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Retrieval and Use of the Balance Set in Multiobjective Global Optimization

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Abstract—It is shown, on examples, how to compute the balance set and the balance number in Vector Optimization Problems (VOPs) of different nature. New developments are presented concerning possible interrelation between the balance set and the balance number, a new notion of the projection of the balance set onto the parameter space, new approaches for solving VOPs with unbounded objective functions, and some approximation techniques in determining the balance set.

Keywords—Global multiobjective optimization, Balance sets, Interactive method, Feasible region.

1. INTRODUCTION

There is extensive literature on multiobjective optimization [1–38]. In a case of conflicting objectives, all known methods try to substitute several objective functions by a single one either for the whole problem (outright scalarization) or for subproblems solved in succession. Whatever the method, it presents a certain solution of some other problem, not the original one, which solution is then offered as a substitute that might be more or less good for the original problem. Once a solution is on hand, the margins $\eta_i = c - c_i$ can be calculated where c_i is the minimum for each partial scalar problem $\min f_i(x)$, $x \in X \subset R^n$, if computed beforehand, and c is the value of the objective function delivered by the solution of a scalarized version of the original problem. If the values η_i are unacceptable, then another scalarized problem should be constructed and solved to present other possible margins $\eta_{i1} = c_1 - c_i$ for approval, etc.

It would be clearly advantageous, if decision makers could have an equation, say,

$$\varphi(\eta) = \varphi(\eta_1, \dots, \eta_m) = 0, \tag{1.1}$$

from which they could choose the best possible values of η_i in advance. Such an equation determines a set in η-space called the *balance set* [39,40].

In this paper we consider vector optimization problems (VOPs)

$$\text{minimize } \{f_1(x), \dots, f_m(x)\}, \qquad \text{subject to} \quad x \in X, \tag{1.2}$$

where X is a compact set in R^n and f_i is a real-valued function over X, $i = 1, \dots, m$.

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In order to simplify the presentation, we briefly review some important concepts and definitions. For more detail, the reader is referred to [39,40].

Let $c_i = \min\{f_i(x),\ x \in X\}$, then $X_i = \{x \in X : f_i(x) = c_i\}$, $i = 1, \dots, m$, are referred to as the sets of exact partial global solutions of the VOP. We define the sets of η-precise partial global solutions as $X_{\eta i} = \{x \in X : f_i(x) - c_i \le \eta\}$; if $c_{\eta i} = \{\max f_i(x),\ x \in X_{\eta i}\}$, then $c_{\eta i} - c_i \le \eta$.

The VOP is said to be balanced if the set $X^0 = \bigcap_{i=1}^m X_i \neq \emptyset$, otherwise it is called unbalanced. Given $\eta > 0$, the VOP is said to be η-balanced if the set

$$X_\eta^0 = \bigcap_{i=1}^{m} X_{\eta i} \neq \emptyset, \tag{1.3}$$

otherwise it is called η-unbalanced. Here η is common for all solution sets $X_{\eta i}$.

The quantity $\eta_0 = \min\{\eta, X_\eta^0 \neq \emptyset\} \ge 0$ is called the balance number of the VOP.

The precision can be different for every objective function reflecting its relative importance. An m-tuple $\eta' = \{\eta_1, \dots, \eta_m\}$, $\eta_i \ge 0$, is called a balance point if the intersection

$$X_{\eta'}^0 = \bigcap_{i=1}^{m} X_{\eta_i} \neq \emptyset, \tag{1.4}$$

and every such intersection with one η_i replaced by a smaller positive number is empty. A set of all balance points is called the balance set. In many cases this set is defined by an equation, such as (1.1).

The examples collected in this paper show how to retrieve the balance set in different problems and what use can be made of it in the natural nonscalarized solution of VOPs. All examples are taken from the literature and present different situations. In this experimental study, we shall also see some new aspects of the balance set approach to multicriteria optimization.

2. TWO CRITERIA IN ONE VARIABLE

We start with a simple bi-objective problem developed by TenHuisen and Wiecek [30] in order to illustrate a scalarization method based on generalized Lagrangian duality

$$\text{minimize } \left\{\sqrt[3]{x+1}, 1-x\right\}, \qquad \text{subject to } \quad x \in X = [0,7]. \tag{2.1}$$

Here, exact partial solutions are $c_1 = 1$, $X_1 = \{0\}$, and $c_2 = -6$, $X_2 = \{7\}$. The problem is unbalanced. The η-precise solutions are

$$\begin{aligned} X_{\eta 1} &= \left\{x \in [0,7], \sqrt[3]{x+1} - 1 \le \eta\right\} = \left\{0 \le x \le (1+\eta)^3 - 1\right\} \cap [0,7], \\ X_{\eta 2} &= \{x \in [0,7], 1 - x + 6 \le \eta\} = \{7 - \eta \le x \le 7, x \ge 0\}. \end{aligned}$$

To determine the balance number η_0, consider the set

$$X_\eta^0 = X_{\eta 1} \cap X_{\eta 2} = \left[0, (1+\eta)^3 - 1\right] \cap [7-\eta, 7],$$

which is nonempty if and only if

$$(1+\eta)^3 - 1 \ge 7 - \eta \qquad \text{or} \qquad (1+\eta)^3 + \eta \ge 8.$$

This yields $\eta \ge 0.92$ and $\eta_0 = 0.92$.

Let (η_1, η_2) be a balance point. The set $X_{\eta'}^0$ is a singleton if and only if

$$7 - \eta_2 = (1+\eta_1)^3 - 1,$$

that is,

$$(1 + \eta_1)^3 + \eta_2 = 8, \tag{2.2}$$

which defines the balance set of this VOP. Using equations (2.2), one can assign appropriate feasible margins η_1, η_2 and see all the possibilities (Figure 1).

If $(1 + \eta_1)^3 + \eta_2 < 8$, no solutions (shaded area).

If $(1 + \eta_1)^3 + \eta_2 = 8$, one single (η_1, η_2)-precise solution.

If $(1 + \eta_1)^3 + \eta_2 > 8$, a continuum of solutions permitting the introduction of a third objective function onto a set of acceptable values of the first two objective functions. Setting $\eta_1 = \eta_2 = \eta_0$, we obtain the balance number $\eta_0 = 0.92$, which in this case belongs to the balance curve, see Figure 1.

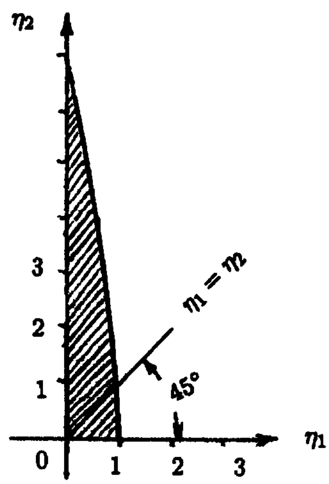

Figure 1.

3. THREE CRITERIA IN TWO VARIABLES

The next example comes from Chankong and Haimes [3, pp. 167, 227, 276] where it was analyzed in detail and used for the explanation of the weighting and ε-constraint scalarizations. Consider the VOP

$$\begin{aligned} &\text{minimize } \left\{(x_1 - 3)^2 + (x_2 - 2)^2, x_1 + x_2, x_1 + 2x_2\right\}, \\ &\text{subject to } \quad x \in X = \{x \in R^2 : x_1 \geq 0, x_2 \geq 0\}. \end{aligned} \tag{3.1}$$

Minimizing each objective function separately over the feasible set X, we get $c_1 = c_2 = c_3 = 0$, and $X_1 = \{(3, 2)\}$, $X_2 = X_3 = \{(0, 0)\}$. The sets X_{η_i} of η_i-precise partial global solutions are as follows:

$$\begin{aligned} X_{\eta_1} &= \left\{x \in X : (x_1 - 3)^2 + (x_2 - 2)^2 \leq \eta_1\right\}, \\ X_{\eta_2} &= \{x \in X : x_1 + x_2 \leq \eta_2\}, \\ X_{\eta_3} &= \{x \in X : x_1 + 2x_2 \leq \eta_3\}. \end{aligned} \tag{3.2}$$

Now, we consider the following system of equations:

$$(x_1 - 3)^2 + (x_2 - 2)^2 = \eta_1, \qquad x_1 + x_2 = \eta_2, \qquad x_1 + 2x_2 = \eta_3, \tag{3.3}$$

which yields the balance set equation in η-space after elimination of state variables x_1, x_2:

$$(2\eta_2 - \eta_3 - 3)^2 + (\eta_3 - \eta_2 - 2)^2 = \eta_1. \tag{3.4}$$

Indeed, for every fixed $(x_1, x_2) \in X$ the values of f_1, f_2, f_3 are equal to η_1, η_2, η_3 which are, thus, related by (3.4) yielding a surface in the three-dimensional η-space on one side of which

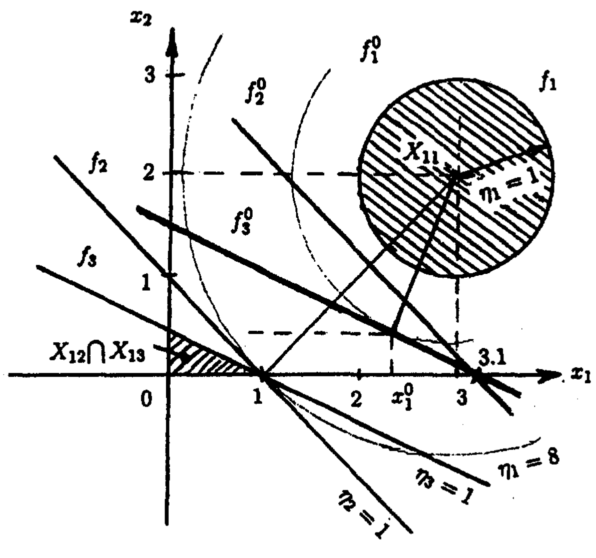

Figure 2.

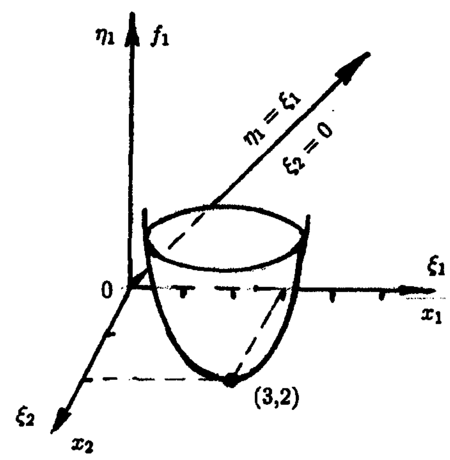

Figure 3.

there is no solution for $(x_1, x_2) \in X$ satisfying (3.2) with some assigned desirable η_1, η_2, η_3 and on the other side there is a continuum of equally good solutions for some augmented η_1, η_2, η_3. Which side is void is easy to determine. For example, let f_2, f_3 be more important than f_1 and take $\eta_2 \le 1$, $\eta_3 \le 1$. Then, by (3.4) we have $\eta_1 \ge 8$, so the side with $\eta_2 = \eta_3 = 1$, $\eta_1 < 8$ is void and the side with $\eta_1 > 8$ has plenty of solutions for inequalities (3.2), see Figure 2.

Returning to equation (3.4), we can get a better insight into the balance set by changing the system of coordinates from (η_1, η_2, η_3) to (η_1, ξ_1, ξ_2). Let

$$\xi_1 = 2\eta_2 - \eta_3, \qquad \xi_2 = \eta_3 - \eta_2,$$

then

$$\eta_2 = \xi_1 + \xi_2, \qquad \eta_3 = \xi_1 + 2\xi_2,$$

and equation (3.4) in the new coordinates becomes the paraboloid equation

$$\eta_1 = (\xi_1 - 3)^2 + (\xi_2 - 2)^2. \tag{3.5}$$

In fact, we can observe that this paraboloid is identical with $f_1 = (x_1 - 3)^2 + (x_2 - 2)^2$, which means that in this case the balance set, up to an affine change of coordinates, represents a surface given by one of the objective functions.

To find a balance number, one may set $\eta_1 = \eta_2 = \eta_3 = \eta$, then equation (3.4) becomes

$$(\eta - 3)^2 - \eta + 4 = 0. \tag{3.6}$$

There is no real η satisfying equation (3.6), so the balance number does not correspond to a point on the balance set. Indeed, the ray $\eta_1 = \eta_2 = \eta_3 \ge 0$ in the new coordinates corresponds to the ray $\xi_1 = \eta_1$, $\xi_2 = 0$, and does not intersect the circular paraboloid (3.5); see Figure 3.

It has to be emphasized that the balance number always exists although cannot be always found from the balance set, as this example shows. Let us find it directly from the definition. As shown in Figure 2, in the plane x_1Ox_2 the point $(1,0)$ corresponds to $\eta_2 = \eta_3 = 1$, $\eta_1 = 8$ which is on the balance set (3.4). However, we are interested in some minimal common value $\eta_1 = \eta_2 = \eta_3 = \eta_0$, for which (1.3) holds for the sets in (3.2). For $\eta_1 = \eta_2 = \eta_3 = 1$, the sets X_{11} and $X_{12} \cap X_{13}$ are shown in Figure 2 (shaded areas). To close the gap, we have to augment η_1, η_2, η_3 by moving f_1 down and f_2, f_3 up. Since we are looking for equal $\eta_2 = \eta_3 = \eta_0$ and since η_2, η_3 are given by

x_1-intercepts of f_2, f_3, the lines f_2, f_3 should be moved up in parallel and always intersecting on the x_1-axis until the first touch of f_3 and f_1 in order to have $X^0_\eta = X_{\eta 1} \cap X_{\eta 2} \cap X_{\eta 3} = \{x^0\}$ nonempty, as shown in Figure 2 for f^0_1, f^0_2, f^0_3. As the lines f^0_1, f^0_3 are tangent, we have

$$(x_1 - 3)dx_1 + (x_2 - 2)\,dx_2 = 0,$$
$$dx_1 + 2\,dx_2 = 0,$$
$$\left.\frac{dx_2}{dx_1}\right|_{f_1} = \frac{3 - x_1}{x_2 - 2} = \left.\frac{dx_2}{dx_1}\right|_{f_3} = -\frac{1}{2}.$$

This yields the system

$$(x_1 - 3)^2 + (x_2 - 2)^2 = \eta, \qquad x_1 + 2x_2 = \eta, \qquad 2(x_1 - 3) = x_2 - 2. \tag{3.7}$$

Eliminating x_1, x_2 from system (3.7), we get the equation for the balance number

$$\eta^2 - 19\eta + 49 = 0, \tag{3.8}$$

of which the smaller positive root yields the balance number $\eta_0 = 3.1$. The corresponding point x^0 in the state space, calculated from (3.7), is $x^0_1 = 2.22$, $x^0_2 = 0.44$ as shown on Figure 2. At this point we have $\eta_1 = \eta_3 = \eta_0 = 3.1$, and the margin on f_2 is even better: $\eta_2 = x^0_1 + x^0_2 = 2.66 < \eta_0$.

This example illustrates a general situation which conforms to the definition of the balance number $\eta_0 = \min \eta$ such that condition (1.3) holds with $X_{\eta i}$ defined by *inequalities*. It also explains the possibility of the balance ray $\eta_1 = \eta_2 = \eta_3 \geq 0$ not intersecting the balance set.

4. PROJECTION OF THE BALANCE SET ONTO THE PARAMETER SPACE AND ITS USE

The VOP studied in this section was considered by Hwang *et al.* [12] using the ε-constraint scalarization:

$$\begin{aligned} &\text{minimize } \left\{-x_1x_2, (x_1 - 4)^2 + x_2^2\right\}, \\ &\text{subject to } \quad x \in X = \left\{x \in R^2 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 25\right\}. \end{aligned} \tag{4.1}$$

In this problem, partial global solutions are $c_1 = -156.25$, $X_1 = \{(12.5, 12.5)\}$ and $c_2 = 0$, $X_2 = \{(4, 0)\}$, respectively. The sets X_{η_i} of η_i-precise global solutions are as follows:

$$X_{\eta_1} = \{x \in X : -x_1x_2 - c_1 \leq \eta_1\}, \tag{4.2}$$
$$X_{\eta_2} = \left\{x \in X : (x_1 - 4)^2 + x_2^2 \leq \eta_2\right\}. \tag{4.3}$$

Sets corresponding to $\eta_1 = \eta_2 = 25$, 64, 100 are presented in Figure 4 (shaded for $\eta_1 = \eta_2 = 25$, 64).

It is evident that increasing η will close the gap. Since for $\eta_1 = \eta_2 = 64$ the intersection $X_{\eta_1} \cap X_{\eta_2}$ is empty and for $\eta_1 = \eta_2 = 100$ nonempty (shaded circular diangle in the center), so the balance number η_0 should be greater than 64 but less than 100. It is also clear that the balance set corresponds to the line of tangent points produced by circles and hyperbolas, which line we call the projection of the balance set onto the parameter space.

DEFINITION. *A set of points in the parameter space X corresponding to balance points in the η-space, i.e., to particular choices of η_i on the balance set, is called the projection of the balance set (in η-space) onto the parameter (or state) space X, simply the projection set.*

The projection set yields a set of solutions which, in some sense, are the best with respect to all given objectives, each solution corresponding to a balance point chosen on the balance set. As was shown in Section 3, the balance number may have no corresponding balance point (for the

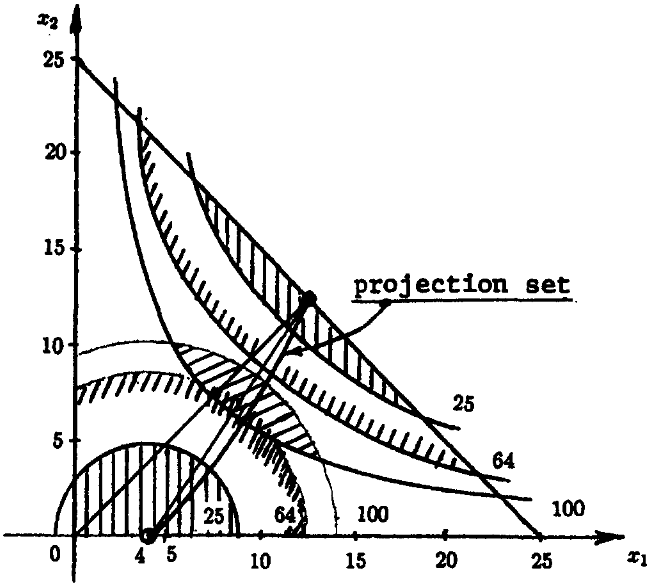

Figure 4.

ray η_i = const not intersecting the balance set) which provides another notion of best solutions corresponding to equal margins η_i = const.

It is worth noting that, in general, dimensions of the η-space and x-space are unrelated, $\dim \eta \neq \dim x$, so the projection set should not be confused with usual projections of points or sets in $X \subseteq R^n$ onto some subsets within the same state space R^n.

It is easy to compute the balance set and its projection onto the state space. Differentiating the equalities (4.2) and (4.3), we get

$$-x_1\, dx_2 - x_2\, dx_1 = 0, \qquad (x_1 - 4)\, dx_1 + x_2\, dx_2 = 0. \tag{4.4}$$

Equating derivatives

$$\left.\frac{dx_2}{dx_1}\right|_{f_1} = -\frac{x_2}{x_1} = \left.\frac{dx_2}{dx_1}\right|_{f_2} = \frac{4 - x_1}{x_2}, \tag{4.5}$$

yields the projection curve of the balance set

$$x_2^2 = x_1^2 - 4x_1. \tag{4.6}$$

To compute the balance set itself, we substitute x_2^2 from (4.6) into the equality of (4.3) to obtain

$$(x_1 - 4)(2x_1 - 4) = \eta_2, \qquad x_1 = 3 + \left(1 + \frac{\eta_2}{2}\right)^{1/2}. \tag{4.7}$$

With this x_1, we get from (4.6)

$$x_2^2 = \left(3 + \left(1 + \frac{\eta_2}{2}\right)^{1/2}\right)\left(\left(1 + \frac{\eta_2}{2}\right)^{1/2} - 1\right), \tag{4.8}$$

yielding, after the substitution into the equality of (4.2), the equation of the balance set

$$(\eta_1 + c_1)^2 = \left(3 + \left(1 + \frac{\eta_2}{2}\right)^{1/2}\right)^3 \left(\left(1 + \frac{\eta_2}{2}\right)^{1/2} - 1\right), \qquad c_1 = -156.25. \tag{4.9}$$

In this problem, one can find the balance number from the equation of the balance set (4.9) by equating $\eta_1 = \eta_2 = \eta$ and solving for η. However, it is simpler to use the projection (4.6) of the balance set. Indeed, taking equalities in (4.2), (4.3) with $\eta_1 = \eta_2 = \eta$, we get the system

$$-x_1x_2 - c_1 = \eta, \tag{4.10}$$
$$(x_1-4)^2 + x_2^2 = \eta, \tag{4.11}$$
$$x_2^2 = x_1^2 - 4x_1.$$

Eliminating η, x_2 we obtain the equation

$$(x_1-4)(2x_1-4) + x_1(x_1(x_1-4))^{1/2} = -c_1 = 156.25, \tag{4.12}$$

of which the first term represents the balance number $\eta_0 = \eta_1 = \eta_2$, due to (4.7). Solving (4.12), we obtain at the same time the point $x_1^0 \approx 9.6$, $x_2^0 \approx 7.3$, and an approximate balance number

$$\eta_0 = \left(x_1^0 - 4\right)\left(2x_1^0 - 4\right) \approx 85.$$

One can approximate the balance set, taking a linear approximation of (4.6) given by the line

$$x_2 = \left(\frac{25}{17}\right)(x_1 - 4), \tag{4.13}$$

passing through the points (4,0) and (12.5, 12.5). Consider again equations (4.2),(4.3), as well as (4.13), which yield an approximate balance set equation, cf. (4.9)

$$\eta_1 = -0.4649\eta_2 - 3.3076(\eta_2)^{1/2} + 156.25. \tag{4.14}$$

Equation (4.14) solved for $\eta_1 = \eta_2 = \eta$ produces the balance number $\eta_0 = 85.754$.

5. TWO OBJECTIVE FUNCTIONS IN TWO VARIABLES WITH ONE OBJECTIVE FUNCTION UNBOUNDED

In this section we analyze again the previous example but drop the feasibility constraint $x_1 + x_2 \le 25$, so that the problem is

$$\begin{aligned} &\text{minimize} && \left\{-x_1x_2, (x_1-4)^2 + x_2^2\right\}, \\ &\text{subject to} && x \in X = \left\{x \in R^2 : x_1 \ge 0, x_2 \ge 0\right\}. \end{aligned} \tag{5.1}$$

Due to this modification, this new problem is unbounded since $c_1 = -\infty$ and $X_1 = \{\infty, x_2 > 0\} \cup \{x_1 > 0, \infty\}$, while c_2 and X_2 do not change. There are two approaches to define the balance set for such problems.

APPROACH 1. For convenience, one can consider an equivalent problem

$$\text{maximize} \quad t = x_1x_2, \tag{5.2}$$
$$\text{subject to} \quad x_1 \ge 0,\ x_2 \ge 0, \tag{5.3}$$
$$(x_1-4)^2 + x_2^2 \le \eta_2, \tag{5.4}$$

with η_2 having the same sense as in (4.3).

Geometry of this problem is the same as depicted in Figure 4 with level curves given by the same circular arcs and hyperbolas. Thus, the solution is located on a circular arc corresponding to some given $\eta_2 > 0$, and computations are obvious:

$$\max t^2 = x_1^2\left[\eta_2 - (x_1-4)^2\right], \qquad x_1 > 0, \quad x_2 > 0, \tag{5.5}$$
$$\frac{dt^2}{dx_1} = 2x_1\left[\eta_2 - (x_1-4)^2\right] - 2x_1^2(x_1-4) = 0, \tag{5.6}$$
$$\eta_2 = (x_1-4)(2x_1-4) = 2(x_1-3)^2 - 2. \tag{5.7}$$

Now, given $\eta_2 \geq 0$, one can calculate $x_1 = 3 + (1 + \eta_2/2)^{1/2}$ from (5.7), then x_2 from the equality in (5.4), and finally $t_{\max}$ from (5.2). Alternatively, given $x_1 \geq 4$, one can calculate η_2 from (5.7), then x_2 and $t_{\max}$ as before.

If one applies the tangent point argument to curves in Figure 4, then the same solution is obtained. Indeed, fixing t, η_2, we have

$$\left.\frac{dx_2}{dx_1}\right|_{f_1} = -\frac{t}{x_1^2} = \left.\frac{dx_2}{dx_1}\right|_{f_2} = -\frac{x_1 - 4}{x_2}, \tag{5.8}$$

yielding

$$\begin{aligned} x_2 &= \frac{x_1^2}{t}(x_1 - 4) = \frac{x_1}{x_2}(x_1 - 4), \\ \eta_2 &= (x_1 - 4)^2 + x_2^2 = (x_1 - 4)(2x_1 - 4), \end{aligned} \tag{5.9}$$

the same as in (5.7).

Since there is no point in introducing the margin η_1 as a difference between $t_{\max} = \infty$ in (5.2), (5.3), cf. (5.1), and some actual finite value of t in (5.2)–(5.4), it is sensible to identify $\eta_1 = -t_{\max}$ corresponding to $\min(-x_1 x_2)$ in (5.1). Then, we can obtain an analog to the balance set by eliminating x_1 from (5.5),(5.7), yielding

$$\begin{aligned} t_{\max}^2 = \eta_1^2 &= \left[3 + \left(1 + \frac{\eta_2}{2}\right)^{1/2}\right]^2 \left[\eta_2 - \left(\left(1 + \frac{\eta_2}{2}\right)^{1/2} - 1\right)^2\right] \\ &= \frac{(\eta_2^2)}{4} + 2\sqrt{2}(2 + \eta_2)^{3/2} + 10\eta_2 - 8. \end{aligned} \tag{5.10}$$

The notion of the balance number is also applicable to this problem if we consider $\eta_2 = -t_{\max} = \eta_1 = \eta$. Then equation (5.10) becomes

$$\left(-\frac{3}{4}\right)\eta^2 + 2\sqrt{2}(2 + \eta)^{3/2} + 10\eta - 8 = 0,$$

and yields the balance number $\eta_0 = 38.224$.

APPROACH 2. Since $c_1 = -\infty$, one can assume a preferred finite value of the objective f_1, say $-M$, where $M > 0$. Then

$$-t - (-M) = \eta_1,$$

hence $t = M - \eta_1$, and the balance set equation (in the usual sense) is given by (5.10) with $t_{\max}^2$ substituted by $(M - \eta_1)^2$.

Let $M = 100$, then the balance set equation is

$$\frac{\eta_2^2}{4} + 2\sqrt{2}(2 + \eta_2)^{3/2} + 10\eta_2 - 8 = (100 - \eta_1)^2. \tag{5.11}$$

For $\eta_1 = \eta_2 = \eta$, equation (5.11) becomes

$$-\frac{3}{4}\eta^2 + 2\sqrt{2}(2 + \eta)^{3/2} + 210\eta - 10008 = 0,$$

and yields the balance number $\eta_0 = 51.961$. If we take $M = 156.25$, as $-c_1$ in the previous section, we obtain the same balance number $\eta_0 = 85.7$ as before. The reader can check that the right-hand sides of equations (4.9) and (5.10) are identical.

6. THREE SPHERICAL COST FUNCTIONS IN TWO VARIABLES

In this section we present an example that leads to some interesting general results on the balance number for specially structured VOPs. Consider the VOP

$$\begin{aligned} &\text{minimize} && \left\{(x_1-1)^2+(x_2-1)^2,(x_1-2)^2+(x_2-3)^2,(x_1-4)^2+(x_2-2)^2\right\}, \\ &\text{subject to} && x \in X = \left\{x \in R^2 : x_1+2x_2 \le 10, x_2 \le 4, x_1 \ge 0, x_2 \ge 0\right\}, \end{aligned} \tag{6.1}$$

which was analyzed in [3, p. 225] by means of the weighting scalarization.

Since the set of exact partial global solutions of this problem is obvious, we proceed to the balance set. From

$$\begin{aligned} (x_1-1)^2+(x_2-1)^2 &= \eta_1, \\ (x_1-2)^2+(x_2-3)^2 &= \eta_2, \\ (x_1-4)^2+(x_2-2)^2 &= \eta_3, \end{aligned} \tag{6.2}$$

we get

$$\begin{aligned} x_1 &= \frac{1}{10}(\eta_1+\eta_2-2\eta_3+25), \\ x_2 &= \frac{1}{10}(2\eta_1-3\eta_2+\eta_3+15), \end{aligned} \tag{6.3}$$

which yields three equivalent equations

$$(\eta_1+\eta_2-2\eta_3+15)^2+(2\eta_1-3\eta_2+\eta_3+5)^2 = 100\eta_1, \tag{6.4}$$

$$(\eta_1+\eta_2-2\eta_3+5)^2+(2\eta_1-3\eta_2+\eta_3-15)^2 = 100\eta_2, \tag{6.5}$$

$$(\eta_1+\eta_2-2\eta_3-15)^2+(2\eta_1-3\eta_2+\eta_3-5)^2 = 100\eta_3, \tag{6.6}$$

each of which reduces to the unique equation of the balance set

$$(\eta_1-\eta_2)^2+(\eta_2-\eta_3)^2-10(\eta_1+\eta_3)+50=0. \tag{6.7}$$

Setting $\eta_1 = \eta_2 = \eta_3 = \eta$, equation (6.7) produces the balance number $\eta_0 = 2.5$.

Using this number in (6.3), we get the global 2.5-minimizer $x_1^0 = 2.5$, $x_2^0 = 1.5$ equally good for every objective function in (6.1). Not surprisingly, it is located in the center of the circumference passing through the centers of all three objective functions. If objective functions in (6.1) are valued differently, say, according to $\eta_1 = 2$, $\eta_2 = 3$, $\eta_3 = 5$, then one has first to check whether they satisfy the balance set equation (6.7). If they do, then plugging them in (6.3) yields the corresponding global minimizer which in this case is unique. If they do not satisfy (6.7) as in the case with the values 2, 3, 5 above, that yield $-15 < 0$ in (6.7), then one has to decide which objectives are more important, fix them, say, $\eta_1 = 2$, $\eta_2 = 3$, and consider (6.7) as an equation for the remaining η_3. If that equation does not have positive real solutions, it means that already $\eta_1 = 2$, $\eta_2 = 3$ are not realizable. If the equation has positive real solutions, take the smallest one η_3^* and compare it with the given η_3. If $\eta_3^* > \eta_3$, then realizable margins are $(\eta_1, \eta_2, \eta_3^*)$ unless one is ready to change all three margins and try again. If, as in our case, $\eta_3^* = 3.1 < 5 = \eta_3$, there is continuum of solutions given by (6.3) for appropriate η_1, η_2, η_3', $\eta_3^* \le \eta_3' \le \eta_3$. In such a case, one can also improve all three margins provided they are on the surface (6.7) or within the feasible side of it.

One can see that the balance set solution is different from the solution presented in [3, pp. 225–227]. It admits an obvious generalization as follows.

THEOREM. *If all spherical objective functions are such that their centers are feasible and lie on a single sphere whose center is in the feasible region, then the balance number η_0 is equal to the square of the radius of the common sphere and the unique global η_0-minimizer is the center of that sphere.*

7. TWO SPHERICAL OBJECTIVE FUNCTIONS IN THREE VARIABLES

In this example we deal with spherical objective functions again. The problem comes from [3, p. 297] where it was used to illustrate Geoffrion's bicriterion method [9] based on the weighting scalarization,

$$\begin{aligned} &\text{minimize } \left\{(x_1-1)^2+x_2^2+(x_3-2)^2,(x_1-2)^2+(x_2-1)^2+x_3^2\right\}, \\ &\text{subject to } \quad x \in X=\left\{x \in R^3 : x_1+x_2+x_3 \leq 6, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\right\}. \end{aligned} \tag{7.1}$$

By the theorem in Section 6, the balance number $\eta_0 = (1/4)[(2-1)^2+1^2+(-2)^2] = 1.5$, and the corresponding 1.5-minimizer is $(x_1^0, x_2^0, x_3^0) = (1.5, 0.5, 1)$. The points of the projection of the balance set are such feasible points of this VOP that are located on the segment joining two centers of the spheres in (7.1) with parametric equations

$$\frac{x_1-1}{2-1}=\frac{x_2}{1}=\frac{x_3-2}{-2}=t, \qquad t \in [0,1]. \tag{7.2}$$

Since $x \in X$, for $t \in [0,1]$, so using

$$\begin{aligned} (x_1-1)^2+x_2^2+(x_2-2)^2 &= \eta_1, \\ (x_1-2)^2+(x_2-1)^2+x_3^2 &= \eta_2, \end{aligned} \tag{7.3}$$

we get the balance set equation in η-space

$$\eta_1 - 2(6\eta_1)^{1/2} + 6 = \eta_2, \tag{7.4}$$

from which, setting $\eta_1 = \eta_2 = \eta$, we obtain again the balance number $\eta_0 = 1.5$.

8. BALANCE SET CORRESPONDING TO A PIECE OF THE BOUNDARY OF THE FEASIBLE REGION

This VOP comes from Yu [36, p. 64], where he illustrated a scalarization technique based on goal programming. Consider the following problem:

$$\text{minimize} \quad \left\{-x_1-x_2, -10x_1+x_1^2-4x_2+x_2^2\right\}, \tag{8.1}$$

$$\begin{aligned} \text{subject to} \quad & x \in X=\left\{x \in R^2 : 3x_1+x_2-12 \leq 0, 2x_1+x_2-9 \leq 0,\right. \\ & \left. x_1+2x_2-12 \leq 0, x_1 \geq 0, x_2 \geq 0\right\}. \end{aligned} \tag{8.2}$$

Solving two constrained minimization subproblems, we get $c_1 = -7$ and $c_2 = -26.5$ with $X_1 = \{(2,5)\}$ and $X_2 = \{(3.5, 1.5)\}$. The first minimizer $X_1 = (2,5)$ is located at the intersection of the second and third constraint (point A), and the second minimizer $X_2 = (3.5, 1.5)$ is located at the first constraint (point C); see Figure 5. To improve the second objective at the expense of the first one without leaving the feasible region, one has to move from A to B and then to C along the segments AB and BC which, thus, represent the projection of the balance set.

Now, the computations are clear and two branches of the balance set corresponding to AB and BC are found by solving the two nonlinear systems

$$\begin{aligned} -x_1-x_2+7 &= \eta_1, \\ -10x_1+x_1^2-4x_2+x_2^2+26.5 &= \eta_2, \qquad \text{branch AB}, \\ 2x_1+x_2 &= 9, \end{aligned} \tag{8.3}$$

and

$$\begin{aligned} -x_1-x_2+7 &= \eta_1, \\ -10x_1+x_1^2-4x_2+x_2^2+26.5 &= \eta_2, \qquad \text{branch BC}, \\ 3x_1+x_2 &= 12, \end{aligned} \tag{8.4}$$

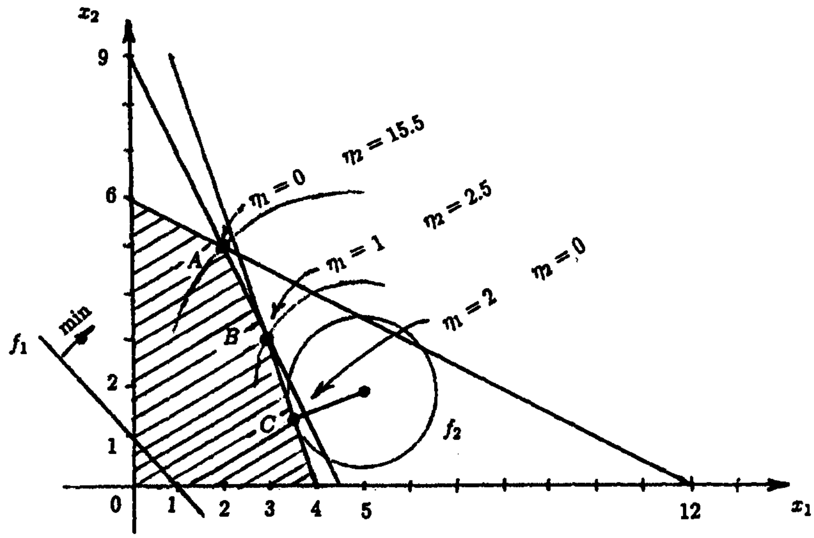

Figure 5.

which result in two curves in the η-space:

$$5\eta_1^2 - 18\eta_1 + 15.5 = \eta_2, \qquad \text{for the branch corresponding to AB,} \tag{8.5}$$

$$\frac{10}{4}\eta_1^2 - 10\eta_1 + 10 = \eta_2, \qquad \text{for the branch corresponding to BC,} \tag{8.6}$$

which compose the entire balance set projected in the state space as ABC. Equation (8.6) with $\eta_1 = \eta_2 = \eta$ gives the balance number $\eta_0 = 1.28$, with the corresponding minimizer $(x_1^0, x_2^0) = (3.14, 2.58)$, determined from (8.4) with $\eta_1 = \eta_2 = 1.28$.

9. A REAL LIFE EXAMPLE

We conclude with a real life example developed by Reid and Vemuri [20] and adapted by Chankong and Haimes [3, pp. 332–334] to illustrate an interactive method based on the concept of trade-offs. A dam is to be constructed such that the cost of construction f_1 and the water loss f_2 (volume/year) are minimized, and the total storage capacity f_3 of the reservoir is maximized. The decision variables are chosen to be total man-hours devoted to building the dam, x_1, and the mean radius of the lake impounded (in miles), x_2. The resulting VOP is

$$\begin{aligned} &\text{minimize} && \left\{e^{0.01x_1} x_1^{0.02} x_2^2, 0.5x_2^2, -e^{0.005x_1} x_1^{0.001} x_2^2\right\}, \\ &\text{subject to} && x \in X = \left\{x \in R^2 : x_1 \geq 0, x_2 \geq 0\right\}. \end{aligned} \tag{9.1}$$

Since $c_1 = c_2 = 0$ and $c_3 = -\infty$, our analysis is analogous to that in Section 5. Following Approach 1 of that section, this VOP leads to the following problem:

$$\begin{aligned} &\text{maximize} && t = e^{0.005x_1} x_1^{0.001} x_2^2, \\ &\text{subject to} && x \in X = \left\{x \in R^2 : x_1 \geq 0, x_2 \geq 0\right\}, \\ &&& e^{0.01x_1} x_1^{0.02} x_2^2 \leq \eta_1, \\ &&& 0.5x_2^2 \leq \eta_2. \end{aligned} \tag{9.2}$$

With $\eta_1 > 0$ and $\eta_2 > 0$, certainly $x_1 > 0$ and $x_2 > 0$, so that the balance set equation can be obtained from the following system:

$$e^{0.005x_1} x_1^{0.001} x_2^2 = t, \tag{9.3}$$

$$e^{0.01x_1} x_1^{0.02} x_2^2 = \eta_1, \tag{9.4}$$

$$0.5x_2^2 = \eta_2. \tag{9.5}$$

Eliminating x_2^2 from (9.3), (9.4) and then taking the root of degree 20 from both sides of (9.4), we obtain a parametric representation of the balance set in the (t, η_1, η_2)-space in the form

$$\frac{t}{2\eta_2} = e^{0.005x_1} x_1^{0.001}, \tag{9.6}$$

$$\sqrt[20]{\frac{\eta_1}{2\eta_2}} = e^{0.0005x_1} x_1^{0.001}. \tag{9.7}$$

It is not possible to obtain an explicit formula-like equation by eliminating x_1 from (9.6), (9.7). However, the surface given by (9.6), (9.7) can be easily tabulated or used otherwise to solve the problem by obtaining a picture of the interplay for best possible values of t, η_1, η_2. In our case, noting that the value of $x_1^{0.001}$ is close to 1 for very large x_1, we can drop this factor from (9.6), (9.7) to obtain an approximate balance set equation (by excluding x_1 after taking logarithms)

$$t^2 = 2\eta_1\eta_2. \tag{9.8}$$

Since storage capacity $f_3 = t$ of a reservoir always has a natural finite limit, a certain preferred capacity $M > 0$ can be assigned in practical cases. It is therefore sensible to use the second approach of Section 5 in the analysis of this problem, substituting t by $M - \eta_3$, which yields, instead of (9.8), an approximate balance set equation in the usual three-dimensional η-space

$$(M - \eta_3)^2 = 2\eta_1\eta_2. \tag{9.9}$$

Setting $\eta_1 = \eta_2 = \eta_3 = \eta$ in (9.9), one can determine the usual balance number from the resulting equation

$$\eta^2 + 2M\eta - M^2 = 0, \tag{9.10}$$

which has the only positive root $\eta_0 = M(\sqrt{2} - 1) \approx 0.4M$. Obviously, one has to pay attention to the units involved.

10. CONCLUSIONS

A collection of vector optimization problems (VOPs) is examined and the balance set is analytically derived for each problem. Bounded as well as unbounded problems with two or three objective functions are solved. All the problems have been previously studied in the literature in support of various scalarization techniques developed for VOPs. In this paper, these same examples serve to illustrate the concept of the balance set, whose equation can be an important tool for multiple criteria decision making.

New developments are presented concerning possible interrelation between the balance set and the balance number, a new notion of the projection of the balance set onto the parameter space, new approaches for solving VOPs with unbounded objective functions, and some approximation techniques in determining the balance set.

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