Jost Solutions and the Spectrum of the System of Difference Equations

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Abstract—In this paper, we investigate the analytical properties of the Jost solutions of non-self-adjoint system of difference equations of first order. Using the analytical properties of the Jost solutions, we also study the discrete spectrum of this system. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Let us consider the Sturm-Liouville equation

$$-y'' + q(x)y = \lambda^2 y, \quad 0 \leq x < \infty,$$

where $q$ is a real valued function and $\lambda$ is a spectral parameter. The bounded solution of (1.1) satisfying the condition

$$\lim_{x \to \infty} y(x, \lambda)e^{-i\lambda x} = 1, \quad \lambda \in \overline{\mathbb{C}}_+ := \{\lambda : \lambda \in \mathbb{C}, \ \text{Im} \ \lambda \geq 0\},$$

will be denoted by $e(x, \lambda)$. The solution $e(x, \lambda)$ is called Jost solution of (1.1). The Jost solution satisfies the equation

$$e(x, \lambda) = e^{i\lambda x} + \int_x^\infty \frac{\sin \lambda(t-x)}{\lambda} q(t)e(t, \lambda) \, dt.$$

It has been shown that, under the condition $\int_0^\infty x|q(x)| \, dx < \infty$, the Jost solution has the representation

$$e(x, \lambda) = e^{i\lambda x} + \int_x^\infty K(x, t)e^{i\lambda t} \, dt,$$

where the function $K(x, t)$ is defined by $q$. The representation (1.2) of the Jost solution of equation (1.1) plays an important role in the solutions of direct and inverse problems of quantum
scattering theory (see [1 Chap. 3; 2 Chap. 4]). Therefore, the similar representations for the Jost solutions of Dirac systems, quadratic pencil of Schrödinger, Klein-Gordon, discrete Schrödinger, and discrete Dirac equations have been obtained ([3-6]). Also, the spectral analysis of non-self-adjoint Sturm-Liouville, quadratic pencil of Schrödinger, Klein-Gordon, discrete Schrödinger, and discrete Dirac equations with spectral singularities have been investigated in detail, using the analytical properties of the Jost solutions ([7-14]).

Let us consider the non-self-adjoint system of difference equations of first order

\[
\begin{align*}
a_{n+1}y_{n+1}^{(2)} + b_n y_n^{(2)} &= p_n y_n^{(1)}, \\
a_{n-1} y_{n-1}^{(1)} + b_n y_n^{(1)} &= q_n y_n^{(2)},
\end{align*}
\]

where \(\{(y_n^{(1)}, y_n^{(2)})\}_{n \in \mathbb{Z}}\) is a vector sequence, \(\{a_n\}_{n \in \mathbb{Z}}, \{b_n\}_{n \in \mathbb{Z}}, \{p_n\}_{n \in \mathbb{Z}},\) and \(\{q_n\}_{n \in \mathbb{Z}}\) are complex sequences and \(a_n \neq 0, b_n \neq 0,\) for all \(n \in \mathbb{Z}.

In this paper, we study the analytical properties of the Jost solutions of (1.3) and using these properties we obtain eigenvalues and spectral singularities of (1.3). Note that, in the case of \(a_n > 0, b_n < 0,\) and \(p_n\) and \(q_n\) are real, for all \(n \in \mathbb{Z}\) (i.e., self-adjoint case of (1.3)), the Jost solutions of (1.3) have been examined in [15]. Also, various problems of difference equations of second order have been investigated by several authors (for the relevant references one may consult [16] or [17]).

2. JOOST SOLUTIONS OF (1.3)

We will assume that the complex sequences \(\{a_n\}_{n \in \mathbb{Z}}, \{b_n\}_{n \in \mathbb{Z}}, \{p_n\}_{n \in \mathbb{Z}},\) and \(\{q_n\}_{n \in \mathbb{Z}}\) satisfy

\[
\sum_{n \in \mathbb{Z}} |n| \left(|1 - a_n| + |1 + b_n| + |p_n| + |q_n|\right) < \infty.
\]

THEOREM 2.1. Under the condition (2.1) for \(\lambda = 2 \sin z/2\) and \(\text{Im} z = 0,\) equation (1.3) has the solutions \(f(z) = \left\{(f_n^{(1)}(z), f_n^{(2)}(z))\right\}_{n \in \mathbb{Z}}\) and \(g(z) = \left\{(g_n^{(1)}(z), g_n^{(2)}(z))\right\}_{n \in \mathbb{Z}}\) having the representations

\[
\begin{align*}
(f_n^{(1)}(z), f_n^{(2)}(z)) &= \alpha_n \left( E_2 + \sum_{m=1}^{\infty} A_{nm} e^{imz} \right) \left( e^{iz/2} - i \right) e^{inz}, \quad n \in \mathbb{Z}, \\
(g_n^{(1)}(z), g_n^{(2)}(z)) &= \beta_n \left( E_2 + \sum_{m=-\infty}^{m=-1} B_{nm} e^{-imz} \right) \left( e^{-iz/2} i \right) e^{-inz}, \quad n \in \mathbb{Z},
\end{align*}
\]

where

\[
\begin{align*}
\alpha_n &= \left( \begin{array}{cc} \alpha_n^{11} & \alpha_n^{12} \\ \alpha_n^{21} & \alpha_n^{22} \end{array} \right), \\
\beta_n &= \left( \begin{array}{cc} \beta_n^{11} & \beta_n^{12} \\ \beta_n^{21} & \beta_n^{22} \end{array} \right), \\
E_2 &= \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \\
A_{nm} &= \left( \begin{array}{cc} A_{nm}^{11} & A_{nm}^{12} \\ A_{nm}^{21} & A_{nm}^{22} \end{array} \right), \\
B_{nm} &= \left( \begin{array}{cc} B_{nm}^{11} & B_{nm}^{12} \\ B_{nm}^{21} & B_{nm}^{22} \end{array} \right).
\end{align*}
\]

PROOF. Substituting the vector valued function \(f\) defined by (2.2) in (1.3) taking \(\lambda = 2 \sin z/2\) and \(\text{Im} z = 0,\) we get the following

\[
\begin{align*}
\alpha_n^{11} &= \left[ \prod_{k=n+1}^{\infty} (-1)^{n-k} b_k a_{k-1} \right]^{-1}, \quad \alpha_n^{12} = 0, \\
\alpha_n^{22} &= \left[ b_n \prod_{k=n+1}^{\infty} (-1)^{n-k+1} b_k a_{k-1} \right]^{-1}, \quad \alpha_n^{21} = \alpha_n^{22} \left( p_n + \sum_{k=n+1}^{\infty} (p_k + q_k) \right),
\end{align*}
\]
\[ A_{n1}^{12} = - \sum_{k=n+1}^{\infty} (p_k + q_k), \]
\[ A_{n1}^{11} = \sum_{k=n+1}^{\infty} \left[ a_k a_{k+1} + b_k^2 - p_k q_k + (p_k + q_k) A_{k1}^{12} - 2 \right], \]
\[ A_{n1}^{22} = -1 + a_{n+1} a_n + (A_{n1}^{12})^2 + A_{n1}^{11}, \]
\[ A_{n1}^{21} = - \sum_{k=n}^{\infty} \left\{ (q_{k+1} + A_{k1}^{12}) \left[ a_{k+1} a_k + q_{k+1} (p_{k+1} + q_{k+1}) + q_{k+1} A_{k1}^{12} + b_{k+1}^2 + A_{k+1,11} - 1 \right] \right. \]
\[ - A_{k1}^{12} (1 + A_{k1}^{11}) \right\} + \sum_{k=n+1}^{\infty} (q_k A_{k1}^{22} - b_k^2 p_k), \]
\[ A_{n2}^{12} = -a_{n+1} a_n (q_{n+1} + A_{n1}^{12}) + A_{n1}^{12} A_{n1}^{11} + A_{n1}^{12} - A_{n1}^{21}, \]
\[ A_{n2}^{11} = \sum_{k=n+1}^{\infty} \left\{ (b_k^2 - 1) A_{k1}^{11} - a_{k+1} a_k \left[ (q_{k+1} + A_{k1}^{12}) A_{k1}^{12,1} - A_{k1}^{22,1} \right] \right. \]
\[ - (p_k - A_{k1}^{12}) \left( q_k A_{k1}^{11,1} + A_{k1}^{12,1} - A_{k1}^{22,1} \right) - q_k A_{k1}^{21} + A_{k1}^{12} A_{k1}^{22} - A_{k1}^{22} \right\}, \]
\[ A_{n2}^{22} = -a_{n+1} a_n (q_{n+1} + A_{n1}^{12}) A_{n1}^{12,1,1} + a_{n+1} a_n A_{n1}^{12,21} + A_{n1}^{21} A_{n1}^{12} - A_{n1}^{11} + A_{n1}^{21}, \]
\[ A_{n2}^{21} = \sum_{k=n}^{\infty} \left\{ A_{k1}^{22} A_{k2}^{11} + A_{k1}^{21} - a_{k+1} a_k \left[ (q_{k+1} + A_{k1}^{12}) A_{k1}^{12,1,1} - A_{k1}^{21} \right] \right\} \]
\[ - \sum_{k=n+1}^{\infty} \left( (q_k A_{k1}^{22} - A_{k1}^{11}) \left[ (q_k A_{k1}^{12,2} - A_{k1}^{11,2}) + b_k^2 A_{k2}^{21} - p_k A_{k2}^{22} + A_{k1}^{21} \right] \right), \]

where \( n \in \mathbb{Z} \). For \( m \geq 3 \) and \( n \in \mathbb{Z} \) we obtain that,
\[ A_{n,m}^{12} = -a_{n+1} a_n \left[ (q_{n+1} + A_{n1}^{12}) A_{n1}^{11,1,m-2} + A_{n1}^{21,1,m-2} \right] + A_{n1}^{12} A_{n1}^{11,1,m-1} + A_{n1}^{12} A_{n1}^{11,1,m-1} - A_{n1}^{21} A_{n1}^{11,1,m-1}, \]
\[ A_{n,m}^{11} = - \sum_{k=n+1}^{\infty} a_{k+1} a_k \left[ (q_{k+1} + A_{k1}^{12}) A_{k1}^{12,1,m-1} - A_{k1}^{22,1,m-1} \right] \]
\[ - \sum_{k=n+1}^{\infty} (p_k - A_{k1}^{12}) \left( q_k A_{k1}^{11,1,m-1} + A_{k1}^{12,1,m-1} - A_{k1}^{22,1,m-1} \right) + \sum_{k=n+1}^{\infty} (b_k^2 - 1) A_{k1}^{11,1,m-1} \]
\[ - \sum_{k=n+1}^{\infty} q_k A_{k1}^{21,1,m-1} + \sum_{k=n+1}^{\infty} A_{k1}^{12,1,m-1} - \sum_{k=n+1}^{\infty} A_{k1}^{22,1,m-1}, \]
\[ A_{n,m}^{22} = -a_{n+1} a_n \left[ (q_{n+1} + A_{n1}^{12}) A_{n1}^{12,1,m-1} - A_{n1}^{22,1,m-1} \right] + A_{n1}^{12} A_{n1}^{12,1,m-1} + A_{n1}^{12} A_{n1}^{11,1,m-1} - A_{n1}^{11} A_{n1}^{11,1,m-1}, \]
\[ A_{n,m}^{21} = - \sum_{k=n}^{\infty} a_{k+1} a_k \left[ (q_{k+1} + A_{k1}^{12}) A_{k1}^{12,1,m-1} - A_{k1}^{21,1,m-1} \right] \]
\[ - \sum_{k=n+1}^{\infty} (q_k + A_{k1}^{12}) (q_k A_{k1}^{21,1,m-1} + A_{k1}^{11,1,m-1} - A_{k1}^{22,1,m-1}) - \sum_{k=n+1}^{\infty} (b_k^2 - 1) A_{k1}^{12,1,m-1} \]
\[ + \sum_{k=n}^{\infty} A_{k1}^{22} A_{k2}^{21} + \sum_{k=n+1}^{\infty} q_k A_{k1}^{22} A_{k2}^{21} + \sum_{k=n}^{\infty} A_{k1}^{21} A_{k2}^{22} - \sum_{k=n+1}^{\infty} A_{k1}^{21} A_{k2}^{22} \]

Due to the condition (2.1), the infinite products and the series in the definition of \( \alpha_{ij}^m \) and \( A_{nm}^{ij} (i,j = 1,2) \) are absolutely convergent. Therefore, \( \alpha_{ij}^m \) and \( A_{nm}^{ij} (i,j = 1,2) \) can uniquely be defined by \( a_n, b_n, p_n, \) and \( q_n (n \in \mathbb{Z}) \), i.e., the system (1.3) for \( \lambda = 2 \sin z/2 \) and \( \text{Im} z = 0 \), has the solution \( f(z) = \left\{ \left( f_1^{(1)}(z), f_2^{(1)}(z) \right) \right\}_{n \in \mathbb{Z}} \) given by (2.2).

A similar result is valid for the solution \( g(z) = \left\{ \left( g_1^{(1)}(z), g_2^{(1)}(z) \right) \right\}_{n \in \mathbb{Z}} \) given by (2.3).
The solutions $f$ and $g$ are called Jost solutions of (1.3). It is obvious that the Jost solutions are defined and continuous on the real axis.

**THEOREM 2.2.** The Jost solutions of (1.3) have an analytic continuation from the real axis to the open upper half-plane.

**PROOF.** Using the equalities for $A^{ij}_{nm}$ ($i, j = 1, 2$) given in Theorem 2.1, we find

$$|A^{ij}_{nm}| \leq C \sum_{k=n+[m/2]}^{\infty} (|1 - a_k| + |1 + b_k| + |p_k| + |q_k|), \quad i, j = 1, 2, \quad (2.4)$$

by induction, where $[m/2]$ is the integer part of $m/2$ and $C > 0$ is a constant. In a similar way, we get

$$|B^{ij}_{nm}| \leq C \sum_{k=-\infty}^{n+[m/2]+1} (|1 - a_k| + |1 + b_k| + |p_k| + |q_k|), \quad i, j = 1, 2. \quad (2.5)$$

From (2.4) and (2.5), we obtain that the series in (2.2) and (2.3) are absolutely convergent in $\mathbb{C}_+$. This shows that the Jost solutions have an analytic continuation from the real axis to the open upper half-plane $\mathbb{C}_+ := \{ z : z \in \mathbb{C}, \text{Im} z > 0 \}$. We will also denote the analytic continuation of the Jost solutions again by (2.2) and (2.3).

Theorems 2.1 and 2.2 give the following.

**COROLLARY 2.3.** The Jost solutions of (1.3) are analytic functions in the open upper half-plane and continuous up to the real axis.

**THEOREM 2.4.** The following asymptotics hold:

$$f(z + 4\pi) = f(z), \quad g(z + 4\pi) = g(z).$$

### 3. EIGENVALUES AND SPECTRAL SINGULARITIES OF (1.3)

The Wronskian of the solutions $y(\lambda) = \{y^{(1)}_n(\lambda)\}_{n \in \mathbb{Z}}$ and $u(\lambda) = \{u^{(1)}_n(\lambda)\}_{n \in \mathbb{Z}}$ of (1.3) are defined by

$$W[y(\lambda), u(\lambda)] = a_n \left[ y^{(1)}_n(\lambda) u^{(2)}_{n+1}(\lambda) - y^{(2)}_n(\lambda) u^{(1)}_{n+1}(\lambda) \right].$$

Analogous to Sturm-Liouville and Dirac equations this Wronskian has the following properties:

1. $W[y(\lambda), u(\lambda)]$ is independent of $n$;
2. the necessary and sufficient condition for $W[y(\lambda), u(\lambda)] = 0$ is the linear dependence of the solutions $y(\lambda)$ and $u(\lambda)$.
Let us suppose that

\[ F(z) := W[f(z), g(z)] , \]

where \( f \) and \( g \) are the Jost solutions of (1.3). Using the properties of the Jost solutions, it is obtained that the function \( F \) is analytic in \( C_+ \), continuous in \( \bar{C}_+ \) and

\[ F(z + 4\pi) = F(z) . \]

Let \( P_0 \) and \( P \) denote the following infinite semistrip

\[ P_0 = \{ z : z = \xi + i\tau, \ 0 \leq \xi \leq 4\pi, \ \tau > 0 \} , \]
\[ P = \{ z : z = \xi + i\tau, \ 0 \leq \xi \leq 4\pi, \ \tau \geq 0 \} . \]

We also denote the set of eigenvalues and spectral singularities of (1.3) by \( \sigma_d \) and \( \sigma_{ss} \), respectively. By the definition of eigenvalues and spectral singularities, it can be written

\[ \sigma_d = \left\{ \lambda : \lambda = 2\sin \frac{z}{2}, \ z \in P_0, \ F(z) = 0 \right\} , \quad (3.1) \]
\[ \sigma_{ss} = \left\{ \lambda : \lambda = 2\sin \frac{z}{2}, \ z \in [0, 4\pi], \ F(z) = 0 \right\} . \quad (3.2) \]

Assume that

\[ M_1 = \{ z : z \in P_0, \ F(z) = 0 \} , \quad M_2 = \{ z : z \in [0, 4\pi], \ F(z) = 0 \} . \]

From (3.1) and (3.2), we get

\[ \sigma_d = \left\{ \lambda : \lambda = 2\sin \frac{z}{2}, \ z \in M_1 \right\} , \quad (3.3) \]
\[ \sigma_{ss} = \left\{ \lambda : \lambda = 2\sin \frac{z}{2}, \ z \in M_2 \right\} . \quad (3.4) \]

**Lemma 3.1.**

i) The set \( M_1 \) is bounded and has, at most, a countable number of elements and its limit points can lie only in \([0, 4\pi]\).

ii) The set \( M_2 \) is compact and \( \mu(M_2) = 0 \), where \( \mu \) denotes the Lebesque measure in the real axis.

**Proof.** Since the function \( F \) is of \( 4\pi \) periodic, it is enough to investigate the zeros of \( F \) in \( P \), instead of its zeros in \( \bar{C}_+ \). We have

\[ F(z) = \left[ \prod_{k \in \mathbb{Z}} (-1)^k a_k b_k \right]^{-1} [1 + o(1)] , \quad z \in P, \ \text{Im} z \to \infty , \quad (3.5) \]

by (2.7) and (2.8). Now, (3.5) shows the boundedness of \( M_1 \) and \( M_2 \). Since \( F \) is analytic function in \( C_+ \), consequently, we obtain that \( F \) has a countable number of zeros in \( P_0 \). By the uniqueness of analytic functions, we find that the limit points of \( M_1 \) can lie only in \([0, 4\pi]\). The closedness and the property of having Lebesque measure zero of \( M_2 \) can be obtained from the boundary uniqueness theorem of analytic functions [18].

From (3.3),(3.4) and Lemma 3.1, we get the following.

**Theorem 3.2.** Under the condition (2.1),

i) the set of eigenvalues of (1.3) is bounded and countable and its limit points can lie only in \([-2, 2]\);

ii) \( \sigma_{ss} \subset [-2, 2] \), \( \sigma_{as} = \sigma_{ss} \) and \( \mu(\sigma_{as}) = 0 \).
Lemma 3.3. If for some $\varepsilon > 0$

$$\sum_{n \in \mathbb{Z}} e^{\varepsilon|n|} \left( |1-a_n| + |1+b_n| + |p_n| + |q_n| \right) < \infty \quad (3.6)$$

holds, the function $F$ has a finite number of zeros in $P$ with a finite multiplicity.

Proof. From (2.4) and (2.5), we find that

$$|A_{nm}^{ij}| \leq Ce^{-\frac{3}{2}|n|(|n|+|m|)}, \quad i, j = 1, 2; \quad n \in \mathbb{Z}, \quad m = 1, 2, \ldots \quad (3.7)$$

$$|B_{nm}^{ij}| \leq Ce^{-\frac{3}{2}|n|(|n|+|m|)}, \quad i, j = 1, 2; \quad n \in \mathbb{Z}, \quad m = -1, -2, \ldots \quad (3.8)$$

where $C > 0$ is a constant. From (2.2), (2.3), (3.7) and (3.8), we observe that Jost solutions $f$ and $g$ have an analytic continuation to the half-plane $\text{Im} z > -\varepsilon/4$. Also, the function $F$ has the same property. So, the limit points of zeros of $F$ in $P$ cannot lie in $[0, 4\pi]$. Using Lemma 3.1, we get that the bounded sets $M_1$ and $M_2$ have a finite number of elements. From analyticity of $F$ in $\text{Im} z > -\varepsilon/4$, we obtain that all zeros of $F$ in $P$ have a finite multiplicity.

Definition 3.1. The multiplicity of a zero of $F$ in $P$ is called the multiplicity of the corresponding eigenvalue or spectral singularity of (1.3).

Using Lemma 3.3 and Definition 3.1, we have the following.

Theorem 3.4. Under condition (3.6), equation (1.3) has a finite number of eigenvalues and spectral singularities, and each is of finite multiplicity.

Now, we can ask the following question.

What is a weaker condition than (3.6), that guarantees the finiteness of eigenvalues and spectral singularities of (1.3)?

References

