$. Thus $F_{i,\mathbf{j}} \cong X_{i,\mathbf{j}}\, F_1^{(j_1)} F_2^{(j_2)} \cdots F_m^{(j_m)}$, so summing on $\mathbf{j}$ we have

$$F_i \cong \sum_{\mathbf{j}} X_{i,\mathbf{j}} F_1^{(j_1)} \cdots F_m^{(j_m)}. \tag{3.2}$$

Now set $f_i = \Gamma(F_i)$. Applying Γ to (3.2), we have

(ii) The generating functions f_i for trees with root of color i satisfy

$$f_i = x_i g_i(f_1, \ldots, f_m); \qquad i = 1, \ldots, m.$$

A digraph of type $\mathbf{k}$ has k_i components which are trees of root color i for each i, and the other components are unicyclic digraphs. Thus if $u(\mathbf{x})$ is the generating function for unicyclic digraphs, the generating function for digraphs of type $\mathbf{k}$ is $(f^{\mathbf{k}}/\mathbf{k}!)\, e^u$. We shall show

(iii) $u(\mathbf{x}) = \operatorname{trace} \log J^{-1}$, where J is the $m \times m$ matrix $(\delta_{ij} - x_i g_i^{(j)}(\mathbf{f}))$.

A unicyclic digraph may be viewed as a set of trees, together with additional arcs which join the roots of the trees into a cycle. Since the additional arcs change the weights of the roots, we must work with "modified trees": We define the structure T_{ij} of modified trees of specification (i, j) so that $T_{ij}(A)$ is the set of all trees on A of root color i, but with a modified weight. The weights of the nonroot vertices are as before, but a root of type (i, k) is assigned the weight $g_{i,k+\mathbf{e}_j}$. Thus the weight of the root is what it should be when we add an entering arc from a vertex of color j. We have

$$\Gamma(T_{ij}) = x_i \sum_{\mathbf{k}} g_{i,\mathbf{k}+\mathbf{e}_j} \frac{f^{\mathbf{k}}}{\mathbf{k}!}$$
$$= x_i g_i^{(j)}(\mathbf{f}). \tag{3.3}$$

Let us say that a sequence of s modified trees is *compatible* if their specifications are of the form (i_1, i_2), (i_2, i_3),..., (i_{s-1}, i_s), (i_s, i_1). A unicyclic digraph is obtained with the correct weight by connecting cyclically the roots of a compatible sequence of modified trees. The generating function for compatible sequences of s modified trees is easily seen to be trace M^s, where M is the $m \times m$ matrix $(x_i g_i^{(j)}(\mathbf{f}))$. But each cyclic ordering of s modified trees corresponds to s sequences (obtained by choosing one of the modified trees as the first), so the generating function for unicyclic digraphs is

$$u(\mathbf{x}) = \sum_{s=1}^{\infty} \frac{1}{s} \operatorname{trace} M^s = \operatorname{trace} \log (\delta_{ij} - x_i g_i^{(j)}(\mathbf{f}))^{-1},$$

and (iii) is proved.

Thus the generating function for digraphs of type $\mathbf{k}$ is $(\mathbf{f}^{\mathbf{k}}/k!)$ $\exp(\operatorname{trace} \log J^{-1})$. From the matrix identity

$$\det \exp A = \exp \operatorname{trace} A, \tag{3.4}$$

we have $\exp(\operatorname{trace} \log J^{-1}) = \exp(\log \det J^{-1}) = (\det J)^{-1}$. The theorem now follows from (i) and (ii).

We note that a combinatorial proof of (3.4) has been given by Foata [13]. Jackson [28] and Foata and Garsia [14] have applied this identity to other combinatorial problems.

The identity (3.4) has been attributed to Jacobi. See, for example, Želobenko [51, p. 27]. Although Želobenko gave no reference to Jacobi, he probably had in mind Jacobi's 1844 paper on differential equations [34]. In Section 17 of this paper, Jacobi expressed the determinant of a matrix M associated with a system of linear homogeneous differential equations as the exponential of the trace of a matrix N. In the case in which the differential equations have constant coefficients, it is true that $M = \exp N$, but Jacobi did not use the concept of exponentiation of matrices in this paper. The historical notes in Wedderburn [50, p. 171] suggest that exponentiation of matrices was not considered until after Jacobi's death in 1851.

An accessible account of Jacobi's determinant for differential equations can be found in Poole [44, pp. 7–8]. (I am grateful to John Lew for this reference.) For an account of the early history of the theory of matrices (as opposed to determinants), see Hawkins [27].

The earliest occurrence of (3.4) I have found is in an 1887 paper of Peano [43], on the same subject as Jacobi [34] (although Peano was apparently unaware of Jacobi's work). Since there was little work on exponentiation of matrices before this time (see Wedderburn [50, p. 171] and MacDuffee [4, pp. 97–99]) it seems quite likely that (3.4) was first discovered by Peano.

PART II

We now show that the theorem proved in Section 3 is equivalent to other forms of multivariable Lagrange inversion. It is convenient to change our notation. We shall use ordinary, rather than exponential generating functions, and we shall work with formal Laurent series in the variables $x_1, x_2, \ldots, x_m$ in which only finitely many terms with negative exponents may appear. We write $[\mathbf{x}^n]f(\mathbf{x})$ for the coefficient of $\mathbf{x}^\mathbf{n}$ in $f(\mathbf{x})$. We shall also use "dummy variables" $t_1, t_2, \ldots, t_m$. We write $|M|$ for the determinant of the matrix M.

Henrici [28] gives a comprehensive historical account of multivariable Lagrange inversion formulas.

4. *The Formulas of Good*

THEOREM 2. *Let $g_i(x_1, \ldots, x_m)$, $i = 1, \ldots, m$, be formal power series with coefficients in a field, and with nonzero constant terms. Then there exist unique formal power series $f_i(x)$ satisfying*

$$f_i(\mathbf{x}) = x_i g_i(\mathbf{f}); \qquad i = 1, 2, \ldots, m. \tag{4.1}$$

Let $g_i^{(j)}(\mathbf{t}) = (\partial/\partial t_j)\, g_i(\mathbf{t})$ and let the $m \times m$ matrices $J(\mathbf{t})$ and $K(\mathbf{t})$ be defined by

$$J(\mathbf{t}) = (\delta_{ij} - x_i g_i^{(j)}(\mathbf{t}))$$

and

$$K(\mathbf{t}) = (\delta_{ij} - (t_i/g_i(\mathbf{t}))\, g_i^{(j)}(\mathbf{t})).$$

Then for any formal Laurent series $\Phi(\mathbf{t})$ (not involving $\mathbf{x}$) we have for all $\mathbf{n}$ in $\mathbb{Z}^m$

$$[\mathbf{x}^\mathbf{n}]\ \Phi(\mathbf{f})/|J(\mathbf{f})| = [\mathbf{t}^\mathbf{n}]\ \Phi(\mathbf{t})\,\mathbf{g}^\mathbf{n}(\mathbf{t}) \tag{4.2}$$

$$\Phi(\mathbf{f}) = \sum_\mathbf{n} \mathbf{x}^\mathbf{n}[\mathbf{t}^\mathbf{n}]\ |J(\mathbf{t})|\ \Phi(\mathbf{t})\,\mathbf{g}^\mathbf{n}(\mathbf{t}) \tag{4.3}$$

$$[\mathbf{x}^\mathbf{n}]\ \Phi(\mathbf{f})/|K(\mathbf{f})| = [\mathbf{t}^\mathbf{n}]\ \Phi(\mathbf{t})\,\mathbf{g}^\mathbf{n}(\mathbf{t}) \tag{4.4}$$

$$[\mathbf{x}^\mathbf{n}]\ \Phi(\mathbf{f}) = [\mathbf{t}^\mathbf{n}]\ |K(\mathbf{t})|\ \Phi(\mathbf{t})\,\mathbf{g}^\mathbf{n}(\mathbf{t}). \tag{4.5}$$

Proof. Theorem 1 is equivalent to the special case of (4.2) in which $\Phi(\mathbf{t}) = \mathbf{t}^\mathbf{k}$ for some $\mathbf{k}$ in $\mathbb{N}^m$. (We can clearly assign any values to the $g_{i,\mathbf{j}}$ in Theorem 1, and once the factorials are removed, the formula is seen to be valid over any commutative ring of coefficients.) Now let c_i be the constant term of $g_i(\mathbf{t})$. It follows from (4.1) that c_i is also the constant term of $f_i(\mathbf{x})/x_i$. Then the case $\Phi(\mathbf{t}) = \mathbf{t}^\mathbf{k}$ of (4.2) may be written (with $\mathbf{n} = \mathbf{r} + \mathbf{k}$) as

$$[\mathbf{x}^\mathbf{r}]\ \frac{(\mathbf{f}/\mathbf{c}\mathbf{x})^\mathbf{k}}{|J(\mathbf{f})|} = [\mathbf{t}^\mathbf{r}]\ \mathbf{g}^\mathbf{r}(\mathbf{t}) \cdot (\mathbf{g}(\mathbf{t})/\mathbf{c})^\mathbf{k}. \tag{4.6}$$

It is clear that for fixed $\mathbf{r} \in \mathbb{N}^m$, both sides of (4.6) are polynomials in $\mathbf{k}$. Thus since (4.6) holds for $\mathbf{k}$ in $\mathbb{N}^m$, it holds for all $\mathbf{k}$. Going back, we find that (4.2) holds for $\Phi(\mathbf{t}) = \mathbf{t}^\mathbf{k}$ for all $\mathbf{n}$ and $\mathbf{k}$ in $\mathbb{Z}^m$. Then by linearity it holds for all $\Phi(\mathbf{t})$.

Since $J(\mathbf{t})$ involves $\mathbf{x}$, we cannot replace $\Phi(\mathbf{t})$ by $\Phi(\mathbf{t})|J(\mathbf{t})|$ in (4.2). However, we may rewrite (4.2) as

$$\Phi(\mathbf{f})/|J(\mathbf{f})| = \sum_\mathbf{n} \mathbf{x}^\mathbf{n}[\mathbf{t}^\mathbf{n}]\ \Phi(\mathbf{t})\,\mathbf{g}^\mathbf{n}(\mathbf{t}). \tag{4.7}$$

It is easy to see that we may substitute $\phi(\mathbf{t})\,|J(\mathbf{t})|$ for $\Phi(\mathbf{t})$ in (4.7), to obtain (4.3).

From (4.1), we have $x_i = f_i/g_i(\mathbf{f})$. Thus $J(\mathbf{f}) = K(\mathbf{f})$, so (4.4) is the same as (4.2). Since $K(\mathbf{t})$ does not involve $\mathbf{x}$, we may replace $\Phi(\mathbf{t})$ by $\Phi(\mathbf{t})|K(\mathbf{t})|$ in (4.4) to obtain (4.5).

These formulas were found by Good [21]. (Good's conditions were slightly different, since he worked with analytic functions rather than formal power series.) Formal power series proofs have been given by Tutte [49] and de Bruijn [5].

Good also gave the following generalization of (4.5): Under the hypotheses of Theorem 2, for any Laurent series $\Psi(\mathbf{t}, \mathbf{x})$, we have

$$[\mathbf{x}^\mathbf{n}]\ \Psi(\mathbf{f}, \mathbf{x}) = [\mathbf{t}^\mathbf{n}]\ |K(\mathbf{t})|\ \Psi(\mathbf{t}, \mathbf{t}/\mathbf{g}(\mathbf{t}))\,\mathbf{g}^\mathbf{n}(\mathbf{t}). \tag{4.7}$$

To prove this, we reduce by linearity to the case $\Psi(\mathbf{t}, \mathbf{x}) = \Phi(\mathbf{t})\,\mathbf{x}^{\mathbf{k}}$; this case is equivalent to (4.5) with $\mathbf{n}-\mathbf{k}$ substituted for $\mathbf{n}$. Analogous generalizations of the other formulas of Theorem 2 follow similarly.

5. *The Formula of Jacobi*

For any Laurent series $h(\mathbf{x})$, let us define the *residue* of $h(\mathbf{x})$ in the variables x_i by

$$\operatorname{Res}_{\mathbf{x}} h(\mathbf{x}) = [x_1^{-1} x_2^{-1} \cdots x_m^{-1}]\, h(\mathbf{x}). \tag{5.1}$$

In 1830 Jacobi [33] proved the following theorem for $m \leqslant 3$:

THEOREM 3. *Let $f_1(\mathbf{x}),\ldots, f_m(\mathbf{x})$ be Laurent series, and let $\mathbf{n}^{(i)} \in \mathbb{Z}^m$ be such that $f_i(\mathbf{x})/\mathbf{x}^{\mathbf{n}^{(i)}}$ is a formal power series with nonzero constant term. Then for any Laurent series $\Phi(\mathbf{t})$*

$$\operatorname{Res}_{\mathbf{x}} \left|\frac{\partial f_i}{\partial x_j}\right| \Phi(\mathbf{f}) = |n_j^{(i)}|\, \operatorname{Res}_{\mathbf{t}} \Phi(\mathbf{t}). \tag{5.2}$$

Before discussing Jacobi's proof, let us see why Theorem 2 is equivalent to the case $\mathbf{n}^{(i)} = \mathbf{e}_i$ of Jacobi's theorem. With the notation of Theorem 2, let us differentiate (4.1) with respect to x_k. We obtain

$$\frac{\partial f_i}{\partial x_k} = \delta_{ik} g_i(\mathbf{f}) + x_i \sum_{j=1}^{m} g_i^{(j)}(\mathbf{f}) \frac{\partial f_j}{\partial x_k},$$

which we may write as

$$\sum_{j=1}^{m} [\delta_{ij} - x_i g_i^{(j)}(\mathbf{f})] \frac{\partial f_j}{\partial x_k} = \delta_{ik} g_i(\mathbf{f}).$$

Writing this as a matrix identity, and taking determinants, we have

$$|J(\mathbf{f})| \cdot \left|\frac{\partial f_i}{\partial x_j}\right| = |\delta_{ij} g_i(\mathbf{f})| = (\mathbf{g}(\mathbf{f}))^{\mathbf{1}} = (\mathbf{f}/\mathbf{x})^{\mathbf{1}},$$

where $\mathbf{1} = (1, 1,\ldots, 1)$. Then (4.2) may be written

$$[\mathbf{x}^{\mathbf{n}}] \left|\frac{\partial f_i}{\partial x_j}\right| (\mathbf{x}/\mathbf{f})^{\mathbf{1}}\, \Phi(\mathbf{f}) = [\mathbf{t}^{\mathbf{n}}]\, \Phi(\mathbf{t})\, \mathbf{g}^{\mathbf{n}}(\mathbf{t}),$$

and if we replace $\mathbf{n}$ by $\mathbf{n}+\mathbf{1}$, and $\Phi(\mathbf{t})$ by $\mathbf{t}^{\mathbf{1}}\Phi(\mathbf{t})$, we may write this as

$$[\mathbf{x}^{\mathbf{n}}] \left|\frac{\partial f_i}{\partial x_j}\right| \Phi(\mathbf{f}) = [\mathbf{t}^{\mathbf{n}}]\, \Phi(\mathbf{t})\, \mathbf{g}^{\mathbf{n}+\mathbf{1}}(\mathbf{t}). \tag{5.3}$$

Now if $f_i(\mathbf{x})$ are any formal power series such that f_i/x_i has nonzero constant term, then there exist formal power series g_i such that $f_i = x_i g_i(\mathbf{f})$. Then Jacobi's formula (5.2), with $n_j^{(i)} = \delta_{ij}$, is just the special case $\mathbf{n} = -\mathbf{1}$ of (5.3). Conversely, assuming Jacobi's theorem, if $f_i = x_i g_i(\mathbf{f})$, then we have

$$[\mathbf{x}^{\mathbf{n}}] \left|\frac{\partial f_i}{\partial x_j}\right| \Phi(\mathbf{f}) = \operatorname{Res}_{\mathbf{x}} \left|\frac{\partial f_i}{\partial x_j}\right| \Phi(\mathbf{f})/\mathbf{x}^{\mathbf{n}+\mathbf{1}}$$

$$= \operatorname{Res}_{\mathbf{x}} \left|\frac{\partial f_i}{\partial x_j}\right| \Phi(\mathbf{f})[\mathbf{g}(\mathbf{f})/\mathbf{f}]^{\mathbf{n}+\mathbf{1}}$$

$$= \operatorname{Res}_{\mathbf{t}} \Phi(\mathbf{t})[\mathbf{g}(\mathbf{t})/\mathbf{t}]^{\mathbf{n}+\mathbf{1}} = [\mathbf{t}^{\mathbf{n}}]\, \Phi(\mathbf{t})\, \mathbf{g}^{\mathbf{n}+\mathbf{1}}(\mathbf{t}),$$

and this is (5.3).

It may be noted that Good proved the formulas of Section 4 by first proving a version of Jacobi's formula.

We now give a sketch of Jacobi's proof. Similar proofs were found by Garsia [15] and Goldstein [20]. See also Goulden and Jackson [26, pp. 19–22] and Henrici [28]. The first step is to show that for any Laurent series $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$,

$$\operatorname{Res}_{\mathbf{x}} \left|\frac{\partial f_i}{\partial x_j}\right| = 0. \tag{5.4}$$

This will follow from the formula

$$\left|\frac{\partial f_i}{\partial x_j}\right| = \sum_{i=1}^{m} \frac{\partial (f_1 A_i)}{\partial x_i}, \tag{5.5}$$

where A_i is the cofactor of $\partial f_1/\partial x_i$ in $|\partial f_i/\partial x_j|$, since for any $h(\mathbf{x})$, $\partial h/\partial x_i$ has no terms in x_i^{-1}.

In his 1830 paper, Jacobi proved (5.5) for $m = 2$ and $m = 3$ by explicit computation. He proved it for all m in 1844 [34, Sect. 2] in his work on differential equations mentioned earlier. Here he derived it from the formula

$$\sum_{i=1}^{m} \frac{\partial A_i}{\partial x_i} = 0, \tag{5.6}$$

from which it follows easily. Jacobi proved (5.6) by showing that when the left side is expanded, its terms cancel in pairs.

Garsia proved (5.4) more easily by using multilinearity to reduce to the case in which the f_i are monomials.

Next, Jacobi observed that if no component of $\mathbf{k}$ is -1, then

$$\mathbf{f}^{\mathbf{k}} \left| \frac{\partial f_i}{\partial x_j} \right| = (\mathbf{k}+\mathbf{1})^{-1} \left| \frac{\partial f_i^{k_i+1}}{\partial x_j} \right| \tag{5.7}$$

and hence has no residue. For $\mathbf{k} = -\mathbf{1}$, if $f_i(\mathbf{x})/\mathbf{x}^{\mathbf{n}^{(i)}} = c_i h_i(\mathbf{x})$ is a formal power series with constant term $c_i \neq 0$, then

$$\mathbf{f}^{-1} \left| \frac{\partial f_i}{\partial x_j} \right| = \left| f_i^{-1} \frac{\partial f_i}{\partial x_j} \right| = \left| \frac{n_j^{(i)}}{x_j} + \frac{\partial}{\partial x_j} \log h_i \right| \tag{5.8}$$

and the residue is $|n_j^{(i)}|$. Jacobi dealt with the remaining cases, in which $\mathbf{k} \neq -\mathbf{1}$, but some components of $\mathbf{k}$ are -1, by special formulas for $m=2$ and $m=3$, and as far as I know, never gave a complete proof for arbitrary m. However, this case is not hard to deal with. Following Garsia, we may combine (5.7) and (5.8) to get a determinant which may be reduced by multilinearity to a sum of determinants of the form of (5.7) and (5.8). Alternatively, by (5.4), for any $\mathbf{k}^{(1)},\dots,\mathbf{k}^{(n)}$ in $\mathbb{Z}^m$,

$$\operatorname{Res}_{\mathbf{x}} \left| \frac{\partial \mathbf{f}^{\mathbf{k}^{(i)}}}{\partial x_j} \right| = 0.$$

By the chain rule for Jacobians, this yields

$$\operatorname{Res}_{\mathbf{x}} \mathbf{f}^{\mathbf{k}^{(1)}+\cdots+\mathbf{k}^{(m)}-\mathbf{1}} \, |k_j^{(i)}| \cdot \left| \frac{\partial f_i}{\partial x_j} \right| = 0.$$

We then need only show that as long as $\mathbf{k} \neq -\mathbf{1}$, we can find $\mathbf{k}^{(1)},\dots,\mathbf{k}^{(m)}$ with $\mathbf{k}^{(1)}+\cdots+\mathbf{k}^{(m)}-\mathbf{1}=\mathbf{k}$ and $|k_j^{(i)}| \neq 0$.

Goldstein's proof is similar, but instead of starting with (5.4), he writes $\mathbf{f}^{\mathbf{k}}\, |\partial f_i/\partial x_j|$ as a constant times a determinant in which every entry is either $\partial f_i^{k_i+1}/\partial x_j$ or $n_j^{(i)}/x_j + (\partial/\partial x_j) \log h_i$. By multilinearity, this reduces to determinants in which the (i,j) entry is either $\partial B_i/\partial x_j$ or $1/x_j$ for some B_i; by multilinearity again, this reduces to the case in which B_i is a monomial, and these determinants are easy to evaluate.

6. *The Formulas of Stieltjes, Abhyankar, and Joni*

We now look at Theorem 2 from another point of view. Let us assume, as before, that the coefficients of the g_i are indeterminates. Then we may assume that our ring of constants contains the formal power series ring in these indeterminates and that the constant terms of the g_i are invertible.

Consider the system of equations

$$F_i = g_i(\mathbf{F}); \qquad i=1,\dots,m. \tag{6.1}$$

It is not hard to see that (6.1) determines the F_i uniquely as formal power series in the coefficients of the g_i. If we set $\mathbf{x}=\mathbf{1}$ in (4.1), we obtain

$$f_i(\mathbf{1})=g_i(\mathbf{f}(\mathbf{1})), \tag{6.2}$$

and it is clear that $g_i(\mathbf{f}(\mathbf{1}))$ exists as a formal power series in the coefficients of the g_i. Thus since the F_i are unique, we must have $F_i=f_i(\mathbf{1})$.

Now (4.2) may be written

$$\Phi(\mathbf{f})/|J(\mathbf{f})| = \sum_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}[\mathbf{t}^{\mathbf{n}}]\,\Phi(\mathbf{t})\,\mathbf{g}^{\mathbf{n}}(\mathbf{t}). \tag{6.3}$$

We may set $\mathbf{x}=\mathbf{1}$ in (6.3) to get

$$\Phi(\mathbf{F})/\,|L(\mathbf{F})| = \sum_{n} [\mathbf{t}^{\mathbf{n}}]\,\Phi(\mathbf{t})\,\mathbf{g}^{\mathbf{n}}(\mathbf{t}), \tag{6.4}$$

where $L(\mathbf{t})=(\delta_{ij}-(\partial/\partial t_j)\,g_i)$ is the matrix obtained by setting $\mathbf{x}=\mathbf{1}$ in $J(\mathbf{t})$. Similarly we may set $\mathbf{x}=\mathbf{1}$ in the other formulas of Theorem 2:

THEOREM 4. *Let $g_i(\mathbf{t})$, $i=1,\ldots,m$ be formal power series whose coefficients are indeterminates. Then there is a unique solution $F_1,\ldots,F_m$ to the system*

$$F_i=g_i(\mathbf{F});\qquad i=1,\ldots,m, \tag{6.5}$$

as power series in the coefficients of the g_i, and for all Laurent series $\Phi(\mathbf{t})$ we have

$$\Phi(\mathbf{F})/|L(\mathbf{F})| = \sum_{\mathbf{n}} [\mathbf{t}^{\mathbf{n}}]\,\Phi(\mathbf{t})\,\mathbf{g}^{\mathbf{n}}(\mathbf{t}), \tag{6.6}$$

$$\Phi(\mathbf{F}) = \sum_{\mathbf{n}} [\mathbf{t}^{\mathbf{n}}]\,|L(\mathbf{t})|\,\Phi(\mathbf{t})\,\mathbf{g}^{\mathbf{n}}(\mathbf{t}), \tag{6.7}$$

$$\Phi(\mathbf{F})/|K(\mathbf{F})| = \sum_{\mathbf{n}} [\mathbf{t}^{\mathbf{n}}]\,\Phi(\mathbf{t})\,\mathbf{g}^{\mathbf{n}}(\mathbf{t}), \tag{6.8}$$

$$\Phi(\mathbf{F}) = \sum_{\mathbf{n}} [\mathbf{t}^{\mathbf{n}}]\,|K(\mathbf{t})|\,\Phi(\mathbf{t})\,\mathbf{g}^{\mathbf{n}}(\mathbf{t}), \tag{6.9}$$

where the $m\times m$ matrices $L(\mathbf{t})$ and $K(\mathbf{t})$ are given by

$$L(\mathbf{t})=\left(\delta_{ij}-\frac{\partial g_i(\mathbf{t})}{\partial t_j}\right)\qquad \textit{and}\qquad K(\mathbf{t})=\left(\delta_{ij}-\frac{t_i}{g_i(\mathbf{t})}\frac{\partial g_i(\mathbf{t})}{\partial t_j}\right).$$

It is clear that if the coefficients of the g_i are not all indeterminates, these formulas will still hold as long as the F_i are uniquely determined and the sums exist as formal Laurent series.

As an example of Theorem 4, consider the system

$$F_1 = (1 + aF_1)(1 + bF_2),$$
$$F_2 = (1 + cF_1)(1 + dF_2),$$

where a, b, c, and d are indeterminates. We find that

$$|L(t_1, t_2)| = \{1 - a(1 + bt_2)\}\{1 - d(1 + ct_1)\} - bc(1 + at_1)(1 + dt_2)$$

and

$$|K(t_1, t_2)| = \frac{1}{(1 + at_1)(1 + dt_2)} - \frac{bct_1 t_2}{(1 + ct_1)(1 + bt_2)}.$$

Since $|K(t_1, t_2)|$ is simpler, we use (6.9). Setting $\Phi(t_1, t_2) = t_1^{r_1} t_2^{r_2}$ in (6.9), we obtain

$$\begin{aligned} F_1^{r_1} F_2^{r_2} &= \sum_{n_1, n_2} [t_1^{n_1 - r_1} t_2^{n_2 - r_2}]\{(1 + at_1)^{n_1 - 1}(1 + bt_2)^{n_1}(1 + ct_1)^{n_2}(1 + dt_2)^{n_2 - 1} \\ &\qquad - bct_1 t_2 (1 + at_1)^{n_1}(1 + bt_2)^{n_1 - 1}(1 + ct_1)^{n_2 - 1}(1 + dt_2)^{n_2}\} \\ &= \sum_{i,j,k,l} a^i b^j c^k d^l \left\{\binom{i+k+r_1-1}{i}\binom{i+k+r_1}{j}\binom{j+l+r_2}{k}\binom{j+l+r_2-1}{l}\right. \\ &\qquad \left. -\binom{i+k+r_1}{i}\binom{i+k+r_1-1}{j-1}\binom{j+l+r_2-1}{k-1}\binom{j+l+r_2}{l}\right\} \\ &= \sum_{i,j,k,l} \frac{r_1 j + r_2 k + r_1 r_2}{(i+k+r_1)(j+l+r_2)} \\ &\qquad \times \binom{i+k+r_1}{i}\binom{i+k+r_1}{j}\binom{j+l+r_2}{k}\binom{j+l+r_2}{l} a^i b^j c^k d^l. \end{aligned} \tag{6.10}$$

(This formula must be modified if a denominator vanishes, i.e., if r_1 or r_2 is a negative integer.) In the special case $a = d = 0$, we have $F_1 = (1 + b)/(1 - bc)$ and $F_2 = (1 + c)/(1 - bc)$, and (4.22) reduces to

$$\frac{(1 + b)^{r_1}(1 + c)^{r_2}}{(1 - bc)^{r_1 + r_2}} = \sum_{j,k} \frac{r_1 j + r_2 k + r_1 r_2}{(k + r_1)(j + r_2)} \binom{k + r_1}{j}\binom{j + r_2}{k} b^j c^k,$$

which is equivalent to Theorem 1 of [11]. (See also [18].)

The special case of (6.5) in which $g_i(\mathbf{t})$ is of the form $x_i + G_i(\mathbf{t})$, where $\mathbf{G}(\mathbf{t})$ does not involve $\mathbf{x}$, is of particular interest since the equations $x_i = F_i - G_i(\mathbf{F})$ assert that the system $\mathbf{F}$ is the compositional inverse (in the variables x_i) of the system $\mathbf{x} - \mathbf{G}(\mathbf{x})$. The inversion formulas of Stieltjes, Abhyankar, Joni, and Labelle are concerned with this case. The following is Joni's formula [35]. (See also Garsia and Joni [16].) Another proof has been given by Hofbauer [29].

THEOREM 5. *Let the formal power series $F_i(\mathbf{x})$ be defined by*

$$F_i(\mathbf{x}) = x_i + G_i(\mathbf{F}(\mathbf{x})); \qquad i = 1, ..., m, \tag{6.11}$$

where the G_i are formal power series with no terms of degrees 0 *or* 1. *Then for any formal power series $\Phi(\mathbf{x})$, we have*

$$\Phi(\mathbf{F})/|M(\mathbf{F})| = \sum_{\mathbf{k}} \frac{\mathbf{D}^{\mathbf{k}}}{\mathbf{k}!} \Phi(\mathbf{x})\, \mathbf{G}^{\mathbf{k}}(\mathbf{x}) \tag{6.12}$$

and

$$\Phi(\mathbf{F}) = \sum_{\mathbf{k}} \frac{\mathbf{D}^{\mathbf{k}}}{\mathbf{k}!} |M(\mathbf{x})|\, \Phi(\mathbf{x})\, \mathbf{G}^{\mathbf{k}}(\mathbf{x}), \tag{6.13}$$

where $M(\mathbf{x})$ is the $m \times m$ matrix $(\delta_{ij} - \partial G_i(\mathbf{x})/\partial x_j)$ and $D_i = \partial/\partial x_i$.

Proof. We first prove (6.12). From (6.6), with $g_i(\mathbf{t}) = x_i + G_i(\mathbf{t})$, we have

$$\Phi(\mathbf{F})/|M(\mathbf{F})| = \sum_{\mathbf{n} \geq 0} [\mathbf{t}^{\mathbf{n}}]\, \Phi(\mathbf{t})(\mathbf{x} + \mathbf{G}(\mathbf{t}))^{\mathbf{n}}. \tag{6.14}$$

Now since $(\mathbf{D}^{\mathbf{k}}/\mathbf{k}!)\, \mathbf{x}^{\mathbf{n}} = \binom{n_1}{k_1} \cdots \binom{n_m}{k_m} x^{\mathbf{n}-\mathbf{k}}$, the right side of (6.12) is

$$\begin{aligned}
&\sum_{\mathbf{k}} \frac{\mathbf{D}^{\mathbf{k}}}{\mathbf{k}!} \sum_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} [\mathbf{t}^{\mathbf{n}}]\, \Phi(\mathbf{t})\, \mathbf{G}^{\mathbf{k}}(\mathbf{t}) \\
&\quad = \sum_{\mathbf{k},\mathbf{n}} [\mathbf{t}^{\mathbf{n}}]\, \Phi(\mathbf{t}) \binom{n_1}{k_1} \cdots \binom{n_m}{k_m} \mathbf{x}^{\mathbf{n}-\mathbf{k}} \mathbf{G}^{\mathbf{k}}(\mathbf{t}) \\
&\quad = \sum_{\mathbf{n}} [\mathbf{t}^{n}]\, \Phi(\mathbf{t})(\mathbf{x} + \mathbf{G}(\mathbf{t}))^{\mathbf{n}},
\end{aligned}$$

and (6.12) follows from (6.14). We can prove (6.13) the same way from (6.7), or derive it from (6.12).

Abhyankar [1, Theorem 2.1] proved the following formula. Suppose that $f_i(\mathbf{x}) = x_i +$ terms of degree $\geqslant 2$, for $i = 1,..., m$. Then for any formal power series $U(\mathbf{x})$,

$$U(\mathbf{x}) = \sum_{\mathbf{k}} \frac{\mathbf{D}^{\mathbf{k}}}{\mathbf{k}!} \left| \frac{\partial f_i}{\partial x_j} \right| U(\mathbf{f})(\mathbf{x} - \mathbf{f})^{\mathbf{k}}. \tag{6.15}$$

To derive (6.15) from (6.13), we set $f_i = x_i - G_i$. Then the right side of (6.15) is

$$\sum_{\mathbf{k}} \frac{D^{\mathbf{k}}}{\mathbf{k}!} |M(\mathbf{x})| \, U(\mathbf{f}(\mathbf{x})) \, \mathbf{G}^{\mathbf{k}}(\mathbf{x}).$$

By (6.13), this is $U(\mathbf{f}(\mathbf{F}))$, where $\mathbf{F}$ satisfies (6.11), i.e., $\mathbf{f}(\mathbf{F}) = \mathbf{x}$, and (6.15) follows. Similarly, (6.13) can be obtained by taking $U = \Phi(F)$ in (6.15).

Henrici [28] proved Abhyankar's formula directly, using residues with more general Laurent series than those considered here.

Stieltjes [48] found a variant of (6.12) in 1885, in what is probably the earliest complete proof of a general multivariable Lagrange inversion formula. (The two-variable case of this formula had been found in 1869 by Darboux [9].) Stieltjes proved the following:

Let $G_i(\mathbf{x})$ be arbitrary formal power series, let a_i be variables, and define formal power series $F_i(\mathbf{x})$, with coefficients which may be formal power series in the a_i, by

$$F_i(\mathbf{x}) = x_i + a_i G_i(\mathbf{F}(\mathbf{x})); \qquad i = 1,..., m. \tag{6.16}$$

Then for any formal power series $\Phi(\mathbf{x})$, we have

$$\Phi(\mathbf{F}) \, \Delta = \sum_{\mathbf{k}} \mathbf{a}^{\mathbf{k}} \frac{\mathbf{D}^{\mathbf{k}}}{\mathbf{k}!} \Phi(\mathbf{x}) \, \mathbf{G}^{\mathbf{k}}(\mathbf{x}), \tag{6.17}$$

where $\Delta = |\partial F_i(\mathbf{x})/\partial x_j|$.

It is easily verified that $\Delta \cdot |N(\mathbf{F})| = 1$, where $N(\mathbf{x})$ is the $m \times m$ matrix $(\delta_{ij} - a_i(\partial G_i(\mathbf{x})/\partial x_j))$, so Stieltjes's formula reduces to (6.12) when $a_i = 1$. On the other hand, the variables a_i can be absorbed into the G_i, so (6.17) can be recovered from (6.12) as long as the G_i have no terms of degree 0 or 1. However, our restriction on terms of degree 0 or 1 in G_i is made only so that the series will converge; the same proof works for Stieltjes's variant.

The variables a_i do play an essential role in Stieltjes's proof; his key lemma is the formula

$$\frac{\partial}{\partial a_i} [\Delta \Phi(\mathbf{F})] = \frac{\partial}{\partial x_i} [\Delta \Phi(\mathbf{F}) \, G_i(\mathbf{F})]. \tag{6.18}$$

A similar approach was used much earlier by Laplace [41] in the one- and two-variable cases.

Labelle [39] has given a different formula for $\Phi(\mathbf{F})$, where $\mathbf{F}$ satisfies (6.11). His formula involves "Lie series" and does not seem to be closely related to the formulas we have discussed here.

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