Sufficient conditions for graphs to have \((g,f)\)-factors

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Abstract

We give sufficient conditions for a graph to have a \((g,f)\)-factor. For example, we prove that a graph \(G\) has a \((g,f)\)-factor if

\[ g(v) < f(v) \]

for all vertices \(v\) of \(G\) and

\[ \frac{g(x)}{\deg_G(x)} < \frac{f(y)}{\deg_G(y)} \]

for all adjacent vertices \(x\) and \(y\) of \(G\).

We consider finite graphs which may have loops and multiple edges. A graph without loops or multiple edges is called a simple graph. Let \(G\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\). For a vertex \(v\) of \(G\), we denote by \(\deg_G(v)\) the degree of \(v\) in \(G\). We write \(\delta(G)\) for the minimum degree of \(G\). For a subset \(X\) of \(V(G)\), we denote by \(G(X)\) the subgraph of \(G\) induced by \(X\). For two disjoint subsets \(X\) and \(Y\) of \(V(G)\), we denote by \(e_G(X,Y)\) the number of edges of \(G\) joining a vertex in \(X\) to a vertex in \(Y\). Let \(g\) and \(f\) be integer-valued functions defined on \(V(G)\) (this situation is denoted by \(g,f : V(G) \rightarrow \mathbb{Z}\)) such that \(g(v) < f(v)\) for all \(v \in V(G)\). Then a spanning subgraph \(F\) of \(G\) is called a \((g,f)\)-factor of \(G\) if \(g(v) \leq \deg_G(v) \leq f(v)\) for all \(v \in V(G)\). If \(g(v) = a\) and \(f(v) = b\) for all vertices \(v\), then a \((g,f)\)-factor is called an \([a,b]\)-factor. A graph \(G\) is called an \(r\)-regular graph if \(\deg_G(v) = r\) for all vertices \(v\) of \(G\). A graph \(G\) is said to be locally \(s\)-almost regular if \(|\deg_G(x) - \deg_G(y)| \leq s\) for any two adjacent vertices \(x\) and \(y\) of \(G\).

A criterion for a graph to have a \((g,f)\)-factor was found by Lovasz [4] (see lemma). Subsequently, many results about \((g,f)\)-factors and \([a,b]\)-factors have been obtained. For example, Joentgen and Volkmann [2] recently proved that a locally \(s\)-almost regular graph \(G\) has a \([k,k + t]\)-factor if \(1 \leq t\), \(0 \leq k \leq \delta(G)\) and \(sk \leq \delta(G)t\) (see Corollary 2). Other results related to our paper can be found in [1]. In this paper we shall prove the following theorem, which contains some known results, for instance, the above result of Joentgen and Volkmann.

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Theorem. Let $G$ be a connected graph and $g,f : V(G) \to \mathbb{Z}$ such that $g(v) \leq f(v)$, $g(v) \leq \deg_G(v)$ and $0 \leq f(v)$ for all vertices $v$ of $G$. Suppose that $G$, $g$ and $f$ satisfy the following three conditions:

(i) $G$ contains a vertex $w$ with $g(w) < f(w)$, or $g(x) = f(x)$ for all vertices $x$ of $G$ and $\sum_{x \in V(G)} f(x) = 0 \pmod{2}$.

(ii) For any two adjacent vertices $x$ and $y$ of $G$,

$$\frac{g(x)}{\deg_G(x)} \leq \frac{f(y)}{\deg_G(y)}.$$

(iii) For every proper non-empty subset $X$ of $V(G)$ such that $g(x) = f(x)$ for all $x \in X$ and $\langle X \rangle_G$ is connected, we have

$$\sum_{z \in V(G) \setminus X} e_G(z,X) \min \left\{ \frac{f(z)}{\deg_G(z)}, 1 - \frac{g(z)}{\deg_G(z)} \right\} \geq 1.$$

Then $G$ has a $(g,f)$-factor.

We first give some corollaries of this theorem. Note that if a graph in a corollary is not connected, we apply this theorem to each of its components. If $g(v) < f(v)$ for all vertices $v$, then Conditions (i) and (iii) of the theorem always hold, and thus we can obtain the following corollary.

Corollary 1. Let $G$ be a graph and $g,f : V(G) \to \mathbb{Z}$ such that $g(v) \leq \deg_G(v)$, $0 \leq f(v)$ and $g(v) < f(v)$ for all vertices $v$ of $G$. If

$$\frac{g(x)}{\deg_G(x)} \leq \frac{f(y)}{\deg_G(y)}$$

for any two adjacent vertices $x$ and $y$ of $G$, then $G$ has a $(g,f)$-factor.

Corollary 2 (Joentgen and Volkmann [2]). Let $k, s$ and $t$ be integers with $k, t \geq 1$ and $s \geq 0$, and let $G$ be a locally $s$-almost regular graph. If $\delta(G) \geq k$ and $sk \leq \delta(G)t$, then $G$ has a $[k,k+t]$-factor.

Proof. We define $g,f : V(G) \to \mathbb{Z}$ by $g(v) = k$ and $f(v) = k + t$ for all $v \in V(G)$. Let $\delta = \delta(G)$, and $x$ and $y$ be any two adjacent vertices of $G$. Since $sk \leq \delta t$, $\deg_G(y) \leq \deg_G(x) + s$ and $\delta \leq \deg_G(x)$, we have

$$\frac{g(x)}{f(y)} = \frac{k}{k+t} \leq \frac{\delta}{\delta+s} \leq \frac{\deg_G(x)}{\deg_G(y)}.$$

Hence $G$ has a $[k,k+t]$-factor by Corollary 1. □

The following corollary is also an immediate consequence of Corollary 1.
Corollary 3 (Kano and Saito [3]). Let $G$ be a graph and $g, f : V(G) \rightarrow \mathbb{Z}$ such that $g(v) < f(v)$ for all $v \in V(G)$. If there exists a real number $0 \leq \theta \leq 1$, such that $g(v) \leq \theta \deg_G(v) \leq f(v)$ for all $v \in V(G)$, then $G$ has a $(g,f)$-factor.

Corollary 4. Let $G$ be an $r$-regular simple graph and $g, f : V(G) \rightarrow \mathbb{Z}$ such that $g(v) < \deg_G(v)$, $0 < f(v)$ and $g(v) \leq f(v)$ for all $v \in V(G)$. Suppose that $g(x) < f(y)$ for any two adjacent vertices $x$ and $y$ and that for every vertex $u$ with $g(u) = f(u)$, there exists at least one vertex $w$ which is adjacent to $u$ and satisfies $g(w) < f(w)$. Then $G$ has a $(g,f)$-factor.

Proof. Let $X$ be a subset of $V(G)$ such that $g(x) = f(x)$ for all $x \in X$ and $X_G$ is connected. If $|X| > r$, then $e_G(X, V(G) \setminus X) > 1$ since for every vertex $x$ of $X$, there exists $y \in V(G) \setminus X$ such that $x$ and $y$ are adjacent. If $|X| < r$, then $e_G(X, V(G) \setminus X) > (r - |X| + 1) |X| > r$ since $G$ is a simple graph. Therefore,

$$\sum_{z \in V(G) \setminus X} e_G(z, X) \min \left\{ \frac{f(z)}{\deg_G(z)}, 1 - \frac{g(z)}{\deg_G(z)} \right\} \geq \frac{1}{r} = 1.$$

Consequently, $G$ has a $(g,f)$-factor by the theorem.

In [5, Theorem 4.1], Tutte proved that if $r, k$ are integers with $r > k > 0$, then every $r$-regular graph has a $[k, k + 1]$-factor. For simple graphs, we obtain the following refinement of this result of Tutte as an immediate consequence of Corollary 4.

Corollary 5. Let $r$ and $k$ be integers with $r > k > 0$, $G$ be an $r$-regular simple graph, and $W$ be a maximal independent subset of $V(G)$. Then $G$ has a $[k, k + 1]$-factor $F$ such that $\deg_F(v) = k$ for all $v \in V(G) - W$, as well as a $[k, k + 1]$-factor $H$ such that $\deg_H(v) = k + 1$ for all $v \in V(G) - W$.

We now prove our theorem by making use of the following lemma, which is known as the $(g,f)$-factor theorem.

Lemma (Lovász [4]). Let $G$ be a graph and $g, f : V(G) \rightarrow \mathbb{Z}$ such that $g(v) \leq f(v)$ for all vertices $v$ of $G$. Then $G$ has a $(g,f)$-factor if and only if

$$\gamma(S, T) := \sum_{s \in S} f(s) + \sum_{t \in T} (\deg_G(t) - g(t)) - e_G(S, T) - h(S, T) \geq 0$$

for all disjoint subsets $S$ and $T$ of $V(G)$, where $h(S, T)$ denotes the number of components $C$ of $G - (S \cup T)$ such that $g(x) = f(x)$ for all $x \in V(C)$ and

$$e_G(T, V(C)) + \sum_{x \in V(C)} f(x) \equiv 1 \pmod{2}.$$

Proof of the Theorem. Let $S$ and $T$ be disjoint subsets of $V(G)$. In view of the lemma, it suffices to show that $\gamma(S, T) \geq 0$. It is clear that $\gamma(\emptyset, \emptyset) = 0$ by Condition (i) of the
theorem. Thus assume that $S \cup T \neq \emptyset$, and let $C_1, \ldots, C_m \,(m = h(S, T))$ be the components of $G - (S \cup T)$ satisfying the conditions on $h(S, T)$ in the lemma. Then we get

$$
\gamma(S, T) = \sum_{s \in S} \deg_G(s) \frac{f(s)}{\deg_G(s)} + \sum_{t \in T} \deg_G(t) \left(1 - \frac{g(t)}{\deg_G(t)}\right) - e_G(S, T) - m
$$

$$
\geq \sum_s \sum_t e_G(s, t) \frac{f(s)}{\deg_G(s)} + \sum_t \sum_s e_G(s, t) \left(1 - \frac{g(t)}{\deg_G(t)}\right) - \sum_s \sum_t e_G(s, t)
$$

$$
+ \sum_{i=1}^m \left( \sum_s e_G(s, V(C_i)) \frac{f(s)}{\deg_G(s)} + \sum_t e_G(t, V(C_i)) \left(1 - \frac{g(t)}{\deg_G(t)}\right) - 1 \right).
$$

Since we have

$$
\sum_s \sum_t e_G(s, t) \left(\frac{f(s)}{\deg_G(s)} + \left(1 - \frac{g(t)}{\deg_G(t)}\right) - 1\right) \geq 0,
$$

by (ii), it now follows from (iii) that

$$
\gamma(S, T) \geq \sum_{i=1}^m \left( \sum_{z \in V(G) \setminus V(C_i)} e_G(z, V(C_i)) \min \left\{ \frac{f(z)}{\deg_G(z)}, 1 - \frac{g(z)}{\deg_G(z)} \right\} - 1 \right) \geq 0,
$$

as desired.

References


