On Convergence of Double Series

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For the convergence of double series and iterated series, a sufficient condition is obtained. This result provides a test for the convergence of double series and iterated series.

Key Words: double series; iterated series; uniform convergence; Antosik–Miku–sinski matrix theorem.

For \((n, m), (n', m') \in \mathbb{N}^2\), let \((n, m) \leq (n', m')\) if and only if \(n \leq n'\) and \(m \leq m'\). Then \(\mathbb{N}^2\) is a directed set and for elements \(x_{ij}\) of an abelian topological group, \(\{ \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} : (n, m) \in \mathbb{N}^2 \}\) is a net. If \(\lim_{(n, m)} \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij}\) exists, then we say that the double series \(\sum_{i,j} x_{ij}\) converges and

\[
\sum_{i,j} x_{ij} = \lim_{(n, m)} \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij}.
\]

For \(\sum_{i,j} x_{ij}\), there is an interesting result owing to Antosik:

**Theorem 1** [1, 6, 7]. Let \(G\) be a Hausdorff abelian topological group and let \(x_{ij} \in G\) for \(i, j \in \mathbb{N}\). If

(I) \(\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} x_{ijk}\) converges for every \(j_1 < j_2 < \cdots\), then
(II) the three series \(\sum_{i,j} x_{ij}, \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} x_{ij},\) and \(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}\) all converge and are equal.
Using this result Swartz [7] established a perfect Stiles-type Orlicz–Pettis theorem. However, as was stated in [6], condition (I) is so strong that the matrix

\[
\begin{bmatrix}
\frac{1}{ij} (-1)^{i+j}
\end{bmatrix}_{i,j}
\]

fails to satisfy it, though

\[
\sum_{i,j} \frac{(-1)^{i+j}}{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \frac{(-1)^{i+j}}{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^{i+j}}{ij} = \left[ \sum_{j=1}^{\infty} \frac{(-1)^{j}}{j} \right]^2.
\]

**Corollary 2.** Condition (I) is equivalent to

(III) For every sequence \( j_1 < j_2 < \cdots \) in \( \mathbb{N} \), the three series \( \sum_{i,k} x_{ij_k} \), \( \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} x_{ij_k} \), and \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij_k} \) all converge and are equal.

Usually, we are interested in (II) but, indeed, (III) is too much and, equivalently, (I) is too strong for (II). Therefore, in this note we will find a weaker condition which implies (II).

Recall that a net \( \{ x_d : d \in D \} \) is Cauchy if for every neighborhood \( U \) of \( 0 \in G \) there is a \( d_0 \in D \) such that \( x_d - x_{d'} \in U \) whenever \( d \geq d_0 \) and \( d' \geq d_0 \). \( G \) is complete if every Cauchy net in \( G \) converges to a point in \( G \). For complete metric spaces, there is a Moore lemma [3, p. 281] which gives a sufficient condition for the existence of double and iterated limits. It is clear that the Moore lemma is also valid for complete abelian topological groups.

**Lemma 3 (Moore lemma).** Let \( D_1 \) and \( D_2 \) be directed sets, and suppose that \( D_1 \times D_2 \) is directed by the relation \((d_1, d_2) \leq (d'_1, d'_2)\), which is defined to mean that \( d_1 \leq d'_1 \) and \( d_2 \leq d'_2 \). Let \( f : D_1 \times D_2 \to G \) be a net in the complete Hausdorff abelian topological group \( G \). Suppose that

(a) for each \( d_2 \in D_2 \), \( \lim_{d_1} f(d_1, d_2) \) exists, and

(b) \( \lim_{d_2} f(d_1, d_2) \) exists uniformly for \( d_1 \in D_1 \).

Then the three limits \( \lim_{(d_1, d_2)} f(d_1, d_2) \), \( \lim_{d_1} \lim_{d_2} f(d_1, d_2) \), and \( \lim_{d_2} \lim_{d_1} f(d_1, d_2) \) all exist and are equal.

**Theorem 4.** Let \( G \) be a complete Hausdorff abelian topological group and let \( x_{ij} \in G \) for \( i, j \in \mathbb{N} \). Suppose that

(1) \( \sum_{i=1}^{\infty} x_{ij} \) converges for each \( j \),

(2) \( \sum_{j=1}^{\infty} x_{ij} \) converges for each \( i \), and

(3) for every sequence \( p_1 < q_1 < p_2 < q_2 < \cdots \) in \( \mathbb{N} \) there is a sequence \( k_1 < k_2 < \cdots \) in \( \mathbb{N} \) such that the iterated series \( \sum_{i=1}^{\infty} \sum_{r=1}^{\infty} \sum_{j=p_{k_r}}^{q_{k_r}} x_{ij} \) converges.
Then the three series $\sum_{i,j} x_{ij}, \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} x_{ij},$ and $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}$ all converge and are equal.

Proof. By (1) and (2),

$$\lim_{n} \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} = \sum_{j=1}^{m} \sum_{i=1}^{\infty} x_{ij}$$

exists for each $m$ and

$$\lim_{m} \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{\infty} x_{ij}$$

exists for each $n$. If the series $\sum_{i=1}^{n} \sum_{j=1}^{\infty} x_{ij}$ converges uniformly for $n \in \mathbb{N}$, then the desired conclusion follows from Lemma 3.

Suppose that the convergence of $\sum_{i=1}^{n} \sum_{j=1}^{\infty} x_{ij}$ is not uniform with respect to $n \in \mathbb{N}$. Then there is a neighborhood $U$ of 0 $\in G$ for which the following holds:

for every $m_0 \in \mathbb{N}$, there is $m > m_0$ and $n \in \mathbb{N}$ with $\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} \notin U$. (*)

Pick a neighborhood $V$ of 0 $\in G$ with $V + V \subseteq U$. By (*), there exist $p_1 > 1$ and $n_1 \in \mathbb{N}$ such that

$$\sum_{i=1}^{n_1} \sum_{j=p_1}^{\infty} x_{ij} \notin U \quad \text{but} \quad \sum_{i=1}^{n_1} \sum_{j=q_1+1}^{\infty} x_{ij} \in V$$

for sufficiently large $q_1 > p_1$ and hence,

$$\sum_{i=1}^{n_1} \sum_{j=p_1}^{q_1} x_{ij} \notin V.$$

Let $p_0 > q_1$ be such that $\sum_{i=1}^{n} \sum_{j=p}^{\infty} x_{ij} \in U$ for all $1 \leq n \leq n_1$ and $p > p_0$. Then, by (*) again, there exist $p_2 > p_0 (> q_1)$ and $n_2 > n_1$ such that

$$\sum_{i=1}^{n_2} \sum_{j=p_2}^{\infty} x_{ij} \notin U$$

and, hence,

$$\sum_{i=1}^{n_2} \sum_{j=p_2}^{q_2} x_{ij} \notin V$$

for some $q_2 > p_2$. Continuing this construction we have integer sequences $n_1 < n_2 < \cdots$ and $p_1 < q_1 < p_2 < q_2 < \cdots$ such that

$$\sum_{i=1}^{n_k} \sum_{j=p_k}^{q_k} x_{ij} \notin V, \quad k = 1, 2, 3, \ldots \quad (***)$$
Consider the matrix \([\Sigma_{i=1}^{n_k} \Sigma_{j=p_m}^{q_m} x_{ij}]_{k,m}\). For each \(m\),
\[
\lim_{k} \sum_{i=1}^{n_k} \sum_{j=p_m}^{q_m} x_{ij} = \sum_{i=1}^{q_m} \sum_{j=p_m}^{\infty} x_{ij}
\]
evaluates by (1). Let \(m_1 < m_2 < \cdots \) in \(\mathbb{N}\). Then \(p_{m_1} < q_{m_1} < p_{m_2} < q_{m_2} < \cdots\) and, by condition (3), there is a sequence \(v_1 < v_2 < \cdots\) in \(\mathbb{N}\) such that the iterated series \(\sum_{i=1}^{\infty} \sum_{j=p_{m_i}}^{q_{m_i}} x_{ij}\) converges and, hence,
\[
\lim_{k} \sum_{i=1}^{n_k} \sum_{j=p_{m_k}}^{q_{m_k}} x_{ij} = \sum_{i=1}^{\infty} \sum_{j=p_{m_i}}^{q_{m_i}} x_{ij}.
\]

Now, by the Antosik–Mikusinski matrix theorem [2, 4],
\[
\lim_{k} \sum_{i=1}^{n_k} \sum_{j=p_k}^{q_k} x_{ij} = 0.
\]
This contradicts (***) and hence
\[
\lim_{m} \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}
\]
uniformly for \(n \in \mathbb{N}\).

Note that, without completeness of \(G\), conditions (1), (2), and (3) imply that \(\sum_{i=1}^{n} \sum_{j=1}^{\infty} x_{ij}\) converges uniformly for \(n \in \mathbb{N}\). Consider the condition

(2') the iterated series \(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}\) converges.

Clearly, (2') is stronger than (2).

**Theorem 5.** Let \(G\) be a Hausdorff abelian topological group which need not be complete, \(x_{ij} \in G\) for \(i, j \in \mathbb{N}\). If conditions (1), (2'), and (3) hold, then the three series \(\sum_{i=1}^{n} \sum_{j=1}^{\infty} x_{ij}, \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} x_{ij},\) and \(\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}\) all converge and are equal.

**Proof.** Since (2') \(\Rightarrow\) (2), as in the proof of Theorem 4,
\[
\lim_{m} \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}
\]
uniformly for \(n \in \mathbb{N}\).

Let \(U\) and \(V\) be neighborhoods of \(0 \in G\) such that \(V\) is symmetric and \(V + V \subseteq U\). There is an \(m_0 \in \mathbb{N}\) for which
\[
\sum_{i=1}^{n} \sum_{j=1}^{\infty} x_{ij} - \sum_{i=1}^{m} \sum_{j=1}^{m} x_{ij} \in V
\]
for all $n \in \mathbb{N}$ and $m \geq m_0$. Since the iterated series $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}$ converges, there is an $n_0 \in \mathbb{N}$ such that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{\infty} x_{ij} \in V$$

for all $n \geq n_0$. Therefore, if $n \geq n_0$ and $m \geq m_0$, then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{\infty} x_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{\infty} x_{ij} + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{\infty} x_{ij}$$

$$\in V + V$$

$$\subseteq U.$$

This shows that

$$\sum_{i,j} x_{ij} = \lim_{(n,m)} \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}.$$

Note that condition (I) implies (1), (2), (2'), and (3). Hence Theorem 4 implies Theorem 1.

**Corollary 6.** Let $G$ be a Hausdorff abelian topological group and let $x_{ij} \in G$ for $i, j \in \mathbb{N}$ be such that conditions (1) and (3) hold. Then (2') and (II) are equivalent, i.e., the three series $\sum_{i,j} x_{ij}$, $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}$, and $\sum_{i=j}^{\infty} \sum_{j=1}^{\infty} x_{ij}$ all converge and are equal if and only if the iterated series $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}$ converges.

A pseudometric is a real function $d$ on a set $X \times X$ such that $d(x, y) = d(y, x) \geq 0$, $d(x, x) = 0$, $d(x, z) \leq d(x, y) + d(y, z)$ [9, p. 10]. For a vector space $X$, a function $\|\cdot\|: X \to [0, +\infty)$ is a paranorm on $X$ if and only if there is a pseudometric $d$ on $X$ such that $d(x, y) = d(x - y, 0)$ for $x, y \in X$ and $\|x\| = d(x, 0)$ for $x \in X$ [9, p. 16]. A series $\sum x_j$ on a paranormed space $(X, \|\cdot\|)$ is said to be absolutely convergent if $\sum_{j=1}^{\infty} \|x_j\| < +\infty$. A paranormed space $X$ is complete if and only if every absolutely convergent series on $X$ is convergent in $X$ [9, p. 57].

Consider the condition

$$(3') \text{ for every sequence } p_1 < q_1 < p_2 < q_2 < \cdots \text{ in } \mathbb{N} \text{ there is a sequence } k_1 < k_2 < \cdots \text{ in } \mathbb{N} \text{ such that } \lim_{r} \sum_{i=1}^{n} \sum_{j=p_{k_{r}}+1}^{q_{k_{r}}} x_{ij} = 0 \text{ uniformly with respect to } n \in \mathbb{N}.$$
THEOREM 7. Let \((X, \| \cdot \|)\) be a complete Hausdorff paranormed space and let \(x_{ij} \in X\) for \(i, j \in \mathbb{N}\). If conditions (1), (2), and \((3')\) hold, then the three series \(\sum_{i,j} x_{ij}, \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij}\), and \(\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} x_{ij}\) all converge and are equal.

Proof. Let \(p_1 < q_1 < p_2 < q_2 < \cdots \in \mathbb{N}\). There is a sequence \(k_1 < k_2 < \cdots \in \mathbb{N}\) such that

\[
\lim_{n} \left\| \sum_{i=1}^{n} \sum_{j=p_{k_n}}^{q_{k_n}} x_{ij} \right\| = 0
\]

uniformly for \(n \in \mathbb{N}\). Hence there exists a sequence \(\nu_1 < \nu_2 < \cdots \in \mathbb{N}\) for which

\[
\sup_{n} \left\| \sum_{i=1}^{n} \sum_{j=p_{k_{\nu}}^{q_{k_{\nu}}}} x_{ij} \right\| \leq 2^{-r}, \quad r = 1, 2, 3, \ldots .
\]

Observing that \((X, \| \cdot \|)\) is complete,

\[
\sum_{r=1}^{\infty} \sup_{n} \left\| \sum_{i=1}^{n} \sum_{j=p_{k_{\nu}}^{q_{k_{\nu}}}} x_{ij} \right\| \leq \sum_{r=1}^{\infty} 2^{-r} = 1
\]

shows that the series \(\sum_{r=1}^{\infty} \sum_{i=1}^{n} \sum_{j=p_{k_{\nu}}^{q_{k_{\nu}}}} x_{ij}\) converges uniformly for \(n \in \mathbb{N}\). This shows that the series

\[
\sum_{i=1}^{n} \sum_{j=p_{k_{\nu}}^{q_{k_{\nu}}}} x_{ij} \quad \left( = \sum_{r=1}^{\infty} \sum_{i=1}^{n} \sum_{j=p_{k_{\nu}}^{q_{k_{\nu}}}} x_{ij} \right)
\]

converges uniformly for \(n \in \mathbb{N}\), i.e.,

\[
\lim_{m} \sum_{i=1}^{n} \sum_{r=1}^{m} \sum_{j=p_{k_{\nu}}^{q_{k_{\nu}}}} x_{ij} = \sum_{i=1}^{n} \sum_{r=1}^{\infty} \sum_{j=p_{k_{\nu}}^{q_{k_{\nu}}}} x_{ij}
\]

uniformly for \(n \in \mathbb{N}\).

On the other hand, by condition (1),

\[
\lim_{n} \sum_{i=1}^{n} \sum_{r=1}^{m} \sum_{j=p_{k_{\nu}}^{q_{k_{\nu}}}} x_{ij} = \sum_{r=1}^{m} \sum_{j=p_{k_{\nu}}^{q_{k_{\nu}}}}^{\infty} \sum_{i=1}^{n} x_{ij}
\]

exists for each \(m \in \mathbb{N}\).
Therefore, by Lemma 3, the iterated series \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} \) converges; i.e., condition (3) holds. Now the desired conclusion follows from Theorem 4.

**Example 1.** Let

\[
e_j = \left( \frac{j-1}{0, \ldots, 0, 1, 0, 0, \ldots} \right) \quad \text{for all } j.
\]

Then the double series \( \sum_{i,j} \left( \frac{(-1)^i}{\sqrt{i} \ln(j + 1)} e_{i+j} \right) \) converges in \( (C_0, \| \cdot \|_\infty) \). In fact, if \( p_1 < q_1 < p_2 < q_2 < \cdots \) in \( \mathbb{N} \), then for each \( n \in \mathbb{N} \), we have that

\[
\left\| \sum_{i=1}^{n} \sum_{j=p_k}^{q_k} \frac{(-1)^i}{\sqrt{i} \ln(j + 1)} e_{i+j} \right\|_\infty
= \left\| \sum_{j=p_k}^{q_k} \frac{1}{\ln(j + 1)} \sum_{i=1}^{n} \frac{(-1)^i}{\sqrt{i}} e_{i+j} \right\|_\infty
= \left\| \left( \frac{p_k}{0, \ldots, 0}, \frac{-1}{\ln(p_k+1)}, \frac{1}{\sqrt{2} \ln(p_k+1)}, \ldots, \right.ight.
+ \left. \left( \frac{p_k+1}{0, \ldots, 0, 0}, \frac{-1}{\ln(p_k+2)}, \frac{1}{\sqrt{2} \ln(p_k+2)}, \ldots, \right.ight.
+ \left. \left. \ldots \right. \right)
+ \left. \left( \frac{q_k}{0, \ldots, 0, 0}, \frac{-1}{\ln(q_k+1)}, \frac{1}{\sqrt{2} \ln(q_k+1)}, \ldots, \right. \right.
+ \left. \left. \ldots \right. \right) \right\|_\infty
\leq 3 \sup_m \left| \sum_{i=1}^{m} \frac{(-1)^i}{\sqrt{i}} \right| \left| \frac{1}{\ln(p_k+1)} \right|
\]
by the Abel transformation. Therefore,

$$\lim_{k} \left\| \sum_{i=1}^{n} \sum_{j=p_k}^{q_k} \frac{(-1)^i}{\sqrt{i} \ln(j+1)} e_{i+j} \right\|_{\infty} = 0$$

uniformly for \( n \in \mathbb{N} \).

Finally, we show that, sometimes, condition (3') is necessary for the convergence of \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_{ij} \).

**Example 2.** Let \( X \) be a Banach space with an unconditional basis \( \{e_j\} \) and \( x_j = \sum_{i=1}^{\infty} t_{ij} e_j \in X \) with \( t_{ij} \geq 0 \) for all \( i, j \in \mathbb{N} \). If the series \( \sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} t_{ij} e_j \) converges, then for every sequence \( p_1 < q_1 < p_2 < q_2 < \cdots \) in \( \mathbb{N} \),

$$\lim_{k} \left\| \sum_{i=1}^{n} \sum_{i=p_k}^{q_k} t_{ij} e_j \right\| = 0$$

uniformly for \( n \in \mathbb{N} \); i.e., (3') holds and \( \sum_{i=1}^{\infty} t_{ij} e_j = \sum_{i=1}^{\infty} x_i \).

To see this, observe that the continuity of the coordinate functionals with respect to the basis \( \{e_j\} \) implies that

$$\sum_{i=1}^{\infty} x_i = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} t_{ij} e_j = \lim_{n} \sum_{i=1}^{n} \sum_{j=1}^{\infty} t_{ij} e_j$$

$$= \lim_{n} \sum_{j=1}^{\infty} \left( \sum_{i=1}^{n} t_{ij} \right) e_j = \sum_{j=1}^{\infty} \left( \sum_{i=1}^{\infty} t_{ij} \right) e_j$$

because if \( \sum_{i=1}^{\infty} x_i = \sum_{j=1}^{\infty} t_{ij} e_j \), then \( t_j = \lim_{n} \sum_{i=1}^{n} t_{ij} = \sum_{i=1}^{\infty} t_{ij} \) for all \( j \).

Since \( \{e_j\} \) is unconditional, \( \sum_{i=1}^{\infty} a_j e_j \) converges if \( |a_j| \leq \sum_{i=1}^{\infty} t_{ij} \) for all \( j \) and, in particular, \( \sum_{j=1}^{\infty} \sum_{i=1}^{n} t_{ij} e_j \) converges for all \( n \) and \( j_1 < j_2 < \cdots \). Let \( D_j = \{ z \in \mathbb{C} : |z| \leq \sum_{i=1}^{\infty} t_{ij} \}, \ j = 1, 2, 3, \ldots \). If the convergence of \( \sum_{j=1}^{\infty} a_j e_j \) is not uniform for \( \{a_j\} \in \prod_{j=1}^{\infty} D_j \), then there exist an \( \epsilon > 0 \) and \( \{(a_{k,j})_{j=1}^{\infty} : k = 1, 2, 3, \ldots \} \subseteq \prod_{j=1}^{\infty} D_j \) and an integer sequence \( m_1 < n_1 < m_2 < n_2 < \cdots \) such that

$$\left\| \sum_{j=m_k}^{n_k} a_{kj} e_j \right\| \geq \epsilon, \quad k = 1, 2, 3, \ldots$$

Therefore, letting

$$a_j = \begin{cases} a_{kj}, & m_k \leq j \leq n_k \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$
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\{a_j\} \in \prod_{j=1}^\infty D_j \text{ but } \sum_{j=1}^\infty a_j e_j \text{ diverges. This contradiction shows that } 
\sum_{j=1}^\infty a_j e_j \text{ converges uniformly with respect to } \{a_j\} \in \prod_{j=1}^\infty D_j \text{ and hence, if } 
(a_{nj})_{j=1}^\infty, (a_j)_{j=1}^\infty \in \prod_{j=1}^\infty D_j \text{ are such that } \lim_n a_{nj} = a_j \text{ for all } j, \text{ then }

\lim_n \left\| \sum_{j=1}^\infty a_j e_j - \sum_{j=1}^\infty a_{nj} e_j \right\| = 0.

In particular,

\lim_n \left\| \sum_{k=1}^\infty \left( \sum_{i=1}^\infty t_{ij} \right) e_j - \sum_{k=1}^n \left( \sum_{i=1}^\infty t_{ij} \right) e_j \right\| = 0

for every \(j_1 < j_2 < \cdots\).

Now let \(p_1 < q_1 < p_2 < q_2 < \cdots\) in \(\mathbb{N}\). If

\lim_k \left\| \sum_{i=1}^n \sum_{j=p_k}^q t_{ij} e_j \right\| = 0

is not uniform with respect to \(n \in \mathbb{N}\), then there exist \(\epsilon > 0\) and integer sequences \(n_1 < n_2 < \cdots\) and \(k_1 < k_2 < \cdots\) such that

\[ \left\| \sum_{i=1}^{n_r} \sum_{j=p_{k_r}}^{q_{k_r}} t_{ij} e_j \right\| \geq \epsilon, \quad r = 1, 2, 3, \ldots \] \hspace{1cm} (\alpha)

Consider the matrix \([\sum_{i=1}^{n_r} \sum_{j=p_{k_r}}^{q_{k_r}} t_{ij} e_j]_{r,s}\). For each \(s \in \mathbb{N}\),

\lim_r \sum_{i=1}^{n_r} \sum_{j=p_{k_r}}^{q_{k_r}} t_{ij} e_j = \sum_{j=p_{k_r}}^{q_{k_r}} \left( \sum_{i=1}^\infty t_{ij} \right) e_j

exists. Suppose that \(s_1 < s_2 < \cdots\) in \(\mathbb{N}\). Then the series

\[ \sum_{v=1}^\infty \sum_{j=p_{k_v}}^{q_{k_v}} \left( \sum_{i=1}^\infty t_{ij} \right) e_j \]

converges and

\lim_r \left\| \sum_{v=1}^\infty \sum_{j=p_{k_v}}^{q_{k_v}} \left( \sum_{i=1}^\infty t_{ij} \right) e_j - \sum_{v=1}^\infty \sum_{j=p_{k_v}}^{q_{k_v}} \left( \sum_{i=1}^\infty t_{ij} \right) e_j \right\| = 0.

Therefore, the Antosik–Mikusinski matrix theorem shows that

\lim_r \left\| \sum_{i=1}^{n_r} \sum_{j=p_{k_r}}^{q_{k_r}} t_{ij} e_j \right\| = 0.
This contradicts (α) and hence

$$\lim_{k} \left\| \sum_{i=1}^{n} \sum_{j=p_k}^{q_k} t_{ij} e_j \right\| = 0$$

uniformly for $n \in \mathbb{N}$.

Note that Stuart [5] has also given a generalization of Antosik’s result but this generalization belongs to a different direction [8, p. 99].

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