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Geometric disintegration and star-shaped distributions

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Abstract

Geometric and stochastic representations are derived for the big class of p-generalized elliptically contoured distributions, and (generalizing Cavalieri's and Torricelli's method of indivisibles in a non-Euclidean sense) a geometric disintegration method is established for deriving even more general star-shaped distributions. Applications to constructing non-concentric elliptically contoured and generalized von Mises distributions are presented.

Keywords: p-generalized elliptically contoured distributions; Non-concentric elliptically contoured distributions; Star-generalized von Mises distributions; p-generalized ellipsoids and ellipsoidal coordinates; Stochastic random vector representation; Geometric measure representation; Star-generalized surface measure; Star-generalized uniform distribution; Intersection-proportion function; Ball number function

AMS subject classification: Primary 60E05; 60D05; secondary 28A50; 28A75; 51F99

1 Introduction

The needs of statistical practice and challenging probabilistic questions in the interplay of measure theory and several other mathematical disciplines stimulate the development of statistical distribution theory. In part, well established mathematical strategies are followed to enlarge known families of distributions, and partly new types of distributions are derived by new methods.

Numerous studies on multivariate probability distributions with a view towards their statistical applications are closely connected with notions like decomposition or disintegration of probability laws, invariant measures on groups and related manifolds, and cross sections. Among the basic references in this field, we refer to (Barndorff-Nielsen et al. 1989; Eaton 1983, 1989; Farrell 1976, 1985; Koehn 1970; Muirhead 1982; Nachbin 1976; Wijsman 1967, 1986, 1990). In the spirit of those works, generalizations of elliptically contoured distributions where the contours are described by a positive function that is positive homogeneous or are arbitrary cross sections are discussed in (Balkema and Nolde 2010; Fernandez et al. 1995) and in (Kamiya et al. 2008; Takemura and Kuriki 1996) respectively.

Zonoid trimming for multivariate distributions is considered in (Koshevoy and Mosler 1997; Mosler 2002). In (Balkema et al. 2010; Balkema and Nolde 2010; Joenssen and Vogel

2012; Kinoshita and Resnick 1991; Mosler 2013; Nolde 2014) the authors describe the way in which star-shaped sets and, correspondingly, star-shaped distributions occur in limit set theory, in meta density analysis, power studies for goodness-of-fit tests, depth analysis and analysis of residual dependence between extreme values in the presence of asymptotic independence, respectively. Norm-contoured sets and related estimation problems of identifying structures in high-dimensional data sets are dealt with in (Scholz 2002).

The authors in (Fang et al. 1990; Kallenberg 2005; Schindler 2003) present studies of symmetric laws from different points of view. The class of $l_{n,p}$-symmetric distributions, $n \in \mathbb{N} = \{1, 2, \ldots\}$, $p \geq 1$, was introduced in (Osiewalski and Steel 1993) and studied in (Gupta and Song 1997; Song and Gupta 1997; Schechtman and Zinn 1990; Rachev and Rüschendorf 1991; Szablowski 1998). Applications of these distributions are discussed in (Nardon and Pianca 2009; Pogány and Nadarajah 2010).

Geometric measure representations for $l_{n,p}$-symmetric distributions, $n \in \mathbb{N}$, $p > 0$, and for heteroscedastic Gaussian distributions were derived in (Richter 2009, 2013), respectively.

An extension of the class of $l_{n,p}$-symmetric distributions to the class of skewed $l_{n,p}$-symmetric distributions has been derived in (Arellano-Valle and Richter 2012). A general approach to geometric representations of skewed $l_{n,2}$-symmetric distributions can be found for $n = 2$ in (Günzel et al. 2012) and for arbitrary n in (Richter and Venz 2014). Another definition of power exponential distributions than the one used here was given in (Gómez et al. 1998), where a special case of elliptically contoured distributions is dealt with. Densities of p-generalized elliptically contoured distributions and of more general star-shaped distributions have been considered in (Balkema and Nolde 2010; Fernandez et al. 1995).

In the present paper, geometric and stochastic representations are derived for the big class of p-generalized elliptically contoured distributions. Generalizing Cavalieri's and Torricelli's method of indivisibles in a non-Euclidean sense, a geometric disintegration method is established for deriving even more general star-shaped distributions. Basic properties of these distributions are studied, applications of the new representations to constructing non-concentric elliptically contoured distributions and to generalizing the von Mises distribution are discussed, and the necessary background from non-Euclidean metric geometry is developed.

Many authors use iterated integration in distribution theory by first integrating with respect to (w.r.t.) a radius variable and then w.r.t. certain directional coordinates. In the present paper, we shall use basically the inverse order of integration. This way, we shall make use of the star-sphere intersection-proportion function (ipf) of a given set. The ipf is essentially based upon a suitably defined non-Euclidean surface content on a star sphere. The latter notion needs therefore the most effort in the present work. Areas from probability theory and mathematical statistics where the ipf successfully applies are surveyed in (Richter 2012). Applying this function allows to study the contours of mass concentration of a probability distribution independently from the tail behavior of the distribution, and often leads to a numerical stabilization of the evaluation of probability integrals. Further, the ipf allows a non-Euclidean surface measure interpretation of certain sector measures considered in the literature.

The paper is organized as follows. After quoting some preliminary facts in Section 2, we deal with the notion of a star-generalized surface content in Section 3. This notion will be studied both based upon a local definition and in terms of an integral in Section 3.1. The latter definition makes use of a preliminary system of coordinates which moreover enables a generalization of the method of indivisibles. Then Section 3.2. deals exclusively with the new surface measure on p-generalized ellipsoids. After much technical work, Theorem 5 finally proves that the local approach to the star generalized surface content results in the same quantity as a suitably defined non-Euclidean surface content in terms of an integral defined using a modified standard approach of differential geometry. To this end, some more coordinate systems are introduced and exploited. This includes consideration of star-generalized trigonometric functions and several Jacobians. In Section 4, star-shaped distributions are introduced in several steps and some of their basic properties are studied. The most explicit results are presented for the class of p-generalized elliptically contoured distributions in Section 4.7. For this specific class, all of the more general results of the preceding parts of Section 4, including the main results in Theorems 7 and 8, allow an additional interpretation which in each case is based upon a suitable non-Euclidean geometry. Moreover, the ball number function will be extended in Section 4.5 and characteristic functions are discussed in Section 4.6. In the two-dimensional case, some consequences from the preceding sections concerning the new class of non-concentric elliptically contoured distributions and a star generalization of the von Mises distribution are drawn in Sections 5.1 and 5.2, respectively. The paper ends with some concluding remarks in Section 6, basically indicating some possible future work.

2 Preliminaries

The main considerations of this paper are most easily understood by making use of a relatively easy coordinate transformation. For showing the deeper meaning of several results derived this way, we shall make use, however, of different rather technical systems of coordinates which will be introduced in later sections. Here, we begin with some preliminary notions, including the preliminary coordinate system mentioned in the Introduction.

Throughout this paper, $K \subset \mathbb{R}^n$ denotes a star body, i.e. a nonempty star-shaped set that is compact and is equal to the closure of its interior, having the origin 0_n in its interior. Its topological boundary will be denoted by S. The functional $h_K : \mathbb{R}^n \to [0, \infty)$ defined by $h_K(x) = \inf\{\lambda > 0 : x \in \lambda K\}$, $x \in \mathbb{R}^n$ where $\lambda K = \left\{(\lambda x_1, \ldots, \lambda x_n)^T : (x_1, \ldots, x_n)^T \in K\right\}$ is known as the Minkowski functional of the star body K. We assume that h_K is positive-homogeneous of degree one, i.e. $h_K(\lambda x) = \lambda h_K(x), \lambda > 0$, which is the case if, e.g., h_K is a norm or an antinorm. For the latter notion, we refer to (Moszyńska and Richter 2012), and for the role which homogeneous functionals generally play in stochastics, we refer to (Hoffmann-Jorgensen 1994).

Let us consider $K(r) = rK = \{x \in \mathbb{R}^n : h_K(x) \leq r\}$ and its boundary $S(r) = rS$ as the star ball and star sphere of Minkowski radius or star radius $r > 0$, respectively. A countable collection $\mathfrak{F} = \{C_1, C_2, ...\}$ of pairwise disjoint cones C_j with vertex being the origin 0_n and $\mathbb{R}^n = \bigcup\limits_j C_j$ will be called a fan. By $\mathfrak{B}_n$ we denote the Borel-σ-field in $\mathbb{R}^n$. We put $S_j = S \cap C_j$, $S_j \cap \mathfrak{B}_n = \mathfrak{B}_{S,j}$ and $\mathfrak{B}_S = \sigma\{\mathfrak{B}_{S,1}, \mathfrak{B}_{S,2}, ...\}$. We shall consider only star bodies K and sets $A \in \mathfrak{B}_S$ satisfying the following condition.

Assumption 1. *The star body K and the set $A \in \mathfrak{B}_S$ are chosen such that for every j the set*

$$G(A \cap S_j) = \left\{ \vartheta \in \mathbb{R}^{n-1} : \exists \eta \text{ with } \theta = \left(\vartheta^T, \eta\right)^T \in A \cap S_j \right\}$$

is well defined and such that for every $\vartheta = (\vartheta_1, \ldots, \vartheta_{n-1})^T \in G(A \cap S_j)$ there is a uniquely determined $\eta > 0$ satisfying $h_K\left((\vartheta_1, \ldots, \vartheta_{n-1}, \eta)^T\right) = 1$.

The latter quantity will be denoted by $\eta_j(\vartheta)$, $j = 1, 2, \ldots$.

For every $x \in \mathbb{R}^n, x \neq 0$ there are uniquely determined $r > 0$ and $\theta \in S$ such that $x = r\theta$. For $x \in rS_j$, we have $x = r\left(\vartheta^T, \eta_j(\vartheta)\right)^T$, and we will write $r\eta_j(\vartheta) = y_j(\vartheta)$. Consequently, $h_K(x) = h_K(r\theta) = rh_K\left(\left(\vartheta^T, \eta_j(\vartheta)\right)^T\right) = r$.

For $j = 1, 2, \ldots$ the star spherical coordinate transformation $StSph_j : [0, \infty) \times G(S_j) \to C_j$ is defined by $x_i = r\vartheta_i, i = 1, \ldots, n-1, x_n = y_j(\vartheta)$. The equations $r = h_K(x), \vartheta_i = x_i/r$, $i = 1, \ldots, n-1$ define a.e. uniquely the inverse map of $StSph_j$.

Note that if K is convex or an axes aligned p-generalized ellipsoid, $p > 0$, see Section 3.2.1, one may assume the sets $S \cap C_{1(2)}$ to be the upper and lower hemi-spheres, $S^{+(-)} = \left\{\theta = (\theta_1, \ldots, \theta_n)^T \text{with } \theta_n > (<) 0\right\}$, respectively.

Lemma 1. *The absolute value of the Jacobian of the star-spherical coordinate transformation is $J(r, \vartheta) = r^{n-1} J_j^*(\vartheta)$, with $J_j^*(\vartheta) = |\eta_j(\vartheta) - \sum\limits_{i=1}^{n-1} \vartheta_i \frac{\partial}{\partial \vartheta_i} \eta_j(\vartheta)|$ for every $r > 0, \vartheta \in G(S_j), j = 1, 2, \ldots$*

Proof. The formula for $J(r, \vartheta) = |\frac{d(x_1, \ldots, x_n)}{d(r, \vartheta)}|$ given in the lemma can be checked immediately by determining all partial derivatives and evaluating the resulting determinant. □

The coordinate system introduced here will be the basis of our considerations in Section 3.1 dealing with a general local notion of surface content. A specific integral notion of surface content dealt with in Section 3.2.3 will make use of another system of coordinates which will be introduced in Section 3.2.2. For the comparison study of the two seemingly rather different two approaches to measuring surfaces in Section 3.2.4, we will consider again suitable coordinates.

An essential part of the message of Lemma 1 is that the Jacobian allows a factorization into a term not depending on the radius coordinate and one that is independent of the directional coordinates.

Later in this paper, the restriction of the star spherical coordinate transformation to the case $r = 1$ will be denoted by $StSph^*$.

3 The star-generalized surface measure

3.1 Basics

The results in (Richter 2009, 2013) reflect the basic role which a suitable notion of non-Euclidean surface content plays for the study of non-spherical distributions. Here, we give first formally a local definition of a generalized surface measure which allows us to derive geometric and stochastic representations of star-shaped distributions and correspondingly distributed random vectors, respectively. For a more advanced understanding of the notions and results, we refer to Remark 6 below.

For $A \in \mathfrak{B}_S$, we introduce the central projection cone $CPC(A) = \{x \in \mathbb{R}^n : x/h_K(x) \in A\}$ and the star sector of star radius ϱ, $sector(A, \varrho) = CPC(A) \cap K(\varrho)$. We are now ready to introduce the first basic notion of this paper. To this end, let μ be the Lebesgue measure in $\mathbb{R}^n$.

Definition 1. *The star-generalized surface measure is defined on* $\varrho \cdot \mathfrak{B}_S$ *by* $\mathfrak{O}_S(A) = f'(\varrho)$ *where* $f(\varrho) = \mu(sector(A, \varrho))$.

If K is the Euclidean unit ball, and thus S is the Euclidean unit sphere, then $\mathfrak{O}_S$ equals the usual Euclidean surface content measure. The equation in Definition 1 should be well known for this case, but, astonishing enough, numerous authors do not make this very clear to their readers.

In contrast to the usual differential geometric definition of the notion of surface content in terms of an integral, the approach in Definition 1 is based upon a derivative. The equation

$$\mu(sector(A, R)) = \int_0^R \mathfrak{O}_S(rA)dr, A \in \mathfrak{B}_S \tag{1}$$

is an immediate consequence of the fundamental theorem of calculus and might seem therefore to be of no special interest, here. If, however, non-trivial explanations for $\mathfrak{O}_S$ are available as in (Richter 2009, 2013) where K is an $l_{n,p}-$ball or an ellipsoidal ball, respectively, then things change noticably. In both cases, a particular non-Euclidean geometry was identified such that the correspondingly modified integral notion of surface content based upon this non-Euclidean geometry coincides with the locally defined surface measure $\mathfrak{O}_S$. This allows a non-Euclidean interpretable extension of Cavalieri's and Torricelli's method of indivisibles, see (Richter 1985, 2009). Later in this paper, we shall observe this for a bigger class of star bodies. Moreover, we remark that $\mathfrak{O}_S(A) = n\mu(sector(A, 1)), \forall A \in \mathfrak{B}_S$, meaning much more than just $\mathfrak{O}_S(S) = n\mu(K)$.

Theorem 1. *For sets* $A \in \mathfrak{B}_S$ *satisfying Assumption 1 in Section 2, the star-generalized surface measure allows the representation*

$$\mathfrak{O}_S(A) = \sum_j \int_{G(A \cap S_j)} J_j^*(\vartheta)d\vartheta.$$

Proof. Using star-spherical coordinates, and that $G(A \cap S_j) = StSph^{*-1}(A \cap S_j)$, we get according to Lemma 1

$$\mu(sector(A, \varrho)) = \int_{sector(A,\varrho)} dx = \int_0^\varrho \sum_j \int_{StSph^{*-1}(A \cap S_j)} r^{n-1} J_j^*(\vartheta)d(\vartheta, r).$$

Definition 1 applies. □

Remark 1. *(a) With the notations* $\mathfrak{O}_S(A) = \int\limits_A \mathfrak{O}_S(d\theta)$*, and*

$$\sum_j \int\limits_{G(A\cap S_j)} J_j^*(\vartheta)d\vartheta = \int\limits_{G(A)} J^*(\vartheta)d\vartheta,$$

an alternative expression of Theorem 1 is

$$\int\limits_A \mathfrak{O}_S(d\theta) = \int\limits_{G(A)} J^*(\vartheta)d\vartheta.$$

If f is integrable then we write $\int\limits_A f(\theta)\mathfrak{O}_S(d\theta) = \int\limits_{G(A)} f\left(\left(\vartheta^T, \eta(\vartheta)\right)^T\right) J^*(\vartheta)d\vartheta$.

(b) The sector measure on $\mathfrak{B}_S$*, i.e. the measure* $sm_K(A) = \frac{\mu(sector(A,1))}{\mu(K)}$*, satisfies the representation* $sm_K(A) = \frac{\mathfrak{O}_S(A)}{\mathfrak{O}_S(S)}, A \in \mathfrak{B}_S$.

(c) A class of examples where Theorem 1 applies is given by all star bodies K corresponding to norms or antinorms for which there exist countably many pairwise disjoint sets A_j *satisfying Assumption 1 and* $S = \bigcup\limits_j A_j$.

The following consequence of Theorem 1 follows using Fubini's theorem and can be read in the special case $f = 1$ as a disintegration formula for the Lebesgue measure. For a certain survey of such formulas, see (Richter 2012). These representations may also be considered as closely connected with a generalized method of indivisibles with the latter being defined as the intersections of a Borel set B with the star spheres $S(r), r > 0$. Constructions of such type are called cross sections by several authors, see (Eaton 1983; Farrell 1976, 1985; Koehn 1970; Wijsman 1967, 1986, 1990) and (Takemura and Kuriki 1996).

Corollary 1. *Let the star body K satisfy Assumption 1. Then*

(a) For $B \in \mathfrak{B}_n$ *and integrable f,* $\int\limits_B f(x)dx = \int\limits_0^\infty \left[r^{n-1} \int\limits_{[\frac{1}{r}B]\cap S} f(r\theta)\mathfrak{O}_S(d\theta) \right] dr.$

(b) For bounded measurable B, $\int\limits_B dx = \int\limits_0^\infty \mathfrak{O}_S(B\cap S(r))dr.$

Proof. Changing from Cartesian to star spherical coordinates yields

$$\int\limits_B f(x)dx = \int\limits_0^\infty \left[r^{n-1} \sum_j \int\limits_{G([\frac{1}{r}B]\cap S_j)} f(StSph_j(r,\vartheta))J_j^*(\vartheta)d\vartheta \right] dr$$

$$= \int\limits_0^\infty \left[r^{n-1} \sum_j \int\limits_{[\frac{1}{r}B]\cap S_j} f(r\theta)\mathfrak{O}_S(d\theta) \right] dr.$$

The rest follows with $f = 1$ and the notation in Remark 1 □

Corollary 1 may be rewritten using the following second basic notion of this paper.

Definition 2. *The star sphere intersection-proportion function (ipf) of the set* $B \in \mathfrak{B}_n$ *is defined as* $\mathfrak{F}_S(B, r) = \mathfrak{O}_S\left(\left[\frac{1}{r}B\right] \cap S\right) / \mathfrak{O}_S(S),\ r > 0.$

The ipf was first introduced in (Richter 1985, 1987, 1991) for Gaussian and spherical distributions, respectively, i.e. for cases where S is the Euclidean unit sphere, and generalized later in (Richter 2007) to the case that S is an $l_{n,p}-$sphere. Moreover, the ipf corresponding to an asymmetric sphere S was considered for the case that K is the shifted positive part of an $l_{n,1}-$ ball, i.e. a simplex, and for the case that K is a, possibly asymmetric, polygon or Platonic body, respectively.

Corollary 2. *If the conditions of Corollary 1(b) are satisfied,*

$$\int\limits_B dx = \mathfrak{O}_S(S) \int\limits_0^\infty r^{n-1} \mathfrak{F}_S(B,r) dr.$$

Proof. It follows from Corollary 1 that

$$\int\limits_B dx = \int\limits_0^\infty r^{n-1} \mathfrak{O}_S \left(\left[\frac{1}{r} B \right] \cap S \right) dr.$$

The rest follows by Definition 2. □

Remark 2. *According to Remark 1(b) and Definition 2, the ipf allows the sector measure interpretation* $\mathfrak{F}_S(B,r) = sm_K\left(\left[\frac{1}{r}B\right] \cap S\right)$, $r > 0$.

Whether one prefers the interpretation of the ipf according to the definition of the sector measure sm_K in terms of volumes or according to Definition 2 in terms of star-generalized surface contents may depend on several aspects. The authors in (Barthe et al. 2003; Naor 2007; Schechtman and Zinn 1990) use the notion of cone measure in similar situations.

As already mentioned in the first part of the present section, one is naturally interested in a fully differential geometric explanation of the star-generalized surface measure $\mathfrak{O}_S$ in terms of an integral. Such an explanation will be given in Section 3.2.3 when K is an element of a class of generalized ellipsoids which are star-shaped but not necessarily convex.

3.2 The star-generalized surface content of *p*-generalized ellipsoids

3.2.1 Volumes of p-generalized ellipsoids

Because the notion of the star-generalized surface content is derived from that of volumes, we first study volumes of p-generalized ellipsoids in this section. To this end, let $\mathfrak{b} = \{\mathfrak{b}_1, \ldots, \mathfrak{b}_n\}$ be any orthonormal basis (onb) in $\mathbb{R}^n$ and put $x = \sum\limits_{i=1}^n \xi_i \mathfrak{b}_i$ for $x \in \mathbb{R}^n$. Moreover, let $a = (a_1, \ldots, a_n)^T$ be an arbitrary vector having positive components, p a positive real number, $|.|_{a,p} : \mathbb{R}^n \to [0,\infty)$ the function defined by $|x|_{a,p} = \left(\sum\limits_1^n |\frac{\xi_i}{a_i}|^p\right)^{1/p}$, $x \in \mathbb{R}^n$ and $B_{a,p} = \left\{x \in \mathbb{R}^n : |x|_{a,p} \le 1\right\}$ the corresponding unit ball w.r.t. $\mathfrak{b}$. Its topological boundary $E_{a,p}$ is a generalized ellipsoid having form parameter p and main axes being aligned with the coordinate axes and having lengths $2a_i, i = 1, \ldots, n$. One may consider $E_{a,p}$ also as a sphere w.r.t. the function $|.|_{a,p}$ which is a norm if $p \ge 1$ and an antinorm if $0 < p < 1$.

The $\mathfrak{b}_i$-axis may be interpreted in the sense of main axis from principal component analysis. For a discussion of these notions in connection with that of correlation, we refer to (Dietrich et al. 2013). The set $B_{a,p}(R) = RB_{a,p}$ will be called a p-generalized ellipsoidal ball, or simply p-generalized ellipsoid, of $|.|_{a,p}$-radius R, $R > 0$, and w.r.t. the basis $\mathfrak{b}$.

The evaluation of the volume of $B_{a,p}(R)$ may be immediately reduced to that of an $l_{n,p}$-ball having a suitable p-radius. To this end, we denote the $l_{n,p}$-ball of p-radius R by $K_{n,p}(R) = RK_{n,p}$ where $K_{n,p} = B_{\mathbb{1},p}$, $\mathbb{1} = (1, \ldots, 1)^T \in \mathbb{R}^n$, and its topological boundary, the $l_{n,p}-$sphere of p-radius R, by $S_{n,p}(R) = RS_{n,p}$. Moreover, we put $a_i^* = \prod\limits_{j=1, j\neq i}^{n} a_j$, $i = 1, \ldots, n$ and let $diag\left(a_1^*, \ldots, a_n^*\right)$ denote a diagonal $n \times n$-matrix whose diagonal entries are $a_2 \cdot \ldots \cdot a_n, \ \ldots, a_1 \cdot \ldots \cdot a_{n-1}$, respectively. If $\mathfrak{b}$ is the standard onb in $\mathbb{R}^n$, then $diag\left(a_1^*, \ldots, a_n^*\right) B_{a,p}(R) = K_{n,p}(a_1 ... a_n R)$. Changing variables $u = diag\left(a_1^*, \ldots, a_n^*\right) x$ in the integral $\mu(B_{a,p}(R)) = \int\limits_{\{x \in R^n : |x|_{a,p} \leq R\}} dx$ gives

$$\mu(B_{a,p}(R)) = \int\limits_{K_{n,p}(Ra_1 \ldots a_n)} \frac{du}{(a_1 \ldots a_n)^{n-1}}.$$

Hence, $\mu(B_{a,p}(R)) = \frac{\mu(K_{n,p}(Ra_1 \ldots a_n))}{(a_1 \ldots a_n)^{n-1}} = a_1 \ldots a_n \frac{\omega_{n,p}}{n} R^n$ where, in accordance with (Richter 2009), $\omega_{n,p} = \frac{2^n \left(\Gamma(\frac{1}{p})\right)^n}{p^{n-1}\Gamma(\frac{n}{p})} = O_{p,q}(S_{n,p})$ with $\frac{1}{p} + \frac{1}{q} = 1$ is the $l_{n,q}$-surface content of the $l_{n,p}-$unit sphere $S_{n,p}$. The following theorem has thus been proved.

Theorem 2. *The p-generalized ellipsoid of $|.|_{a,p}$-radius R has the volume*

$$\mu(B_{a,p}(R)) = a_1 \ldots a_n \frac{\omega_{n,p}}{n} R^n.$$

Corollary 3. *The star-generalized surface content of a p-generalized ellipsoid with axes of lengths $2a_i, i = 1, \ldots, n$ is $\mathfrak{O}_S(E_{a,p}) = a_1 ... a_n \omega_{n,p}$.*

This corollary is an immediate consequence of Definition 1.

Notice that this formula for the star-generalized surface content of $E_{a,p}$ proves that the parameters p and a have separate influence on the result. Moreover, it makes no use of elliptic integrals, whereas the Euclidean surface content of $E_{a,p}$ does.

Similarly, as Equation (1), the equation

$$\mu(B_{a,p}(R)) = \int\limits_0^R \mathfrak{O}_S(E_{a,p}(r)) dr \tag{2}$$

where $E_{a,p}(r) = rE_{a,p}$, might seem to be of no special interest, at this stage of our study. We shall show, however, later in this paper that $\mathfrak{O}_S$ allows a non-trivial interpretation as the surface measure w.r.t. a well defined, non-Euclidean, metric geometry. This allows us to re-define $\mathfrak{O}_S$ in a well established differential geometric approach. This will be done in the next but one section. We shall make use of a specific coordinate system which will be defined in the next section.

Following the notation in (Richter 2013), we will call the star-generalized surface measure alternatively the $E_{a,p}$-generalized surface measure if K is a p-generalized ellipsoid with axes of lengths $2a_i, i = 1, \ldots, n$.

3.2.2 The p-generalized ellipsoidal coordinates

We recall that $l_{n,p}$-generalized and ellipsoidal generalized trigonometric functions and coordinates have been shown in (Richter 2007, 2009) and (Richter 2011b, 2013) to be powerful tools for studying $l_{n,p}$-symmetric and elliptically contoured distributions, respectively. The coordinates which we define in this section are in some sense combinations and generalizations of the aforementioned ones. They will be used in Section 3.2.4 for showing the equivalence of two approaches to the star-generalized surface measure $\mathfrak{O}_S$: the local one presented already in Definition 1, and an integral one which will be given later.

Let us assume for a moment that $n = 2$ and $(x, y)^T = x\mathfrak{b}_1 + y\mathfrak{b}_2$ where $\{\mathfrak{b}_1, \mathfrak{b}_2\}$ is an onb in $\mathbb{R}^2$. In the following definition, ϕ can be interpreted as the angle between $pos(x, 0)$ and $pos(x, y)$ where $pos(\xi, \eta) = \left\{(\gamma\xi, \gamma\eta)^T : \gamma > 0\right\}$.

Definition 3. *The $E_{a,b;p}$-generalized trigonometric functions are defined as*

$$\cos_{(a,b;p)}(\phi) = \frac{(\cos\phi)/a}{N_{a,b;p}(\phi)} \text{ and } \sin_{(a,b;p)}(\phi) = \frac{(\sin\phi)/b}{N_{a,b;p}(\phi)},\ \phi \in [0, 2\pi)$$

for positive *a, b, p* and where $N_{a,b;p}(\phi) = (|(\cos\phi)/a|^p + |(\sin\phi)/b|^p)^{1/p}$.

Remark 3. *These generalized trigonometric functions may be extended to functions on the whole real line with period 2π. Basic analytical and geometric interpretations of these functions follow from the representations*

$$\cos_{(a,b;p)}(\phi) = \frac{x/a}{(|x/a|^p + |y/b|^p)^{1/p}},\ |\cos_{(a,b;p)}(\phi)| = \frac{|(\cos_{(a,b)}(\phi), 0)|_p}{|(\cos_{(a,b)}(\phi), \sin_{(a,b)}(\phi))|_p},$$

and

$$\sin_{(a,b;p)}(\phi) = \frac{y/b}{(|x/a|^p + |y/b|^p)^{1/p}},\ |\sin_{(a,b;p)}(\phi)| = \frac{|(0, \sin_{(a,b)}(\phi))|_p}{|(\cos_{(a,b)}(\phi), \sin_{(a,b)}(\phi))|_p},$$

where $|.|_p = |.|_{\mathbb{1},p}$ and $\sin_{(a,b)}$ and $\cos_{(a,b)}$ are defined in (Richter 2011b).

Remark 4. *Euler's formula is generalized by* $|\cos_{(a,b;p)}(\phi)|^p + |\sin_{(a,b;p)}(\phi)|^p = 1$.

Remark 5. *For every ϕ,*

$$\cos'_{(a,b;p)}(\phi) = -\frac{\sin_{a,b;p}(\phi)|\sin_{a,b;p}(\phi)|^{p-2}}{abN^2_{(a,b;p)}(\phi)},\ \sin'_{(a,b;p)}(\phi) = \frac{\cos_{a,b;p}(\phi)|\cos_{a,b;p}(\phi)|^{p-2}}{abN^2_{(a,b;p)}(\phi)}.$$

We assume again that $x = x_1\mathfrak{b}_1 + ... + x_n\mathfrak{b}_n$, $x \in \mathbb{R}^n$.

Definition 4. *The p-generalized ellipsoidal coordinate transformation $T^E_{a,p} = T^E_{a,p}(n)$, $T^E_{a,p} : M_n \to \mathbb{R}^n$, with $M_n = [0, \infty) \times M^*_n$, $M^*_n = [0, \pi)^{\times(n-2)} \times [0, 2\pi)$ is defined by*

$$x_1 = a_1 r \cos_{(a_1,a_2;p)}(\phi_1), x_2 = a_2 r \sin_{(a_1,a_2;p)}(\phi_1)\cos_{(a_2,a_3;p)}(\phi_2), \ldots,$$

$$x_{n-1} = a_{n-1} r \sin_{(a_1,a_2;p)}(\phi_1) ... \sin_{(a_{n-2},a_{n-1};p)}(\phi_{n-2})\cos_{(a_{n-1},a_n;p)}(\phi_{n-1}),$$

$$x_n = a_n r \sin_{(a_1,a_2;p)}(\phi_1) ... \sin_{(a_{n-2},a_{n-1};p)}(\phi_{n-2})\sin_{(a_{n-1},a_n;p)}(\phi_{n-1}).$$

Theorem 3. *The map $T^E_{a,p}$ is almost one-to-one, its inverse is a.e. given by*

$$r=\left(\sum_1^n\left(\left|\frac{x_i}{a_i}\right|\right)^p\right)^{1/p},\phi_j=\arccos_{(a_j,a_{j+1};p)}\left(\frac{x_j/a_j}{\left(\sum\limits_{i=j}^n|x_i/a_i|^p\right)^{1/p}}\right), \textit{and, if } x_{n-1}\neq 0,$$

$\arctan|\frac{x_n}{x_{n-1}}| = \phi_{n-1}$ *if* $(x_{n-1},x_n)\in Q_1$, $=\pi-\phi_{n-1}$ *in* Q_2, $=-\pi+\phi_{n-1}$ *in* Q_3, *and* $=2\pi-\phi_{n-1}$ *in* Q_4.

Here, $\arccos_{(a_j,a_{j+1};p)}$ *denotes the function inverse to* $\cos_{(a_j,a_{j+1};p)}$ *and* Q_1 *up to* Q_4 *denote anti-clockwise enumerated quadrants from* $\mathbb{R}^2$.

Proof. The proof of this theorem is quite similar to that of Theorem 1 in (Richter 2007) and is therefore omitted. □

Theorem 4. *The Jacobian of the coordinate transformation $T^E_{a,p}$ is*

$$J\left(T^E_{a,p}\right)(r,\phi_1,\ldots,\phi_{n-1})=r^{n-1}J^*\left(T^E_{a,p}\right)(\phi_1,\ldots,\phi_{n-1}),$$

$$J^*\left(T^E_{a,p}\right)(\phi_1,\ldots,\phi_{n-1})=\frac{1}{a_2\cdot\ldots\cdot a_{n-1}}\prod_{i=1}^{n-1}\frac{\left(\sin_{(a_i,a_{i+1};p)}(\phi_i)\right)^{n-1-i}}{N^2_{(a_i,a_{i+1};p)}(\phi_i)}.$$

Proof. The proof will be given in four steps. First, we change variables $\frac{x_i}{a_i}=y_i$, $i=1,\ldots,n$. The Jacobian of this transformation is $\left|\frac{D(x_1,\ldots,x_n)}{D(y_1,\ldots,y_n)}\right|=a_1\cdot\ldots\cdot a_n$.

Next, we change variables $y_1=\tilde r\mu_1, y_2=\tilde r\left(1-|\mu_1|^p\right)^{1/p}\mu_2,\ldots,$

$$y_{n-1}=\tilde r\left(1-|\mu_1|^p\right)^{1/p}\cdot\ldots\cdot\left(1-|\mu_{n-2}|^p\right)^{1/p}\mu_{n-1},$$

$$y_n=+(-)\tilde r\left(1-|\mu_1|^p\right)^{1/p}\cdot\ldots\cdot\left(1-|\mu_{n-2}|^p\right)^{1/p}\left(1-|\mu_{n-1}|^p\right)^{1/p}.$$

As it was shown in the proof of Theorem 2 in the afore mentioned paper, the Jacobian of this transformation is $\left|\frac{D(y_1,\ldots,y_n)}{D(\tilde r,\mu_1,\ldots,\mu_{n-1})}\right|=\tilde r^{n-1}\prod\limits_{i=1}^{n-1}\left(1-|\mu_i|^p\right)^{(n-p-i)/p}$.

Third, we change variables $\tilde r=r,\mu_i=\cos_{(a_i,a_{i+1};p)}(\phi_i), i=1,\ldots,n-1$. The Jacobian of this transformation is

$$\left|\frac{D(\tilde r,\mu_1,\ldots,\mu_{n-1})}{D(r,\phi_1,\ldots,\phi_{n-1})}\right|=\left|\det diag\left(1,\frac{d}{d\phi_1}\cos_{(a_1,a_2;p)}(\phi_1),\ldots,\frac{d}{d\phi_{n-1}}\cos_{(a_{n-1},a_n;p)}(\phi_{n-1})\right)\right|.$$

It follows from the properties of the $E_{a,b;p}$-generalized trigonometric functions that

$$\left|\frac{D(\tilde r,\mu_1,\ldots,\mu_{n-1})}{D(r,\phi_1,\ldots,\phi_{n-1})}\right|=\prod_{i=1}^{n-1}\frac{\sin_{(a_i,a_{i+1};p)}(\phi_i)|\sin_{(a_i,a_{i+1};p)}(\phi_i)|^{p-2}}{a_ia_{i+1}N^2_{(a_i,a_{i+1};p)}(\phi_i)}.$$

On combining all three transformations, we get finally

$$J\left(T^E_{a,p}\right)(r,\phi_1,\ldots,\phi_{n-1})=a_1\ldots a_n\cdot r^{n-1}\prod_{i=1}^{n-1}\frac{\left(\sin_{(a_i,a_{i+1};p)}(\phi_i)\right)^{d-i-1}}{a_ia_{i+1}N^2_{(a_i,a_{i+1};p)}(\phi_i)}.$$

□

Corollary 4. *If* $n=2$ *then* $J\left(T^E_{a,p}\right)(r,\phi)=\frac{r}{N^2_{(a_1,a_2;p)}(\phi)}$, *and if* $n=3$ *then* $J\left(T^E_{a,p}\right)(r,\phi_1,\phi_2)=\frac{r^2\sin\phi_1}{a_2^2N^2_{(a_2,a_3;p)}(\phi_2)N^3_{(a_1,a_2;p)}(\phi_1)}$.

For the corresponding results in the case $p = 2$, we refer to (Richter 2011b, 2013).

3.2.3 Integral approach to the star-generalized surface measure on p-generalized ellipsoids

Let us recall that measuring the Euclidean surface content of $E_{a,p}(R)$ necessarily involves certain elliptic integrals. In this paper, however, we make use of a non-Euclidean definition of surface content which avoids such integrals. To this end, we shall consider the ellipsoid $E_{a,p}(R)$ as a subset of the generalized Minkowski space $\left(\mathbb{R}^n, |.|_{\frac{1}{a},q}\right)$ where $\frac{1}{a} = \left(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\right)^T$ and p and q are connected with each other by the equation $\frac{1}{p} + \frac{1}{q} = 1$. We will introduce now the notion of the $|.|_{\frac{1}{a},q}$-surface content of $E_{a,p}(R)$ in a similar way as the notion of the $l_{n,q}$-surface content was introduced in (Richter 2009) for $l_{n,p}$-spheres. Notice that effects coming from scaling axes with the help of the parameter vector a and effects being due to the form parameter p are dealt with here in a separate way when introducing the function $|.|_{\frac{1}{a},q}$.

Let $\mathfrak{b}$ be the standard onb, and let y be defined as the positive solution of $\sum\limits_{i=1}^{n-1} |x_i/a_i|^p + |y/a_n|^p = R^p$ where $\sum\limits_{i=1}^{n-1} |x_i/a_i|^p < R^p$. At $(x_1, \ldots, x_{n-1}, y)^T$, $y > 0$, the vector normal to the upper half $E^+_{a,p}(R)$ of the ellipsoid $E_{a,p}(R)$ is $N(x_1, \ldots, x_{n-1}) = (-1)^n \left(\sum\limits_{i=1}^{n-1} \frac{\partial y}{\partial x_i} e_i - e_n\right)$. Since it always will be clear how to deal with the case $y < 0$, we will not further mention this case.

Definition 5. *Let $A \subset E^+_{a,p}(R) \cap \mathfrak{B}_n$. The integral (or $|.|_{\frac{1}{a},q}$-) surface content of the set A is defined by $O_{a,p,q}(A) = \int\limits_{G(A)} |N(x_1, \ldots, x_{n-1})|_{\frac{1}{a},q}\, dx_1 \ldots dx_{n-1}$ where $G(A) = \left\{ (x_1, \ldots, x_{n-1})^T : (x_1, \ldots, x_n)^T \in A \right\}$.*

Later in this paper, this definition will be called the integral approach to the notion of star-generalized surface content. This will be justified in the next section. Let us mention that if $a = \mathbb{1}$ then the surface measure $O_{a,p,q}$ which is based upon the geometry of the ellipsoid $E_{\frac{1}{a},q}$ equals the surface measure $O_{p,q}$ in (Richter 2009) being based upon the geometry of the $l_{n,q}$-ball $K_{n,q}$ which is dual to $K_{n,p}$. Two-dimensional special cases of Definition 5 were dealt with, e.g., in (Richter 2011a, 2011b) for arbitrary star discs and ellipses, respectively.

Lemma 2. *The $|.|_{\frac{1}{a},q}$-surface content of the whole generalized ellipsoid $E_{a,p}(R)$ of $|.|_{a,p}$-radius R is $O_{a,p,q}\left(E_{a,p}(R)\right) = a_1 \ldots a_n \omega_{n,p} R^{n-1}$.*

Proof. It follows from $\frac{\partial y}{\partial x_j} = -\frac{a_n |x_j|^{p-1}}{a_j^p \left(R^p - \sum\limits_{i=1}^{n-1} (|x_i/a_i|)^p\right)^{(p-1)/p}}$, $j = 1, \ldots, n-1$

and with $q = p/(p-1)$ that

$$|N(x_1, \ldots, x_{n-1})|^q_{\frac{1}{a},q} = \sum_{j=1}^{n-1} \frac{a_n^q |x_j/a_j|^p}{R^p - \sum\limits_{i=1}^{n-1} |x_i/a_i|^p)} + a_n^q = \frac{a_n^q R^p}{R^p - \sum\limits_{i=1}^{n-1} |x_i/a_i|^p}.$$

Hence, because of symmetry,

$$O_{a,p,q}(E_{a,p}(R)) = 2a_n R^{p-1} \int\limits_{G(E^+_{a,p}(R))} \frac{d(x_1 ... x_{n-1})}{\left(R^p - \sum\limits_{i=1}^{n-1} |x_i/a_i|^p\right)^{(p-1)/p}}.$$

For suitably transforming this integral, we shall introduce now another system of coordinates. Let the p-generalized $(n-1)$-dimensional standard elliptical coordinate transformation

$$T_{a,p} : M_{n-1} \to \mathbb{R}^{n-1}, M_{n-1} = [0,\infty) \times M^*_{n-1}, M^*_{n-1} = [0,\pi)^{\times(n-3)} \times [0, 2\pi)$$

be defined by

$$\begin{aligned} x_1 &= a_1 r \cos_p(\phi_1), x_2 = a_2 r \sin_p(\phi_1)\cos_p(\phi_2), \ldots, \\ x_{n-2} &= a_{n-2} r \sin_p(\phi_1) \ldots \sin_p(\phi_{n-3}) \cos_p(\phi_{n-2}), \\ x_{n-1} &= a_{n-1} r \sin_p(\phi_1) \ldots \sin_p(\phi_{n-3}) \sin_p(\phi_{n-2}) \end{aligned}$$

where the p-generalized trigonometric functions $\sin_p$ and $\cos_p$ are defined in (Richter 2007). If $a = \mathbb{1} \in \mathbb{R}^{n-1}$ then this transformation coincides with the $l_{n-1,p}$-spherical coordinate transformation $SPH_p^{(n-1)}$, the Jacobian of which is given in (Richter 2007). If we write $J(T)$ for the Jacobian of a coordinate transformation T then $J(T_{a,p})(r,\phi) = \left|\frac{d(x_1,\ldots,x_{n-1})}{d(r,\phi_1,\ldots,\phi_{n-2})}\right| = a_1 \cdot \ldots \cdot a_{n-1} J\left(SPH_p^{(n-1)}\right)(r,\phi)$.

Moreover, let $J^*\left(SPH_p^{(n-1)}\right)(\phi) = J\left(SPH_p^{(n-1)}\right)(1,\phi)$ be the restriction of the Jacobian of $SPH_p^{(n-1)}$ to the sphere defined by $r = 1$.

Changing from Cartesian to p-generalized standard elliptical coordinates gives

$$O_{a,p,q}(E_{a,p}(R)) = 2a_1 \ldots a_n R^{p-1} \int\limits_0^R \frac{r^{n-2}}{(R^p - r^p)^{(p-1)/p}} dr$$

$$\times \int\limits_0^\pi \ldots \int\limits_0^\pi \int\limits_0^{2\pi} J^*\left(SPH_p^{(n-1)}\right)(\phi_1, \ldots, \phi_{n-2}) d\phi_{n-2} \ldots d\phi_1.$$

Because of

$$\int\limits_0^R \frac{r^{n-2} dr}{(R^p - r^p)^{(p-1)/p}} = R^{n-p} \int\limits_0^1 \frac{t^{n-2} dt}{(1-t^p)^{(p-1)/p}} = \frac{1}{p} R^{n-p} B\left(\frac{1}{p}, \frac{n-1}{p}\right)$$

and $\frac{1}{p} B\left(\frac{1}{p}, \frac{n-1}{p}\right) \omega_{n-1,p} = \frac{1}{2}\omega_{n,p}$, it follows that $O_{a,p,q}(E_{a,p}(R)) = a_1 \ldots a_n \omega_{n,p} R^{n-1}$ □

Hence, for the specific sets $E_{a,b}(R)$, the local and the integral approaches to the star-generalized surface content lead to the same result. In the next section, we will generalize this result. When doing this, we will again make use of a modified coordinate system.

3.2.4 Comparing the local and integral approaches to generalized surface measures on p-generalized ellipsoids

In Section 3.2.3, the surface measure $O_{a,p,q}$ was used for measuring the whole p-generalized ellipsoid $E_{a,p}$ following a differential geometric, or integral or global approach. In the present section, however, we compare it for arbitrary $A \in \mathfrak{B}^E_{a,p} = \mathfrak{B}^n \cap E_{a,p}$ with the alternative local approach which makes use of derivatives and which was introduced in Definition 1. In this sense, we continue to follow the general method of analyzing the non-Euclidean geometry underlying a multivariate probability distribution which was developed in (Richter 2009, 2013). The following theorem says that the star-generalized surface measure coincides with the integral surface measure. For a comparison of these surface measures, it is sufficient to consider them for sets $A \in \mathfrak{B}^E_{a,p}$.

Theorem 5. *With* $S = E_{a,p}$ *and* $1/p + 1/q = 1$, $\mathfrak{O}_S(A) = O_{a,p,q}(A), \forall A \in \mathfrak{B}^E_{a,p}$.

Proof. W.l.o.g., we restrict our consideration to sets $A \in E^+_{a,p} \cap \mathfrak{B}^n$ and start from a slight generalization of the first result in the proof of Lemma 2,

$$O_{a,p,q}(A) = a_n \int\limits_{G(A)} \frac{d(x_1, \ldots, x_{n-1})}{\left(1 - \sum\limits_{i=1}^{n-1} |x_i/a_i|^p\right)^{(p-1)/p}}.$$

We change from Cartesian to p-generalized ellipsoidal coordinates in $(n-1)$ dimensions, $T^E_{a,p}(n-1)$: $(x_1, \ldots, x_{n-1}) \longrightarrow (r, \phi_1, \ldots, \phi_{n-2})$. Because of $\sum\limits_{i=1}^{n-1} |x_i/a_i|^p = r^p$,

$$O_{a,p,q}(A) = a_n \int\limits_{\left(T^E_{a,p}(n-1)\right)^{-1}(G(A))} \frac{r^{n-2}}{(1-r^p)^{(p-1)/p}} J^* \, d(r, \phi_1, \ldots, \phi_{n-2}),$$

$$J^* = J^*\left(T^E_{a,p}(n-1)\right)(\phi_1, \ldots, \phi_{n-2}) = a_{n-1} \prod_{i=1}^{n-2} \frac{\left(\sin_{(a_i, a_{i+1};p)}(\phi_i)\right)^{n-1-i}}{a_{i+1} N^2_{(a_i, a_{i+1};p)}(\phi_i)}.$$

$$\text{If } A = A(r_1, r_2, M^*) = \left\{ \left(y_1, \ldots, y_{n-1}, \left(1 - \sum_{i=1}^{n-1} |x_i/a_i|^p\right)^{1/p} \right)^T : \right.$$

$$\left. (y_1, \ldots, y_{n-1})^T = T^E_{a,p}(n-1)([r_1, r_2) \times M^*) \right\},$$

with $M^* = \{(\phi_1, \ldots, \phi_{n-2}) : \phi_{il} \le \phi_i \le \phi_{iu}, i = 1, \ldots, n-2\}$

$$\subset [0, \pi)^{\times(n-3)} \times [0, 2\pi) = M^*_{n-1}$$

then $$O_{a,p,q}(A) = a_n a_{n-1} \int\limits_{r^1}^{r_2} \frac{r^{n-2}}{(1-r^p)^{(p-1)/p}} dr$$

$$\times \int\limits_{M^*} J^*\left(T^E_{a,p}(n-1)\right)(\phi_1, \ldots, \phi_{n-2}) d(\phi_1, \ldots, \phi_{n-2}).$$

In what follows, we use the coordinate transformation $\tilde{T}_{a,p} : (R, r, \phi) \to z[R, r, \phi], \phi = (\phi_1, \ldots, \phi_{n-2})$ defined by

$z_1 = a_1 R r \cos_{(a_1, a_2;p)}(\phi_1), z_2 = a_2 R r \sin_{(a_1, a_2;p)}(\phi_1) \cos_{(a_2, a_3;p)}(\phi_2), \ldots,$
$z_{n-2} = a_{n-2} R r \sin_{(a_1, a_2;p)}(\phi_1) \cdot \ldots \cdot \sin_{(a_{n-3}, a_{n-2};p)}(\phi_{n-3}) \cos_{(a_{n-2}, a_{n-1};p)}(\phi_{n-2})$
$z_{n-1} = a_{n-1} R r \sin_{(a_1, a_2;p)}(\phi_1) \cdot \ldots \cdot \sin_{(a_{n-3}, a_{n-2};p)}(\phi_{n-3}) \sin_{(a_{n-2}, a_{n-1};p)}(\phi_{n-2}),$
$z_n = a_n R (1 - r^p)^{1/p}.$

This transformation allows the representations
$A(r_1, r_2, M^*) = \tilde{T}_{a,p}(1, [r_1, r_2), M^*) = \{z[R, r, \phi] : R = 1, r \in [r_1, r_2), \phi \in M^*\}$ and
$sector(A(r_1, r_2, M^*), \rho) = \tilde{T}_{a,p}([0, \rho) \times [r_1, r_2) \times M^*)$
$= \{z[R, r, \phi] : 0 \le R < \rho, r \in [r_1, r_2), \phi \in M^*\}.$

The volume

$$\mu(sector(A(r_1,r_2,M^*),\rho)) = \int\limits_{sector(A(r_1,r_2,M^*),\rho)} dz$$

may therefore be written as

$$\mu(sector(A(r_1,r_2,M^*),\rho)) = \int\limits_{R=0}^{\rho}\int\limits_{r=r_1}^{r_2}\int\limits_{\phi\in M^*} J(\tilde{T}_{a,p})(R,r,\phi)dRdrd\phi.$$

Here, $J(\tilde{T}_{a,p})(R,r,\phi) = \frac{D(z_1,\ldots,z_n)}{D(R,r,\phi_1,\ldots,\phi_{(n-2)})}$ can be evaluated as in (Richter 2013) where the case $p=2$ was dealt with:

$$\begin{aligned} J(\tilde{T}_{a,p})(R,r,\phi) &= |z_{nr}a_{n-1}(rR)^{n-2}J^*\, r - z_{nR}a_{n-1}(rR)^{n-2}J^*\, R| \\ &= \left|\left(-\frac{a_nRr^{p-1}}{(1-r^p)^{(p-1)/p}}(rR)^{n-2}r - a_n(1-r^p)^{1/p}(rR)^{n-2}R\right)J^*\, a_{n-1}\right| \\ &= a_{n-1}a_nJ^*(T^E_{a,p}(n-1))(\phi)R^{n-1}\frac{r^{n-2}}{(1-r^p)^{1-1/p}}. \end{aligned}$$

It follows that

$$\lambda\big(sector\big(A\left(r_1,r_2,M^*\right),\rho\big)\big) = a_{n-1}a_n\int\limits_0^{\rho}R^{n-1}dR\int\limits_{r_1}^{r_2}\frac{r^{n-2}}{\sqrt{1-r^2}}dr\int\limits_{M^*}J^*\left(T^E_{a,p}(n-1)\right)(\phi)d\phi.$$

We consider now the local approach to the non-Euclidean surface content,

$$\frac{d}{d\rho}\mu(sector(A(r_1,r_2,M^*),\rho))|_{\rho=1} = a_{n-1}a_n\int\limits_{r_1}^{r_2}\frac{r^{n-2}}{\sqrt{1-r^2}}dr\int\limits_{M^*}J^*\left(T^E_{a,p}(n-1)\right)(\phi)d\phi,$$

and observe that $\mathfrak{O}_S(A(r_1,r_2,M^*)) = O_{a,p,q}(A(r_1,r_2,M^*))$.

The measures $\mathfrak{O}_S$ and $O_{a,p,q}$ coincide on the semi-algebra which is generated by the sets of the type $A(r_1,r_2,M^*)$. It follows from the measure extension theorem that these measures coincide on the whole Borel-σ-field $\mathfrak{B}^E_{a,p}$ on $E_{a,p}$, too. □

Remark 6. ***Reformulating the results of Section 3.1***

In the special case that $K = B_{a,p}, S = E_{a,p}$, just considered here, all the statements of Equations (1) and (2), Theorem 1, Corollaries 1-3 and Remarks 1 and 2 remain valid if the integral surface measure $O_{a,p,q}$ is used instead of the star-generalized surface measure $\mathfrak{O}_S$. The same is true for all those statements quoted below which are using the local notions of Section 2.

4 Star-shaped distributions and geometric disintegration

4.1 Star-shaped uniform distributions

In this section, we extend the method of indivisibles which was used so far for the Lebesgue measure to a class of probability laws which contains the families of elliptically contoured and $l_{n,p}$-symmetric distributions as special cases. This method was originally developed in (Richter 1985, 1987) for proving large deviation limit theorems for the multivariate standard Gaussian law.

We continue to use the notations from Section 2. Note that the following general consideration always covers the very well interpretable specific case that $S = E_{a,p}$ and thus $\mathfrak{O}_S = O_{a,p,q}$.

Definition 6. *The star-generalized uniform probability distribution on the Borel σ-field $\mathfrak{B}_S$ is defined as $\omega_S(A) = \mathfrak{O}_S(A)/\mathfrak{O}_S(S)$.*

Remark 7. *(a) Let $A \in \mathfrak{B}_S$. Then $\omega_S(A) = sm_K(A)$.*
(b) Let $B \in \mathfrak{B}_n$. Then $\omega_S\left(\left[\frac{1}{r}B\right] \cap S\right) = \mathfrak{F}_S(B, r)$, $r > 0$.

Let $(\Omega, \mathfrak{A}, P)$ be a probability space and $Y : \Omega \to \mathbb{R}^n$ a random vector being uniformly distributed on K, i.e.

$$P(Y \in B) = \mu(B)/\mu(K), \ \forall B \in K \cap \mathfrak{B}_n.$$

The a.s. defined normalized random vector $U_S = Y/h_K(Y)$ takes its values in $S, P(h_K(U_S) = 1) = 1$. Let us further put $R = h_K(Y)$.

Theorem 6. *(a) U_S follows the star-generalized uniform distribution, $U_S \sim \omega_S$.*
(b) The pdf of R is $f_R(r) = I_{(0,1)}(r)n \cdot r^{n-1}$.
(c) The random elements U_S and R are stochastically independent.

Proof. (a) Let $A \in \mathfrak{B}_S$. Then $P(U_S \in A) = P(Y \in sector(A, 1))$, and
$P(U_S \in A) = \frac{\mu(sector(A,1))}{\mu(K)} = \frac{1}{\mu(K)}\int\limits_0^1 r^{n-1}dr \int\limits_{StSph^{*-1}(A)} J^*(\vartheta)d\vartheta = \frac{\mathfrak{O}_S(A)}{n\mu(K)}$.
Because of $\mathfrak{O}_S(S) = n\mu(K)$, we have $P(U_S \in A) = \omega_S(A)$.
(b) For $0 < r < 1$, we consider the cumulative distribution function (cdf) of R,

$$P(R < r) = P(Y \in K(r)) = \mu(K(r))/\mu(K) = r^n I_{(0,1)}(r).$$

(c) The independence of U_S and R follows from $P(R < \varrho, U_S \in A)$
$= P(Y \in sector(A, \varrho)) = \int\limits_{sector(A,\varrho)} P^Y(dx) = \frac{1}{\mu(K)} \int\limits_{sector(A,\varrho)} d\mu$
$= \frac{1}{\mu(K)} \int\limits_0^{\varrho} \int\limits_{G(A)} r^{n-1}J^*(\vartheta)d\vartheta dr = \frac{1}{\mu(K)} \frac{\varrho^n}{n} \mathfrak{O}_S(A) = P(R < \varrho)P(U_S \in A)$ □

Remark 8. *(a) The pdf of R^2 is $\frac{d}{dr}P(R^2 < r) = I_{(0,1)}(r)\frac{n}{2}r^{n/2-1}$, $r \in R$.*
(b) The probability distribution of the random vector Y allows the representation

$$P(Y \in B) = \int\limits_0^{\infty} P\left(U_S \in \frac{1}{r}B|R = r\right) dP(R < r)$$

which may be considered as a reformulation of Corollary 2 with

$$\mathfrak{F}_S(B, r) = \omega_S\left(\left[\frac{1}{r}B\right] \cap S\right) = P\left(U_S \in \frac{1}{r}B|R = r\right) \quad a.s. \tag{3}$$

That is why the family of probability measures $\mathfrak{P} = \{P_r, r > 0\}$ where P_r is defined on the Borel σ-field $\mathfrak{B}_n$ by $P_r(B) = \omega_S\left(\left[\frac{1}{r}B\right] \cap S\right) = P\left(U_S \in \frac{1}{r}B|R = r\right)$, may be called a geometric disintegration of P^Y w.r.t. P^R. The family $\mathfrak{P}$ may also be considered as a regular conditional probability.

4.2 Continuous star-shaped distributions

There are different ways to introduce more general classes of star-shaped distributions than the uniform ones considered so far. One of the possibilities is to continue with star-shaped distributions having a density, to derive their most basic properties and finally to introduce the class of all star-shaped distributions having just the latter as their defining properties. This way may be considered as formally generalizing the notion of norm-contoured distributions in (Richter, W.-D.: Norm contoured distributions in R^2, submitted), as well as being statistically well motivated by comparing empirical density level sets with level sets of Minkowski functionals of suitably chosen star bodies. This way will be followed in the present and in the following two sections. An alternative possibility would be just to introduce here the general class of star-shaped distributions and to restrict consideration to special classes of distributions like continuous ones only later.

Definition 7. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the assumptions $0 < I(g) < \infty$ where $I(g) = \int_0^\infty r^{n-1} g(r) dr$. We call g a density generating function (dgf), $\varphi_{g,K}(x) = C(g,K) g(h_K(x)), x \in \mathbb{R}^n$ a star-shaped density and K its contour defining star body.*

A probability measure having the density $\varphi_{g,K}$ will be denoted by $\Phi_{g,K}$. Let us emphasize that according to this definition 0_n may be any point from the set of all points w.r.t. which K is star-shaped, hence K needs not to be symmetric w.r.t. 0_n. Densities of such type have been studied already in (Balkema and Nolde 2010; Fernandez et al. 1995). Our more general considerations in Sections 4.3-4.7, however, seem to be new. The following theorem deals with a geometric measure representation of continuous star-shaped distributions.

Theorem 7. *For every $B \in \mathfrak{B}^n$, $\Phi_{g,K}(B) = C(g,K) \mathfrak{O}_S(S) \int_0^\infty r^{n-1} g(r) \mathfrak{F}_S(B, r) dr$.*

Proof. Because of $\Phi_{g,K}(B) = C(g,K) \int_B g(h_K(x)) dx$ it follows from Corollary 1(a) that $\Phi_{g,K}(B) = C(g,K) \int_0^\infty r^{n-1} \int_{[\frac{1}{r}B] \cap S} g(h_K(r\theta)) \mathfrak{O}_S(d\theta) dr$. Hence,

$$\Phi_{g,K}(B) = C(g,K) \int_0^\infty r^{n-1} g(r) \mathfrak{O}_S \left(\left[\frac{1}{r} B \right] \cap S \right) dr. \qquad \square$$

Classes of dgfs are surveyed, e.g., in (Fang et al. 1990) and (Richter 2013). Numerous types of applications of special cases of the geometric measure representation in Theorem 7 are surveyed in (Richter 2009, 2012). Later applications are to be found in (Arellano-Valle and Richter 2012; Batún-Cutz et al. 2013) and (Günzel et al. 2012).

4.3 Stochastic representations

In this section, we consider that property of continuous star-shaped distributions which will serve in the next section to define a general class of star-shaped distributions.

Theorem 8. *If $Y \sim \Phi_{g,K}$ then Y allows the stochastic representation $Y \stackrel{d}{=} R_g U_S$ where R_g and U_S are stochastically independent, $U_S \sim \omega_S$ and R_g follows the density $f(r) = \frac{1}{I(g)} r^{n-1} g(r), r > 0$.*

Proof. $P(R < \varrho) = P(Y \in K(\varrho)) = C(g,K) \int_{K(\varrho)} g(h_K(x)) dx$

$$= 2C(g,K) \int_0^{\rho} \int_{G(S^+)} g\left(h_K\left(r\left(\vartheta^T, \eta^+(\vartheta)\right)^T\right)\right) r^{n-1} J^*(\vartheta) d\vartheta\, dr$$

$$= 2C(g,K) \int_0^{\rho} r^{n-1} g(r) dr \int_{G(S^+)} J^*(\vartheta) d\vartheta$$

□

Remark 9. *The normalizing constant $C(g,K)$ in Definition 7 allows according to Theorem 7 the representation $C(g,K) = \frac{1}{\mathfrak{O}_S(S)I(g)}$ and the statement of Theorem 7 may according to Theorem 8 be written as*

$$\Phi_{g,K}(B) = \int_0^{\infty} \mathfrak{F}_S(B,r) P(R \in dr),\ B \in \mathfrak{B}_n \tag{4}$$

where $\mathfrak{F}_S(B,r)$ may be interpreted as in (3). Hence, (4) may be read as a generalization of Remark 8(b). Moreover,

$$\varphi_{g,K}(x) = \frac{g(h_K(x))}{\mathfrak{O}_S(S)I(g)},\ x \in \mathbb{R}^n. \tag{5}$$

4.4 General star-shaped distributions

The results of the previous section may serve as a starting point for defining general star-shaped distributions. We follow the way in (Fang et al. 1990) and (Richter 2009, 2013) when we use the stochastic representation from Section 4.3 for defining now a large family of star-shaped distributions.

Definition 8. *A random vector $Y : \Omega \to \mathbb{R}^n$ is said to follow a star-shaped distribution if there are a star body K having the origin as an interior point, $0_n \in int K$, a vector $\nu \in \mathbb{R}^n$, and a random variable $R : \Omega \to [0,\infty)$ such that $Y - \nu$ satisfies the stochastic representation $Y - \nu \stackrel{d}{=} R \cdot U_S$ where U_S follows the star-generalized uniform distribution on the boundary S of K, $U_S \sim \omega_S$, and R and U_S are stochastically independent. If Y has a density with dgf g then by $\Phi_{g,K,\nu}$ the distribution law of Y is denoted, $Y \sim \Phi_{g,K,\nu}$, and K is called a density contour defining star body of the star-shaped distribution $\Phi_{g,K,\nu}$.*

The set of all star-shaped distributions on $\mathfrak{B}_n$ will be denoted $StSh^{(n)}$ and its subset of continuous distributions by

$$CStSh^{(n)} = \{\Phi_{g,K,\nu} : \nu \in \mathbb{R}^n,\ K \text{ is a star body with } 0 \in \text{ int } K, g \text{ is a dgf}\}.$$

We recall that star-shaped sets are associated with multivariate stable distributions in (Molchanov 2009) to describe characteristic functions, thus playing there another role than in Definition 8. To finish this section, we remark that both the set of all star bodies having the origin as an interior point and the set $StSh^{(n)}$ are invariant w.r.t. any orthogonal transformation.

4.5 Extension of the ball number function

In (Richter 2011), the ball number function was defined for $l_{n,p}$-balls and the problem of extending it to balls being as general as possible was stated. It follows from the results in Section 3 that both the ratios $\frac{\mu(B_{a,p}(r))}{r^n}$ and $\frac{O_{a,p,q}(E_{a,p}(r))}{nr^{n-1}}$ do not depend on the star radius r, and that their constant values are one and the same number. This common value will be called the ball number $\pi(B_{a,p})$ of $B_{a,p}(r), r > 0$. Here, $\pi(B_{a,p}) = \mu(B_{a,p}) = a_1 \cdot \ldots \cdot a_n \cdot \frac{\omega_{n,p}}{n}$, hence the region where the ball number function is defined is extended here to all $B_{a,p}$ balls.

4.6 Characteristic functions

Let $Y : \Omega \to \mathbb{R}^n$ be a star-shaped distributed random vector which satisfies the stochastic representation $Y \overset{d}{=} R \cdot U_S$ where the non-negative random variable R is independent of the star-generalized uniformly distributed random vector U_S, and let moreover ϕ_Y and ϕ_{U_S} denote the characteristic functions (ch.f.) of the vectors Y and U_S, respectively. Further, let F_R denote the cdf of R.

Theorem 9. *The ch.f. of the star-shaped distributed random vector Y allows the integral representation* $\phi_Y(t) = \int\limits_0^\infty \phi_{U_S}(rt) dF_R(r),\ t \in \mathbb{R}^n$.

Proof. Because of the independence of R and U_S, Theorem 1.1.6 in (Sasvári 2013) applies. □

This theorem was proved first for spherically distributed vectors in (Schoenberg 1938) and later for $l_{n,1}$-symmetrically distributed random vectors in (Ng and Tian 2001) and for continuous $l_{n,p}$-symmetrically distributed vectors in (Kalke 2013).

Remark 10. *(a) The ch.f.* $\phi_{U_S}(t) = Ee^{it^T U_S},\ t \in \mathbb{R}^n$, *of* U_S *allows the integral representation* $\phi_{U_S}(t) = \int\limits_S \cos\left(t^T\theta\right) \omega_S(d\theta) + i \int\limits_S \sin\left(t^T\theta\right) \omega_S(d\theta)$.

(b) The ch.f. ϕ_Y *of a star-shaped distributed random vector Y having a density with dgf g and contour defining star body K allows the representation*

$$\mathfrak{O}_S(S) I(g) \phi_Y(t) = \int\limits_0^\infty \left[\sum_j \int\limits_{G(S_j)} \cos(t, r \begin{pmatrix} \vartheta \\ y(\vartheta) \end{pmatrix}) J_j^*(\vartheta) d\vartheta \right] r^{n-1} g(r) dr$$
$$+ i \int\limits_0^\infty \left[\sum_j \int\limits_{G(S_j)} \sin(t, r \begin{pmatrix} \vartheta \\ y(\vartheta) \end{pmatrix}) J_j^*(\vartheta) d\vartheta \right] r^{n-1} g(r) dr$$

where $(.,.)$ *denotes the Euclidean scalar product in* $\mathbb{R}^n$.

(c) On combining the representations in (a) and (b), and taking into account Remark 1, we get an alternative direct proof of Theorem 9 if Y has density $\varphi_{g,K}$.

(d) If U_S *is symmetrically distributed w.r.t. the origin,* $U_S \overset{d}{=} -U_S$, *then the imaginary parts of the integral representations in (a) and (b) vanish, and both* ϕ_{U_K} *and* ϕ_Y *are symmetric w.r.t. the origin.*

4.7 The class of *p*-generalized elliptically contoured distributions

The general principle for deriving geometric and stochastic representations of star-shaped distributions developed so far will be proved in this section to successfully apply to a class of distributions considered in Section 3.5 of (Arellano-Valle and Richter 2012) and including both the $l_{n,p}$-symmetric ones, accordingly represented in (Richter 2009), and the elliptically contoured ones, analogously dealt with in (Richter 2013).

According to Definition 6, Theorem 5 and Corollary 3, the p-generalized elliptically contoured uniform distribution on $\mathfrak{B}_{E_{a,p}}$ is defined for arbitrary $p > 0$ by

$$\omega_{a,p,q}(A) = D(n,a,p)O_{a,p,q}(A) \text{ with } D(n,a,p) = \frac{p^{n-1}\Gamma\left(\frac{n}{p}\right)}{a_1 \cdot \ldots \cdot a_n 2^n \left(\Gamma\left(\frac{1}{p}\right)\right)^n}$$

and with q satisfying $1/p + 1/q = 1$.

If a random vector $Y : \Omega \to \mathbb{R}^n$ follows the uniform probability distribution on $B_{a,p}$ then $U \sim \omega_{a,p,q} = \omega_{E_{a,p}}$ where U is a.s. defined as $U = Y/R$ and is independent of $R = h_{B_{a,p}}(Y)$. The latter, non-negative, random variable has the density described in Theorem 6(b).

Let $\mathfrak{O}(n)$ denote the set of all orthogonal $n \times n$ matrices. A random vector $Y : \Omega \to \mathbb{R}^n$ is said to follow a p-generalized elliptically contoured distribution $EC_{a,p,\nu,O}$ with parameters $a = (a_1, \ldots, a_n)^T, a_i > 0, i = 1, \ldots, n,\ p > 0,\ \nu \in \mathbb{R}^n$ and $O \in \mathfrak{O}(n)$ if there exists a random variable $R : \Omega \to [0, \infty)$ such that Y satisfies the stochastic representation

$$O^T(Y - \nu) \stackrel{d}{=} R \cdot U$$

where $U \sim \omega_{E_{a,p}}$ and U and R are stochastically independent. Note that Y has a density f_Y iff R has a density. If $O^T(Y - \nu)$ has the dgf g, i.e. if

$$f_Y(y) = C(g,a,p)g\left(\left|O^T(y - \nu)\right|_{a,p}\right), y \in \mathbb{R}^n$$

with $C(g,a,p) = C(g, B_{a,p})$ then R has the density

$$f_R(r) = I(g)^{-1} r^{n-1} g(r) I_{[0,\infty)}(r).$$

In this case, we write $Y \sim \Phi_{g,a,p,\nu,O}$ and $f_Y = \varphi_{g,a,p,\nu,O}$. Note that $\Phi_{g,a,p,0_n,I_n} = \Phi_{g,B_{a,p},0_n}$ where I_n denotes the $n \times n$-unit matrix. The measure $\Phi_{g,a,p,\nu,O}$ allows the geometric representation

$$\Phi_{g,a,p,\nu,O}(B) = \int_0^\infty \mathfrak{F}_{a,p}\left(O^T(B - \nu, r)\right) dF_R(r) \tag{6}$$

where $\mathfrak{F}_{a,p}(M,r) = \omega_{E_{a,p}}\left(\left[\frac{1}{r}M\right] \cap E_{a,p}\right), r > 0$.

Example 1. *In the case of dimension $n = 2$, Figure 1 shows the density $\varphi_{g,a,p,\nu,O}$ and contours of its superlevel sets where $g(r) = \exp\left\{-\frac{r^p}{p}\right\}, a = (3,1)^T, p$ takes several values, $\nu = (0,0)^T$ and*

$$O = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix},\ \alpha = 5\pi/3.$$

The matrix O causes an anticlockwise rotation through an angle of size $\pi/3$.

Remark 11 (On independent coordinates). *Let $(R, \Phi_1, \ldots, \Phi_{n-1}) = \left(T^E_{a,p}\right)^{-1}(Y)$ be the random p-generalized ellipsoidal coordinates of Y where $Y \sim \Phi_{g,a,p,0_n,O}$. According to Theorem 3, the generalized radius is $\left(\sum_{i=1}^n |Y_i/a_i|^p\right)^{1/p} = R$, and by the density transformation formula and Theorem 4,*

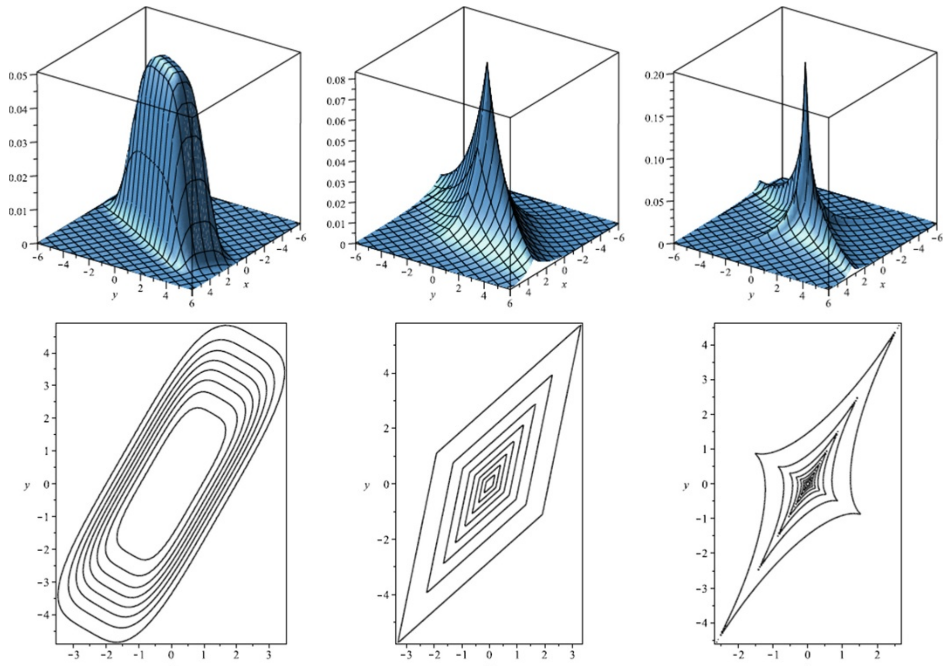

Figure 1 ***p*-generalized elliptically contoured densities for $p = 4$, 1 and 0.6, from left to right.**

$$f_{(R,\Phi_1,\ldots,\Phi_{n-1})}(r,\phi_1,\ldots,\phi_{n-1}) = \frac{C(g,a,p)}{a_2\cdot\ldots\cdot a_{n-1}} g(r) r^{n-1}\prod_{i=1}^{n-1} h_i(\phi_i)$$

with $h_i(\phi_i) = \frac{\left(\sin_{(a_i,a_{i+1};p)}(\phi_i)\right)^{n-i-1}}{N^2_{(a_i,a_{i+1};p)}(\phi_i)}$, $i = 1,\ldots,n-1$ *being integrable functions. Thus, suitably normalized, the functions* $h_0, h_1,\ldots,h_{n-1}$ *with* $h_0(r) = rg(r)I_{(0,\infty)}(r)$ *are the densities of* $R, \Phi_1,\ldots,\Phi_{n-1}$*, respectively.*

Hence, the random coordinates $R, \Phi_1,\ldots,\Phi_{n-1}$ *are stochastically independent.*

This result opens new perspectives for various applications which are not considered in the present paper.

5 Applications

5.1 The non-concentric elliptically contoured distribution class

The general distribution classes considered in the present paper include various interesting special cases to be studied only in detail in the future. Just to start with, let $0 < b < a$ and

$$K_{a,b} = \left\{(x,y)^T \in \mathbb{R}^2 : (x/a)^2 + (y/b)^2 \le 1\right\}.$$

A point $(e,f)^T$ is from the inner part of $K_{a,b}$ iff it satisfies the inequality $(e/a)^2 + (f/b)^2 < 1$. For such points, we put $K_{a,b,e,f} = K_{a,b} - (e,f)^T$. As because $rK_{a,b,e,f} = K_{ar,br,er,fr}$, for arbitrary dgf g, the level sets of the density

$$\varphi^*_{g,a,b,e,f}(x,y) = C\left(g, K_{a,b,e,f}\right) g\left(h_{K_{a,b,e,f}}\left((x,y)^T\right)\right), (x,y)^T \in \mathbb{R}^2$$

are the boundaries of the sets $K_{ar,br,er,fr}$, $r > 0$. Note that if $(e, 0)^T$ is a focal point of the elliptical disc $K_{a,b}$ then the origin $(0, 0)^T$ is a focal point of $E_{ar,br,er,0}$, for all $r > 0$. In this case, we call the origin a position of the density $\varphi^*_{g,a,b,e,0}$. Similarly, the origin may also be called a position of the density $\varphi^*_{g,a,b,0,f}$ if $(0, f)^T$ is a focal point of $K_{a,b}$. The distribution class

$$NCEC = \left\{\varphi^*_{g,a,b,e,f}\ ,\ g \text{ is a dgf}, 0 < b < a, (e/a)^2 + (f/b)^2 < 1\right\}$$

will be called a non-concentric elliptically contoured distribution class. It is left to the reader to derive explicit expressions for the Minkowski functional $h_{K_{a,b,e,f}}$ and the normalizing constant $C(g, K_{a,b,e,f})$.

5.2 Circular distributions

In this section, we study directional distributions on further using the results of the present work. For a recent overview on circular distributions we refer to (Pewsey et al. 2013).

In the case of dimension two, the density of $\Phi_{g,K,\nu}$ is

$$\varphi_{g,K,\nu}(x, y) = C(g, K)g\left(h_K\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}\right)\right),\ (x, y)^T \in \mathbb{R}^2.$$

We recall that star-generalized trigonometric functions and random polar coordinates are defined in (Richter 2011a) by $\cos_K(\phi) = \frac{\cos\phi}{h_K(\cos\phi,\sin\phi)}$, $\sin_K(\phi) = \frac{\sin\phi}{h_K(\cos\phi,\sin\phi)}$ and $X = R\cos_K(\Phi), Y = R\sin_K(\Phi)$, respectively. The K-generalized radius coordinate is $R = h_K(X, Y)$, and the angle Φ satisfies the representation of the usual polar angle,

$$\arctan(|Y/X|) = \Phi \text{ in } Q_1,\ = \pi - \Phi \text{ in } Q_2,\ = \Phi - \pi \text{ in } Q_3, =\ 2\pi - \Phi \text{ in } Q_4.$$

The Jacobian of this transformation is $rR_S^2(\phi)$ where the function $R_S(\phi) = 1/h_K(\cos\phi, \sin\phi)$ describes the boundary S of K:

$$S = \left\{R_S(\phi)\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, 0 \le \phi < 2\pi\right\} = \left\{(x, y)^T : h_K(x, y) = 1\right\}.$$

With uniquely determined $\mu \in [0, 2\pi)$ and $\lambda > 0$, the location vector $(\nu_1, \nu_2)^T$ can be represented as

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \lambda\begin{pmatrix} \cos_K(\mu) \\ \sin_K(\mu) \end{pmatrix}.$$

Thus, the density of $(R, \Phi)^T$ is

$$f_{(R,\Phi)}(r, \phi) = C(g, K)rR_S^2(\phi)g\left(h_K\left(\begin{pmatrix} r\cos_K(\phi) - \lambda\cos_K(\mu) \\ r\sin_K(\phi) - \lambda\sin_K(\mu) \end{pmatrix}\right)\right)I_{[0,2\pi)}(\phi)I_{[0,\infty)}(r).$$

Integrating $f_{(R,\Phi)}$ w.r.t. ϕ, and dividing $f_{(R,\Phi)}$ by the latter result, gives $f_{\Phi|R}(\phi|r) = vMd_{g,K,r,\lambda,\mu}(\phi)$ where

$$vMd_{g,K,r,\lambda,\mu}(\phi) = \frac{R_S^2(\phi)g\left(h_K\left(\begin{pmatrix} r\cos_K(\phi) - \lambda\cos_K(\mu) \\ r\sin_K(\phi) - \lambda\sin_K(\mu) \end{pmatrix}\right)\right)}{\int_0^{2\pi} R_S^2(\phi)g\left(h_K\left(\begin{pmatrix} r\cos_S(\phi) - \lambda\cos_S(\mu) \\ r\sin_S(\phi) - \lambda\sin_S(\mu) \end{pmatrix}\right)\right)d\phi}\ . \tag{7}$$

This function will be called a star-shaped generalization of the von Mises density which itself appears as a special case for K being the Euclidean unit disc in $\mathbb{R}^2$ and $g(r) = \exp\left\{-r^2/2\right\}$, $r > 0$, c.f. (von Mises 1918).

Example 2. *For illustrating our principle of constructing generalized von Mises densities at the hand of a concrete example, let P_n denote the polygon having the n vertices $I_{n,i} = \left(\cos\left(\frac{2\pi}{n}(i-1)\right), \sin\left(\frac{2\pi}{n}(i-1)\right)\right)^T, i = 1, \ldots, n, n \geq 3$, and let K_n be the convex body circumscribed by P_n. The Minkowski functional of K_n has been dealt with in (Richter, W-D, Schicker, K: Circle numbers of centered regular convex polygons, submitted). Figure 2 shows, from the left to the right, in each row, the polygonally contoured density $\varphi_{g,K_n,\nu}(x, y), (x, y)^T \in \mathbb{R}^2$, the contours of this density (black shapes) together with the polygonally generalized circle consisting of all points $(x, y)^T \in \mathbb{R}^2$ satisfying the condition $h_{K_n}\left((x, y)^T\right) = r$ (blue shape), and the polygonally generalized von Mises density $vMd_{g,K_n,r,\lambda}$ where $g(\rho) = \exp\left\{-\frac{\rho^2}{2}\right\}, \rho > 0, r = 1, \mu = 3\pi/4$ and $(n, \lambda) = \left(3, \frac{1}{2}\right)$ in the first row, but $(n, \lambda) = \left(4, \frac{3}{2}\right)$ in the second row. Each of the arrows in the middle panel shows the way from the center of the blue drawn polygonally generalized circle to that of the black drawn ones.*

6 Concluding remarks

Exact distributions of numerous statistics like mean value statistic, Student statistic, Chi-squared statistic have been derived in a broad and well known literature under the assumption that the sample vector follows a multivariate normal law, or more generally, an elliptically contoured one. Similarly, exact statistical distributions have been derived when the sample distribution is exponential or one of its geometric generalizations. For

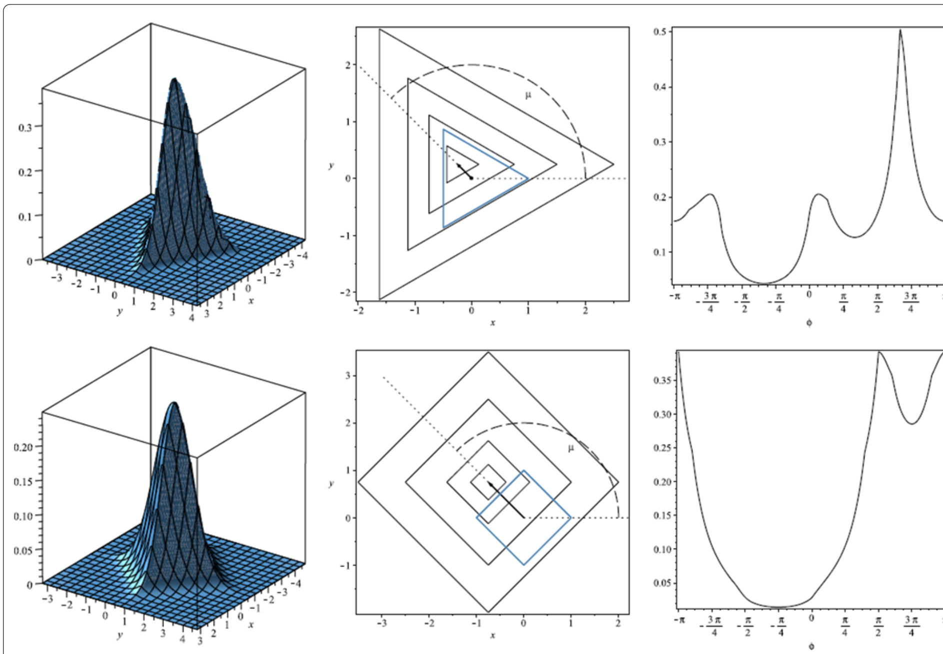

Figure 2 From left to right: polygonally contoured density, density contours with polygonally generalized circle describing the condition $h_{K_n}((x, y)) = 1$, polygonally generalized von Mises density, where $(n, \lambda) = (3, \frac{1}{2})$ in the first row and $= (4, \frac{3}{2})$ in the second row.

results of the latter type, we refer to (Henschel 2001, 2002; Henschel and Richter 2002). However, there seem, in general, not to be as many exact distributional results as in the Gaussian or elliptically contoured cases in the case of samples from any other multivariate distribution. Mathematical research without one of the above assumptions often deals with asymptotic considerations for large sample sizes. Many results of such work are again closely connected with properties of the Gaussian law which occurs often as a limit law.

The present paper provides new possibilities for deriving representations of exact statistical distributions if the sample vector follows a probability law coming from a rather big class of probability laws. It is not the place here to demonstrate in any detail all the possible applications of the present results. Instead, we refer to (Richter 2012) and several references given therein for getting a first overview. A recent example for deriving new results on distributions of functions of Gaussian vectors is given in (Richter 2014). This example might stimulate consideration of exact distributions of new classes of statistics.

Competing interests
The author declares that he has no competing interests.

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