Estimation of quadratic variation for two-parameter diffusions

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Abstract

In this paper we give a central limit theorem for the weighted quadratic variation process of a two-parameter Brownian motion. As an application, we show that the discretized quadratic variations \( \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} |\Delta_i, jY|^2 \) of a two-parameter diffusion \( Y = (Y(s,t))_{(s,t) \in [0,1]^2} \) observed on a regular grid \( G_n \) form an asymptotically normal estimator of the quadratic variation of \( Y \) as \( n \) goes to infinity.

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1. Introduction

In recent years, two-parameter stochastic processes have been used to model and study 2-dimensional spatial phenomena. In particular a great deal of attention was paid to medical images such as X-ray pictures of bones (cf. [12,14,23]) or digital mammograms [10] where one would like to test the presence or absence of anomalies from these data. In these applications, the texture of the image is modeled by a two-parameter stochastic process, as for example the fractional Brownian field (see [12]) or the fractional Brownian sheet [4,23] whose sample paths present fractal (or autosimilar) properties (determined respectively by one or two parameter, called the...
Hurst index(es) representing the roughness features of the phenomenon itself. The estimation of the Hurst index(es) from the data leads to diagnosis of the target anomalies as for example osteoporosis [4,12,14] or leads to a best understanding of the observed structure as e.g. the "Full-Filled Digital Mammograms" texture [10]. A major issue is consequently to obtain an accurate estimation of the Hurst index(es). One way to achieve this goal is to study the asymptotic behavior of the quadratic variations of the discretely observed process as was initially realized in [18]. Similar quantities have been studied in [9,10,12,14,23]. Regarding these applications, a natural next step in order to refine the statistical models and statistical methods in this area is the study of a generalisation of the quadratic variations which is the weighted quadratic variations of two-parameter processes and in particular of the fractional Brownian sheet.

In this paper, we study the asymptotic behavior of the weighted quadratic variation process of the standard Brownian sheet since conclusions obtained in this case can be considered as the most optimistic one can expect for a general fractional Brownian sheet.

First we give a central limit theorem for the weighted quadratic variation process of a two-parameter Brownian motion. More precisely if \( W = (W_{(s,t)})_{(s,t) \in [0,1]^2} \) is a two-parameter Brownian motion and if \( f : \mathbb{R} \to \mathbb{R} \) is a deterministic and regular enough function, we show in Theorem 3.1 that

\[
n \sum_{i=1}^{[nt]} \sum_{j=1}^{[nt]} f \left( W \left( \frac{i-1}{n}, \frac{j-1}{n} \right) \right) \left( |\Delta_{i,j} W|^2 - \frac{1}{n^2} \right) \xrightarrow{\text{law}(S)} \int_{[0,\cdot]\times[0,\cdot]} f \left( W(u,v) \right) dB(u,v),
\]

where \( B \) is a two-parameter Brownian motion independent of \( W \) and \( \Delta_{i,j} W \) denotes the increment of the process \( W \) on the subset \( \Delta_{i,j} := \left[ \frac{i-1}{n}, \frac{i}{n} \right] \times \left[ \frac{j-1}{n}, \frac{j}{n} \right] \) of \([0,1]^2\) defined by

\[
\Delta_{i,j} W := W \left( \frac{i-1}{n}, \frac{i-1}{n} \right) + W \left( \frac{i}{n}, \frac{j}{n} \right) - W \left( \frac{i-1}{n}, \frac{j-1}{n} \right) - W \left( \frac{i}{n}, \frac{j-1}{n} \right).
\]

The notation \( \text{law}(S) \) used above in (1.1) means that the convergence is in the sense of stable convergence in law in the two-parameter Skorohod space. In addition we stress that the limiting process is defined on an extension of the considered probability basis. Secondly, as an application, we deduce a central limit theorem (Theorem 4.2) for the quadratic variation process of a two-parameter diffusion \( Y = (Y_{(s,t)})_{(s,t) \in [0,1]^2} \) observed on a regular grid which allows us to construct an asymptotically normal consistent estimator of the quadratic variation process defined below by (1.5). Indeed, consider a two-parameter stochastic process \( (Y_{(s,t)})_{(s,t) \in [0,1]^2} \) defined by

\[
Y_{(s,t)} := \int_{[0,s] \times [0,t]} \sigma \left( W(u,v) \right) dW(u,v) + \int_{[0,s] \times [0,t]} M(u,v) \, du \, dv, \quad (s, t) \in [0,1]^2,
\]

where \( (W_{(s,t)})_{(s,t) \in [0,1]^2} \) is a two-parameter Brownian motion, \( \sigma : \mathbb{R} \to \mathbb{R} \) is a sufficiently smooth deterministic function and \( (M_{(s,t)})_{(s,t) \in [0,1]^2} \) is a continuous adapted process. Assume \( Y \) is observed on the regular grid \( G_n := \{(i/n, j/n) \mid 1 \leq i, j \leq n\} \). Let

\[
V^n_{(s,t)} := \sum_{i=1}^{[nt]} \sum_{j=1}^{[nt]} |\Delta_{i,j} Y|^2, \quad (s, t) \in [0,1]^2, \quad n \geq 1,
\]
and

\[ C_{(s,t)} := \int_{[0,s] \times [0,t]} \sigma^2(W_{(u,v)}) \, du \, dv, \quad (s, t) \in [0, 1]^2. \] (1.5)

Using (1.1) we show in Theorem 4.2 that

\[ \frac{1}{n} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} |\Delta i, j Y|^2 - \int_{[0,1] \times [0,1]} \sigma^2(W_{(u,v)}) \, du \, dv \right) \xrightarrow{law(S)} \sqrt{2} \int_{[0,1] \times [0,1]} \sigma^2(W_{(u,v)}) \, dB_{(u,v)}. \] (1.6)

Then (1.6) is used to prove that the consistent estimator \( V^n \) of \( C \) is asymptotically normal (cf. Corollary 4.5).

Similar results have been recently established in the one-parameter setting \([1,5,16,20,21]\) and applied to the estimation of the integrated volatility (see for example \([1]\) and references therein), testing for jumps of a process observed at discrete times like for example in \([2]\) or to construct a goodness-of-fit test for the integrated volatility \([16]\). The reader is also referred to \([5–7]\) where the study of the power variation process has been used to solve some financial econometric problems.

Note that in \([20,21]\), functional limit theorems for quite general functions of the variations of an Itô semimartingales are proved. We also mention that the asymptotic behavior of the weighted power variations of a one-parameter fractional Brownian motion was studied in \([26,27]\) (see also \([28]\) for similar questions about the iterated Brownian motion).

We proceed as follows. First we recall in Section 2 some elements of stochastic analysis of two-parameter processes. Actually we present some definitions concerning stochastic calculus of two-parameter processes and the definition of the two-parameter Skorohod space initially introduced in \([25]\) and in \([36]\). Secondly, in Section 3 we establish the central limit theorem (Theorem 3.1) for the weighted quadratic variation process of the two-parameter Brownian motion briefly presented in (1.1). As an application we prove in Section 4 that the consistent estimator \( V^n \) (1.4) of the quadratic variation \( C \) (1.5) is asymptotically normal (Corollary 4.5). Finally we present in an appendix (Appendix) some background on set-indexed processes and on the Malliavin calculus for the two-parameter Brownian motion which are used in Sections 3 and 4.

2. Stochastic analysis of two-parameter processes

In this section we recall some definitions about two-parameter stochastic analysis (we refer to \([15,17,29,30,35]\) for complete explanation about this topic) and about two-parameter Skorohod space (introduced in \([8,25,36]\)) which will be used in Sections 3 and 4.

Let \((\Omega, \mathcal{F}, (\mathcal{F}_z)_{z \in [0,1]^2}, \mathbb{P})\) be a filtered probability space. We denote the partial order relation \( \leq \) on \([0, 1]^2 \) defined by, \( z' \leq z \iff (s' \leq s \text{ and } t' \leq t) \), \( z' = (s', t') \), \( z = (s, t) \). Let \( z = (s, t) \in [0, 1]^2 \). We set \((\mathcal{F}_{(s,t)})_{(s,t) \in [0,1]^2} := \left( \bigvee_{s' \leq s \text{ or } t' \leq t} \mathcal{F}_{(s',t')} \right)_{(s,t) \in [0,1]^2} \) the strong past information filtration on \((\Omega, \mathcal{F}, \mathbb{P})\). Until the end of this paper we assume that the usual conditions (F1)–(F4) of \([13]\) are satisfied. We recall that an \((\mathcal{F}_{z})_{z \in [0,1]^2}\)-adapted process \((Y_z)_{z \in [0,1]^2}\) is said to be a martingale if for every \( z \) and \( z' \) in \([0, 1]^2 \) such that \( z \leq z' \), \( \mathbb{E}[Y_{z'}|\mathcal{F}_z] = Y_z \); a strong martingale if for all \( z \) and \( z' \) in \([0, 1]^2 \) such that \( z \leq z' \), \( \mathbb{E}[Y_{z,z'}|\mathcal{F}_z^*] = 0 \); where \( Y_{[z,z']} \) denotes the increments of \( Y \) on the interval \([z, z']\). As an example, we mention that the
two-parameter Brownian motion \((W_z)_{z \in [0,1]^2}\) is a strong martingale with respect to its natural filtration and a centered Gaussian process with covariance function,

\[
\mathbb{E}[W_{(s,t)} W_{(s',t')}]=(s \wedge s')(t \wedge t'), \quad (s, t), (s', t') \in [0, 1]^2.
\]

The space \(D([0, 1]^2)\) defined in \([25,36]\) is a two-parameter counterpart of the notion of càdlàg functions on \([0, 1]\). The set \(D([0, 1]^2)\) can be equipped with a metric \(d\) which makes it a Polish space and we denote by \(L_2\) the Borel \(\sigma\)-algebra on \((D([0, 1]^2), d)\). Moreover, compact sets on \((D([0, 1]^2), d, L_2)\) can be described thanks to a modulus of continuity \(w\) defined as

\[
w(f, \delta) := \sup_{\|(s,t) - (s',t')\| < \delta} |f(s, t) - f(s', t')|,
\]

where \(\|(s, t) - (s', t')\| := \max\{|s - s'|; |t - t'|\}, \quad (s, t), (s', t') \in [0, 1]^2\) and \(f : [0, 1]^2 \to \mathbb{R}\) is an element of \(D([0, 1]^2)\). This definition enables us to use techniques described in \([11]\) for one-parameter functions.

3. Central limit theorem

In this section we state and prove the functional limit theorem (Theorem 3.1) which will allow us to show in Section 4 that the consistent estimator \(V^n\) (1.4) of the quadratic variation \(C\) (1.5) is asymptotically normal (Corollary 4.5).

Let \(f : \mathbb{R} \to \mathbb{R}\) be a measurable deterministic function. Let a two-parameter Brownian motion \(W = (W_{(s,t)})_{(s,t) \in [0,1]^2}\) defined on a probability basis \(\mathcal{B} := (\Omega, \mathcal{F}, (\mathcal{F}_{(s,t)})_{(s,t) \in [0,1]^2}, \mathbb{P})\). Let also

\[
\xi_{i,j} := n f \left( W(\frac{i-1}{n}, \frac{j-1}{n}) \right) \left( |\Delta i,j W|^2 - \frac{1}{n^2} \right), \quad 1 \leq i, j \leq n, \quad n \geq 1.
\]

The re-normalized weighted quadratic variation process \(X^n = (X^n_{(s,t)})_{(s,t) \in [0,1]^2}\) is defined as

\[
X^n_{(s,t)} := \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} \xi_{i,j}, \quad (s, t) \in [0, 1]^2.
\]  

Stable convergence in law has been introduced in \([33, 34]\). It requires some particular care, since the limiting process \(X\) (see below) is not defined on the probability basis \(\mathcal{B} = (\Omega, \mathcal{F}, (\mathcal{F}_{(s,t)})_{(s,t) \in [0,1]^2}, \mathbb{P})\) on which the \(X^n, \quad n \geq 1\) are defined but on an extension \(\tilde{\mathcal{B}} := (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_{z})_{z \in [0,1]^2}, \tilde{\mathbb{P}})\) of \(\mathcal{B}\). We set the hypothesis we will need in the following theorem.

Hypothesis (H):

\[
\sup_{(s,t) \in [0,1]^2} \mathbb{E}\left[ |f(W_{(s,t)})|^p \right] < \infty, \quad p \in (0, 4].
\]

**Theorem 3.1.** Assume that the deterministic function \(f : \mathbb{R} \to \mathbb{R}\) satisfies hypothesis (H). Then \((X^n)_{n \geq 1}\) defined by (3.1) converges \(\mathcal{F}\)-stably in law in the Skorohod space \((D([0, 1]^2), d, L_2)\) to a non-Gaussian continuous process \(X\) defined below in the proof by (3.2). Moreover, \(X\) is defined on an extension of the probability basis \(\mathcal{B}\).

**Remark 3.2.** The definition of \(\mathcal{F}\)-stable convergence in law will be given in the proof of Theorem 3.1 but heuristically this convergence can be considered as the convergence in law.
of \((X^n, W)_n\) to \((X, W)\). Note also that the limiting process \(X\) is not a Gaussian process but it is conditionally Gaussian given the filtration generated by \(W\), that is \(\mathcal{F}\).

**Proof.** Let us first describe the extension of \(\mathcal{B}\) on which the limiting process \(X\) is defined. We denote by \(\mathcal{B}' := (\Omega', \mathcal{F}', (\tilde{F}_z')_{z \in [0,1]^2}, \mathbb{P}')\) the two-parameter Wiener space, that is \(\Omega' := C^0([0,1]^2)\) is the space of real-valued continuous functions on \([0,1]^2\) vanishing on the set \((s, t) \in [0,1]^2, \ s = 0 \text{ or } t = 0\). Set \(\mathbb{P}'\) the unique measure on \((\Omega', \mathcal{F}')\) under which the canonical process \((B_z)_{z \in [0,1]^2}\) on \(\Omega'\) defined by

\[
B_z(\omega') := \omega'(z), \quad \omega' \in \Omega', \ z \in [0,1]^2,
\]

is a standard two-parameter Brownian motion. Let the extension \(\tilde{\mathcal{B}} := (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{F}_z')_{z \in [0,1]^2}, \tilde{\mathbb{P}})\) defined as

\[
\begin{align*}
\tilde{\Omega} &:= \Omega \times \Omega', \\
\tilde{\mathcal{F}} &:= \mathcal{F} \otimes \mathcal{F}', \\
(\tilde{F}_z')_{z \in [0,1]^2} &:= (\cap_{\rho > z} \mathcal{F}^{\rho}_\rho \otimes \mathcal{F}'_{\rho})_{z \in [0,1]^2}, \\
\tilde{\mathbb{P}}(d\omega, dy) &:= \mathbb{P}(d\omega)\mathbb{P}'(dy).
\end{align*}
\]

We will denote by \(\mathbb{E}\) (respectively \(\tilde{\mathbb{E}}\)) the expectation under \(\mathbb{P}\) (respectively under \(\tilde{\mathbb{P}}\)). On \(\tilde{\mathcal{B}}\) we define the stochastic process \((X_z)_{z \in [0,1]^2}\) as

\[
X_z(\omega, \omega') := \sqrt{2} \left( \int_{[0,z]} f(W_\rho(\omega)) \, dB_\rho \right) (\omega'), \quad z \in [0,1]^2
\]

(3.2)

which is a \(\mathcal{F}\)-progressive conditional Gaussian martingale with independent increments on \(\tilde{\mathcal{B}}\); that is, \(X\) is an \((\tilde{F}_z')_{z \in [0,1]^2}\)-adapted process such that for \(\mathbb{P}\) almost \(\omega\) in \(\tilde{\Omega}\), \(X(\omega, \cdot)\) is a Gaussian process on \(\mathcal{B}'\) with covariance function

\[
\mathbb{E}_{\mathbb{P}} \left[ X(s_1, t_1) (\omega, \cdot) X(s_2, t_2) (\omega, \cdot) \right] \\
= 2 \int_{[s_1 \wedge s_2, s_1 \vee s_2] \times [t_1 \wedge t_2, t_1 \vee t_2]} f^2(W_\rho) \, d\rho, \quad (s_1, t_1), (s_2, t_2) \in [0,1]^2.
\]

Note that \(\tilde{\mathcal{B}}\) is clearly a very good extension of \(\mathcal{B}\) in the sense of a two-parameter counterpart of [22, Definition II.7.1]. Since \(\mathcal{D}([0,1]^2), d, \mathcal{L}_2\) is a Polish space, by [22, Proposition VIII.5.33], \(\mathcal{F}\)-stable convergence in law holds if for every random variable \(Z\) on \((\Omega, \mathcal{F}, \mathbb{P})\) the couple \((Z, X^n)_n\) converges in law. Adapting an argument presented in the proof of [22, Theorem VIII.5.7 b]), the convergence in law of a such couple \((Z, X^n)_n\) will be obtained by first proving in Step (1) that the sequence \((X^n)_n\) is tight (relative to the Skorohod space \(\mathcal{D}([0,1]^2), d, \mathcal{L}_2\)) and then by making in Step (2) an “identification of the limit” via \(\mathcal{F}\)-stable finite-dimensional convergence in law to \(X\). Recall that the latter property means that for every integer \(m \geq 0\), for every continuous and bounded function \(\psi : \mathbb{R}^{m+1} \to \mathbb{R}\) and each of the elements \(z_0, \ldots, z_m\) in a dense subset of \([0,1]^2\),

\[
\mathbb{E} \left[ Z \psi (X^n(z_0), X^n(z_1), \ldots, X^n(z_m)) \right] \xrightarrow{n \to \infty} \tilde{\mathbb{E}} \left[ Z \psi (X(z_0), X(z_1), \ldots, X(z_m)) \right]. \tag{3.3}
\]

**Step (1)**

We show the sequence \((X^n)_n\) is tight in the Skorohod space \(\mathcal{D}([0,1]^2), d, \mathcal{L}_2\).

A complete description of \(\mathcal{D}([0,1]^2), d, \mathcal{L}_2\) can be found in [25]. In particular it is shown in [25] that the set of conditions (3.4) and (3.5) is necessary and sufficient for the sequence \((X^n)_n\)
to be tight in $(D([0, 1]^2), d, \mathcal{L}_2)$,

$$(X^n_0)_n$$ converges in distribution.

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}[w(X^n, \delta) \geq \varepsilon] = 0, \quad \varepsilon > 0,$$

where $w$ is defined in Section 2. Property (3.4) is clear since for every $n \geq 1$, $X^n_0 = X_0 = 0$, $\mathbb{P}$-a.s. We will show (3.5) using a method from [11, p. 89].

Let $\varepsilon > 0$, $\delta > 0$ and $n \geq 1$. Let $m := \left[ \frac{n}{\delta} \right]$ and $v := \left\lceil \frac{n}{m} \right\rceil$. We consider on $[0, 1]^2$ the rectangles $R_{i,j} := \left[ \frac{m_i - 1}{n}, \frac{m_i}{n} \right] \times \left[ \frac{m_j - 1}{n}, \frac{m_j}{n} \right]$, $(i, j) \in \{1, \ldots, v\}^2$ where $m_i := im$, $1 < i < v$ and $m_v = n$. With this notation the length of the shortest side of the rectangles $R_{i,j}$ is greater than $\delta$ and $v \leq 2/\delta$. We can adapt the proof of [11, Theorem 7.4] to our case and we have

$$\mathbb{P}[w(X^n, \delta) \geq 3\varepsilon] \leq \sum_{i=1}^v \sum_{j=1}^v \mathbb{P} \left[ \sup_{z \in R_{i,j}} \left| X^n_z - X^n_z \left( \frac{m_i - 1}{n}, \frac{m_j - 1}{n} \right) \right| \geq \varepsilon \right].$$

(3.6)

Let us give some notations. For $(k, j) \in \{1, \ldots, n\}^2$ let

$$S_{k,l} := \sum_{i=1}^k \sum_{j=1}^l f \left( W \left( \frac{i-1}{n}, \frac{j-1}{n} \right) \right) \left( |\Delta_{i,j} W|^2 - 1/n^2 \right),$$

that is $X^n_{(k,l)} = nS_{k,l}$. For $z$ in $R_{i,j}$ we write $\hat{\tau}_{i,j} := S_{k,l} - S_{m_i - 1, m_j - 1}$. Using these notations we can write (3.6) as

$$\mathbb{P} \left[ w(X^n, \delta) \geq 3\varepsilon \right] \leq \sum_{i=1}^v \sum_{j=1}^v \mathbb{P} \left[ \sup_{m_i - 1 \leq k \leq m_i, m_j - 1 \leq l \leq m_j} |\hat{\tau}_{i,j,k,l}| \geq \frac{\varepsilon}{n} \right].$$

We use [11, Section 10] which provides maximal inequalities for partial sums of non-independent and non-stationary random variables. For $i$, $j$ fixed as above, we re-index the random variables appearing in $\hat{\tau}_{i,j,k,l}$ to obtain

$$\hat{\tau}_{i,j} = \sum_{p=1}^{\eta(i,j,k,l)} \tau_p,$$

with $\tau_p$ equal to some $\xi_{., \cdot}$ divided by $n$ and $\eta(i,j,k,l)$ is an integer. With this notation we can show that

$$\mathbb{E}[|\tau_p|^4] \leq \frac{R}{n^8}, \quad \text{where } R \text{ is a constant.}$$

(3.7)

Actually, $\tau_p = f \left( W \left( \frac{i-1}{n}, \frac{j-1}{n} \right) \right) \left( |\Delta_{i,j} W|^2 - 1/n^2 \right)$ for some $1 \leq i, j \leq n$ and

$$\mathbb{E}[|\tau_p|^4] \leq \mathbb{E} \left[ f^4 \left( W \left( \frac{i-1}{n}, \frac{j-1}{n} \right) \right) \mathbb{E} \left[ \left( |\Delta_{i,j} W|^2 - 1/n^2 \right)^4 \right] \right] \leq \sup_{(s,t) \in [0,1]^2} \mathbb{E} \left[ f^4 \left( W(s,t) \right) \right] \mathbb{E} \left[ \left( |\Delta_{i,j} W|^2 - 1/n^2 \right)^4 \right] \leq \frac{R}{n^8}, \quad \text{by hypothesis (H)}.$$
Let two integers $\alpha \leq \beta$. Let $K$ denote a constant which can differ from one line to another.

\[
P\left[\left\lvert \sum_{p=\alpha+1}^{\beta} \tau_p \right\rvert \geq \lambda \right] \leq \frac{1}{\lambda^4} \mathbb{E}\left[\left\lvert \sum_{p=\alpha+1}^{\beta} \tau_p \right\rvert^4 \right] \leq \frac{1}{\lambda^4} \sum_{p=\alpha+1}^{\beta} \mathbb{E}[|\tau_p|^4]
\]

\[
\leq \frac{K}{\lambda^4 n^8} (\beta - \alpha)^2 \rho, \quad \frac{1}{2} < \rho < 1, \text{ by (3.7).}
\]

Using [11, Theorem 10.2] and (3.8) we obtain

\[
P\left[\max_{i \leq k \leq m, j \leq l \leq m} \hat{S}_{k,l}^{i,j} \geq \lambda \right] \leq \frac{K m^{4\rho}}{n^8 \lambda^4}.
\]

Now injecting inequality (3.9) in (3.6) we have,

\[
P\left[w(X^n, \delta) \geq 3\varepsilon \right] \leq \frac{v^2 K m^{4\rho}}{n^4 \varepsilon^4} \leq \frac{K m^{4\rho}}{\varepsilon^4 n^4 \delta^2}, \quad \text{since } v \leq 2/\delta,
\]

\[
\leq \frac{K m^{4\rho}}{\varepsilon^4 n^4 (\rho - 1)^{2\rho}} - 2, \quad \text{since } m = \lfloor n \delta \rfloor,
\]

which leads to (3.5).

**Step (2)**

Here we choose to consider processes $X^n$ and $X$ as set-indexed processes and we use all the notations and definitions of Appendix A.1. Consequently the $\mathcal{F}$-stable finite-dimensional convergence in law property (3.3) can be rewritten as follows: for every continuous and bounded function $\psi$, for each of the elements $C_0, \ldots, C_m$ in a dense subset of $\mathcal{A}$ (see Appendix A.1 for definitions and notations) and for every random variable $Z$ on $(\Omega, \mathcal{F}, \mathbb{P})$,

\[
\mathbb{E}\left[Z \psi \left( X^n(C_0), X^n(C_1), \ldots, X^n(C_m) \right) \right] \xrightarrow{n \to \infty} \mathbb{E}\left[Z \psi \left( X(C_0), X(C_1), \ldots, X(C_m) \right) \right].
\]

(3.10)

To obtain (3.10) we adapt [19, Proposition 7.3.7] which allows us to replace $\mathcal{F}$-stable finite-dimensional convergence in law with $\mathcal{F}$-stable semi-functional convergence in law that is, for every simple flow $\varphi$ (see Subsection A.1) the sequence of one-parameter processes $(X^n \circ \varphi)_n$ converges $\mathcal{F}$-stably in law to the one-parameter process $X \circ \varphi$. Let us make precise this argument. Assume that stable semi-functional convergence in law holds. We aim at showing (3.10). As in [19, Proposition 7.3.7] since for every $n \geq 1$, $X^n$ is an additive process (cf. Subsection A.1) it is enough to prove (3.10) for elements $C_0, \ldots, C_m$ such that for every $i \in \{1, \ldots, m\}$, $C_i = \varphi(i/m) - \varphi((i - 1)/m)$ where $\varphi$ is an arbitrary simple flow (see Appendix A.1). Since the sequence of one-parameter càdlàg processes $(X^n \circ \varphi)_n$ is supposed to converge $\mathcal{F}$-stably in law to $X \circ \varphi$, and since one can choose a continuous version of $X \circ \varphi$, the projection $\pi_{(0,1/m, \ldots, 1)} : D([0,1]) \to \mathbb{R}^{m+1}$ is continuous and by the mapping theorem,
\[ \mathbb{E} \left[ Z \psi \left( (X^n \circ \varphi)(0), (X^n \circ \varphi)(1/m), \ldots, (X^n \circ \varphi)(1) \right) \right] \]
\[ \longrightarrow_{n \to \infty} \int_{\Omega \times \Omega'_\psi} \left( (X \circ \varphi)(\omega, x)(0), \ldots, (X \circ \varphi)(\omega, x)(1) \right) \mathbb{P}'(dx) \mathbb{P}(d\omega) \]
\[ = \tilde{E} \left[ Z \psi \left( (X \circ \varphi)(0), (X \circ \varphi)(1/m), \ldots, (X \circ \varphi)(1) \right) \right], \quad (3.11) \]

where \( Z \) and \( \psi \) are like in (3.10). Consequently relation (3.10) holds since

\[ X^n(C_i) = (X^n \circ \varphi)(i/m) - (X^n \circ \varphi)((i-1)/m), \quad 1 \leq i \leq m. \]

Using the argument presented above we will now prove \( \mathcal{F} \)-stable semi-functional convergence in law to establish \( \mathcal{F} \)-stable finite-dimensional convergence in law.

Let \( \varphi \) be a simple flow (we write \( \varphi \) as \( (\varphi_1, \varphi_2) \)). We have to show that the sequence of one-parameter càdlàg processes \( (X^n \circ \varphi)_n \) converges \( \mathcal{F} \)-stably in law to the one-parameter process \( X \circ \varphi \). We give some precisions about the extension of probability basis we use. We set \( \mathcal{B}_\varphi := (\Omega, \mathcal{F}, (\mathcal{F}_{\psi(t)})_{t \in [0,1]}, \mathbb{P}) \) and \( \mathcal{B}'_\varphi := (\Omega', \mathcal{F}', (\mathcal{F}'_{\psi(t)})_{t \in [0,1]}, \mathbb{P}') \).

From \( \mathcal{B}_\varphi \) and \( \mathcal{B}'_\varphi \) we define the probability basis \( \tilde{\mathcal{B}}_\varphi := (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_{\psi(t)})_{t \in [0,1]}, \tilde{\mathbb{P}}) \), with,

\[ (\tilde{\mathcal{F}}_{\psi(t)})_{t \in [0,1]} := (\cap_{s > t} \mathcal{F}_{\psi(s)} \otimes \mathcal{F}'_{\psi(s)})_{t \in [0,1]}. \]

Let \( n \geq 1 \), by Lemmas 3.3 and A.1 the one-parameter processes \( ((X^n \circ \varphi)_t)_{t \in [0,1]} \) are martingales on the probability basis \( \mathcal{B}'^n_\varphi := (\Omega', \mathcal{F}', (\mathcal{F}'_{\psi(t)})_{t \in [0,1]}, \mathbb{P}') \) where,

\[ \mathcal{F}'_{t \psi} := \mathcal{F}'(\lfloor n \varphi_1(t) \rfloor n^{-1}, \lfloor n \varphi_2(t) \rfloor n^{-1}), \quad t \in [0, 1]. \]

We also define

\[ \mathcal{F}'_{t \psi} := \mathcal{F}'(\lfloor n \varphi_1(t) \rfloor n^{-1}, \lfloor n \varphi_2(t) \rfloor n^{-1}), \]
\[ \tilde{\mathcal{F}}'_{t \psi} := \mathcal{F}'_{\psi(t)} \otimes \mathcal{F}'_{t \psi}, \]

and the following probability bases,

\[ \mathcal{B}'^n_\varphi := (\Omega', \mathcal{F}', (\mathcal{F}'_{\psi(t)})_{t \in [0,1]}, \mathbb{P}'), \]
\[ \tilde{\mathcal{B}}'_{\varphi} := (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_{\psi(t)})_{t \in [0,1]}, \tilde{\mathbb{P}}). \]

We will now apply [22, Theorem IX.7.3] (or its triangular array formulation [22, Theorem IX.7.28]). This result gives conditions ensuring \( \mathcal{F} \)-stable finite-dimensional convergence in law to a sequence of one-parameter martingales to a continuous conditional martingale with independent increments by identifying the characteristics of these martingales plus some additional conditions. This identification is realized in Lemma 3.4 in which convergences (3.12) and (3.13) can be considered as identification of the characteristics whereas properties (3.14) and (3.15) ensure the \( \mathcal{F} \)-stable feature of the convergence. Consequently from [22, Theorem IX.7.3] and Lemma 3.4, the sequence \( (X^n \circ \varphi)_n \) of one-parameter martingales on \( \mathcal{B}'^n_\varphi \) converges \( \mathcal{F} \)-stably in law to \( X \circ \varphi \) on the extension \( \tilde{\mathcal{B}}_\varphi \) of \( \mathcal{B}_\varphi \) which ends the proof. Note that \( \tilde{\mathcal{B}}_\varphi \) is a very good extension of \( \mathcal{B}_\varphi \) since \( \tilde{\mathcal{B}} \) is a very good extension of \( \mathcal{B} \).  \( \Box \)

Before turning to estimation results in Section 4 we state and prove Lemmas 3.3 and 3.4 which were used in the proof of Theorem 3.1.

**Lemma 3.3.** We use notations of Theorem 3.1 and of its proof. \( X^n \) is a strong martingale.
Proof. Let \( y \) and \( x \) in \([0, 1]^2\) such that \( y = (y_1, y_2) \leq x = (x_1, x_2)\). Let also, \( \mathcal{F}_{y}^{n,*} := \mathcal{F}^{*(n^{-1}[ny_1], n^{-1}[ny_2])}\). We have
\[
\mathbb{E}[X^n((y, x))|\mathcal{F}_{y}^{n,*}] = n \sum_{i=[ny_1]}^{[ny_2]} \sum_{j=[ny_2]} \mathbb{E} \left[ f \left( W\left(\frac{i-1}{n}, \frac{j-1}{n}\right) \left( |\Delta_{i,j} W| - \frac{1}{n^2} \right) \right) |\mathcal{F}_{y}^{n,*} \right]
\]
\[
= n \sum_{i=[ny_1]}^{[ny_2]} \sum_{j=[ny_2]} \mathbb{E} \left[ f \left( W\left(\frac{i-1}{n}, \frac{j-1}{n}\right) \right) \mathbb{E} \left[ \left( |\Delta_{i,j} W| - \frac{1}{n^2} \right) |\mathcal{F}_{y}^{*(i-1)/n,(j-1)/n} \right] |\mathcal{F}_{y}^{n,*} \right]
\]
\[
= 0. \quad \Box
\]

Lemma 3.4. We use notations of Theorem 3.1 and of its proof. In particular we denote a flow \( \varphi \)
as \( (\varphi_1, \varphi_2)\). For every \( n \geq 1 \), \( X^n \circ \varphi \) is a one-parameter martingale with modified second characteristics \((0, \tilde{C}_{X^n \circ \varphi}, \nu_{X^n \circ \varphi})\) on \( B^{n}_{\varphi} \) such that
\[
n_{X^n \circ \varphi}([0, t] \times \{|x| > \varepsilon\}) \xrightarrow[n \to \infty]{} 0, \quad \forall t \in [0, 1], \ \varepsilon > 0,
\]
(3.12)
\[
\tilde{C}_{X^n \circ \varphi}(t) \xrightarrow[n \to \infty]{} 2 \int_{[0, \varphi_1(t)] \times [0, \varphi_2(t)]} f^2(W_{\rho}) \ d\rho, \quad \forall t \in [0, 1].
\]
(3.13)
Furthermore for every bounded martingale \( \tilde{\mathbb{N}} \) orthogonal to \( W \circ \varphi \),
\[
(X^n \circ \varphi, \tilde{\mathbb{N}}t) \xrightarrow[n \to \infty]{} 0, \quad \forall t \in [0, 1].
\]
(3.14)
We also have
\[
(X^n \circ \varphi, W \circ \varphi)t \xrightarrow[n \to \infty]{} 0, \quad \forall t \in [0, 1].
\]
(3.15)

Proof. Let \( n \geq 1 \).

Proof of (3.12). In the following \( \mu_{X^n \circ \varphi} \) denotes the jump measure of \( X^n \circ \varphi \). For \( \omega \in \Omega \),
\[
\mu_{X^n \circ \varphi}(\omega, dt, dx) = \sum_{k=1}^{n} \left[ \delta_{(\varphi_1^{-1}(k/n)) \cap (\varphi_2^{-1}([ln, 1 \leq l \leq n]))}(\omega) \right] H^{1,k,n}(\omega)(dt, dx)
\]
\[
+ \delta_{(\varphi_2^{-1}(k/n)) \cap (\varphi_2^{-1}([ln, 1 \leq l \leq n]))}(\omega) H^{2,k,n}(\omega)(dt, dx)
\]
\[
+ \sum_{k=1}^{n} \sum_{l=1}^{n} \delta_{\varphi_1^{-1}(k/n), \varphi_2^{-1}(l/n)}(\omega) H^{3,k,l,n}(\omega)(dr, dr, dx),
\]
\[
\quad \times \left\{ \begin{array}{ll}
H^{1,k,n}(\omega) &= \sum_{j=1}^{[n\varphi_2^{-1}(k/n)\]} \xi_{k, j}(\omega), \\
H^{2,l,n}(\omega) &= \sum_{i=1}^{[n\varphi_1^{-1}(l/n)\]} \xi_{i, j}(\omega), \\
H^{3,k,l,n}(\omega) &= H^{1,k,n} + H^{2,l,n} + \xi_{k,l}.
\end{array} \right.
\]
We denote by $\nu^n_{\text{comp}}$ the compensator of the measure $\mu^n_{\text{comp}}$. Let $A$ be a Borel set in $\mathbb{R}$. We have

$$
v^n_{\text{comp}}(\omega, [0, t] \times A) = \sum_{k=1}^{[n\varphi_1(t)]} \mathbb{E} \left[ 1_{H^{1,k,n} \in A} |\mathcal{F}_{(k-1)/n} (n\varphi_2(t) - 1/n) \right]
+ \sum_{l=1}^{[n\varphi_2(t)]} \mathbb{E} \left[ 1_{H^{2,l,n} \in A} |\mathcal{F}_{((n\varphi_1(t) - 1)/n, (l-1)/n) \right]
+ \sum_{k=1}^{[n\varphi_1(t)]} \sum_{l=1}^{[n\varphi_2(t)]} \mathbb{E} \left[ 1_{H^{3,k,l,n} \in A} |\mathcal{F}_{((n\varphi_1(t) - 1)/n, (n\varphi_2(t) - 1)/n) \right].
$$

Let $\varepsilon > 0$ and $k, l, n, t$ as above. Denote by $C$ a constant which can differ from one line to another.

$$
\mathbb{P} \left[ |H^{1,k,n}| > \varepsilon |\mathcal{F}_{((k-1)/n, [n\varphi_2(t)]/n) \right]
\leq \frac{1}{\varepsilon^4} \mathbb{E} \left[ \sum_{j=1}^{[n\varphi_2(t)]} |\xi_{k,j}| \right] |\mathcal{F}_{(k-1)/n, [n\varphi_2(t) - 1/n) \right]
\leq \frac{1}{\varepsilon^4} \sum_{j=1}^{[n\varphi_2(t)]} \mathbb{E} \left[ |\xi_{k,j}| \right] |\mathcal{F}_{(k-1)/n, [n\varphi_2(t) - 1/n) \right]
= \frac{n^4}{\varepsilon^4} \sum_{j=1}^{[n\varphi_2(t)]} f^4 \left( W_{(k-1)/n, i-1/n} \right) \mathbb{E} \left[ \left( |\Delta_{k,j} W| - 1/n^2 \right) \right] |\mathcal{F}_{(k-1)/n, [n\varphi_2(t) - 1/n) \right]
\leq C \frac{\varepsilon^4 n^3}{\varepsilon^4} \sup_{\omega \in [0,1]} \frac{1}{n} \sum_{j=1}^{[n\varphi_2(t)]} f^4 \left( W_{(k-1)/n, i-1/n} \right)
\quad \xrightarrow{n \to \infty} 0, \quad \mathbb{P}\text{-a.s.}
$$

Indeed, by hypothesis (H), we have that $\sup_{\omega \in [0,1]} \int_{0}^{[\varphi_2(t)]} f^4 (W_{(s,t)}) dt < \infty \mathbb{P}\text{-a.s.}$ Before giving details about the inequality (\ast) we deduce of the preceding inequalities that

$$
\mathbb{P} \left[ |H^{3,k,l,n}| > \varepsilon |\mathcal{F}_{([n\varphi_1(t) - 1/n, [n\varphi_2(t) - 1/n) \right]
\leq \mathbb{P} \left[ |H^{1,k,n}| > \varepsilon \mathbb{P} \left[ |H^{2,l,n}| > \varepsilon / 3 |\mathcal{F}_{([n\varphi_1(t) - 1/n, [n\varphi_2(t) - 1/n) \right]
+ \mathbb{P} \left[ |\xi_{k,l}| > \varepsilon / 3 |\mathcal{F}_{([n\varphi_1(t) - 1/n, [n\varphi_2(t) - 1/n) \right]
\quad \xrightarrow{n \to \infty} 0, \quad \mathbb{P}\text{-a.s.}
$$

This leads to (3.12). Now we give some details about inequality (\ast).

Let $1 \leq l_2, l_3, l_4 < l_1 \leq [n\varphi_2(t)] - 1$. We have

$$
\mathbb{E} \left[ \xi_{k,l_1} \cdots \xi_{k,l_4} |\mathcal{F}_{(k-1)/n, [n\varphi_2(t) - 1/n) \right]
= \mathbb{E} \left[ \mathbb{E} \left[ \xi_{k,l_1} \cdots \xi_{k,l_4} |\mathcal{F}_{(k-1)/n, [n\varphi_2(t) - 1/n) \right] \mathcal{F}_{(k-l_1-1)/n, [n\varphi_2(t) - 1/n) \right]
= \mathbb{E} \left[ \mathbb{E} \left[ \xi_{k,l_1} \cdots \xi_{k,l_4} |\mathcal{F}_{(k-l_1-1)/n, [n\varphi_2(t) - 1/n) \right] \mathcal{F}_{(k-l_1-1)/n, [n\varphi_2(t) - 1/n) \right)
\cdots
$$
\[
\begin{align*}
\ &= n\mathbb{E}\left[\xi_{k,l_2}\ldots\xi_{k,l_4} f\left(W_{\left(\frac{k-1}{n}, \frac{l_2-1}{n}\right)}^n\right)\mathbb{E}\left[\left(\left|\Delta_{k,l_1} W\right|^2 - 1/n^2\right)\right]\right] \\
\ &= n\mathbb{E}\left[\xi_{k,l_2}\ldots\xi_{k,l_4} f\left(W_{\left(\frac{k-1}{n}, \frac{l_2-1}{n}\right)}^n\right)\mathbb{E}\left[\left(\left|\Delta_{k,l_1} W\right|^2 - 1/n^2\right)\right]\right] = 0.
\end{align*}
\]

**Proof of (3.13).** \(\tilde{C}_{X^n \circ \varphi}(t) = \langle X^n \circ \varphi, X^n \circ \varphi \rangle\) is the compensator of \([X^n \circ \varphi, X^n \circ \varphi]\) with respect to \((\mathcal{F}^n_{t, \varphi})_{t \in [0,1]}\) (see for example [22, Proof of Proposition II.2.17 b]) and we have (cf. [22, (I.4.53)])

\[
[X^n \circ \varphi, X^n \circ \varphi]_t = \sum_{0 \leq s \leq t} (X^n \circ \varphi)(s) - (X^n \circ \varphi)(s-) = \sum_{i=1}^{[n\varphi_1(t)]} \sum_{j=1}^{[n\varphi_2(t)]} \xi_{i,j}^2, \quad t \in [0,1].
\]

Consequently,

\[
\tilde{C}_{X^n \circ \varphi}(t) = \sum_{k=1}^{[n\varphi_1(t)]} \mathbb{E}\left[ (H^{1,k,n})^2 | \mathcal{F}_{(k-1)/n,(\lfloor n\varphi_2(t)\rfloor - 1)/n} \right] \\
\quad + \sum_{l=1}^{[n\varphi_2(t)]} \mathbb{E}\left[ (H^{2,l,n})^2 | \mathcal{F}_{((\lfloor n\varphi_1(t)\rfloor - 1)/n, (l-1)/n} \right] \\
\quad + \sum_{k=1}^{[n\varphi_1(t)]} \sum_{l=1}^{[n\varphi_2(t)]} \mathbb{E}\left[ (H^{3,k,l,n})^2 | \mathcal{F}_{((\lfloor n\varphi_1(t)\rfloor - 1)/n, (\lfloor n\varphi_2(t)\rfloor - 1)/n} \right].
\]

We can show that this sum is equal to

\[
\tilde{C}_{X^n \circ \varphi}(t) = \frac{2}{n^2} \sum_{i=1}^{[n\varphi_1(t)]} \sum_{j=1}^{[n\varphi_2(t)]} f^2 \left(W_{\left(\frac{i-1}{n}, \frac{j-1}{n}\right)}^n\right), \quad t \in [0,1],
\]

(3.16)

since terms of the form \(\mathbb{E}\left[\xi_{k,j}\xi_{k,l} | \mathcal{F}_{\left(\frac{k-1}{n}, \frac{l_2-1}{n}\right)}\right]\) vanish for \(j < l \leq \lfloor n\varphi_2(t)\rfloor\) using the same type an argument described in the proof of (3.12). Furthermore terms of the form \(\mathbb{E}\left[\xi_{k,j}^2 | \mathcal{F}_{\left(\frac{k-1}{n}, \frac{l_2-1}{n}\right)}\right]\) are given by

\[
\mathbb{E}\left[\xi_{k,j}^2 | \mathcal{F}_{\left(\frac{k-1}{n}, \frac{\lfloor n\varphi_2(t)\rfloor - 1}{n}\right)}\right] = n^2 f^2 \left(W_{\left(\frac{k-1}{n}, \frac{j-1}{n}\right)}^n\right) \mathbb{E}\left[\left|\Delta_{k,j}^n W\right|^2 - 1/n^2\right]^2 \\
\quad = \frac{2}{n^2} f^2 \left(W_{\left(\frac{k-1}{n}, \frac{j-1}{n}\right)}^n\right).
\]

We deduce (3.13) from (3.16).

**Proof of (3.14).** Let \(\tilde{N}\) be a martingale orthogonal to \(W \circ \varphi\). Without loss of generality we can assume there exists a strong martingale \(N\) on \(\mathcal{B}\) orthogonal to \(W\) such that \(NW\) is a strong
martingale and such that $\tilde{N} = N \circ \varphi$. Let $n \geq 1$ and $t$ in $[0, 1]$. We have

$$\langle X^n, N \rangle_t = \sum_{i=1}^{\lceil n^2 \rceil} \sum_{j=1}^{\lceil n^2 \rceil} \mathbb{E} \left[ \xi_{i,j} \Delta_i, j N| \mathcal{F}(\frac{i-1}{n}, \frac{j-1}{n}) \right]$$

$$= \sum_{i=1}^{\lceil n^2 \rceil} \sum_{j=1}^{\lceil n^2 \rceil} f \left( (\frac{i-1}{n}, \frac{j-1}{n}) \right) \mathbb{E} \left[ \left( |\Delta_i, j W|^2 - 1/n^2 \right) \Delta_i, j N| \mathcal{F}(\frac{i-1}{n}, \frac{j-1}{n}) \right].$$

Let $1 \leq i, j \leq n$. We use a technique presented in [21, Lemma 6.8]. For $z$ in $[0, 1]^2$ we define $U_z := \mathbb{E} \left[ |\Delta_i, j W|^2 - 1/n^2 \right]$. $(U_z)_{z \geq ((i-1)/n, (j-1)/n)}$ is a martingale for the filtration generated by $U_z - U_z(\frac{i-1}{n}, \frac{j-1}{n})$. Using the representation $|\Delta_i, j W|^2 - 1/n^2 = I_2 \left( 1_{\Delta_i, j} \otimes 1_{\rho, \nu} \right)$ as a multiple stochastic integral (see for example [31, Section 1.1.2] or [32]) we have by [31, Lemma 1.2.5] that for $z \geq ((i-1)/n, (j-1)/n)$, $U_z = I_2 \left( 1_{\Delta_i, j} \otimes 1_{1_{[0,1]}} \right)$. From [24] there exists an adapted process $(\Phi_{(s,t)})_{(s,t) \in [0,1]^2}$ such that

$$U_{(s,t)} = U(\frac{i-1}{n}, \frac{j-1}{n}) + \int_{[(i-1)/n, s] \times [(j-1)/n, t]} \Phi_{(s,t)} dW_{(s,t)}.$$

Define $N'_z = N_z - N(\frac{i-1}{n}, \frac{j-1}{n})$, $z \geq ((i-1)/n, (j-1)/n)$. This process is orthogonal to $(U_z)_{z \geq ((i-1)/n, (j-1)/n)}$. Consequently using a characterization of orthogonal two-parameter martingales given in [13, Proposition 1.6] we have that

$$\mathbb{E} \left[ \Delta_i, j N'| \mathcal{F}(\frac{i-1}{n}, \frac{j-1}{n}) \right] = 0.$$

A straightforward computation gives that

$$\mathbb{E} \left[ \left( |\Delta_i, j W|^2 - 1/n^2 \right) \Delta_i, j N| \mathcal{F}(\frac{i-1}{n}, \frac{j-1}{n}) \right] = \mathbb{E} \left[ \left( \Delta_i, j N'| \mathcal{F}(\frac{i-1}{n}, \frac{j-1}{n}) \right) \Delta_i, j W \right].$$

**Proof of (3.15).** Let $n \geq 1$ and $t \in [0, 1]$. We have

$$\langle X^n \circ \varphi, W \circ f \rangle_t = \sum_{i=1}^{\lceil n^2 \rceil} \sum_{j=1}^{\lceil n^2 \rceil} \mathbb{E} \left[ \xi_{i,j} \Delta_i, j W| \mathcal{F}(\frac{i-1}{n}, \frac{j-1}{n}) \right]$$

$$= \sum_{i=1}^{\lceil n^2 \rceil} \sum_{j=1}^{\lceil n^2 \rceil} f \left( (\frac{i-1}{n}, \frac{j-1}{n}) \right) \mathbb{E} \left[ \left( |\Delta_i, j W|^2 - 1/n^2 \right) \Delta_i, j W| \mathcal{F}(\frac{i-1}{n}, \frac{j-1}{n}) \right]$$

$$= 0. \quad \Box$$

### 4. Estimation of the quadratic variation and asymptotic normality of the estimator

In this section we prove an asymptotic normality property (Corollary 4.5) for the consistent estimator $V^n$ (see (1.4)) of the quadratic variation $C$ (defined in (1.5)).

Consider the following two-parameter stochastic process

$$Y_{(s,t)} := \int_{[0,s] \times [0,t]} \sigma (W_{\rho}) \, dW_{\rho} + \int_{[0,s] \times [0,t]} M_{\rho} \, d\rho, \quad (s, t) \in [0, 1]^2 \quad (4.1)$$
defined on a probability basis \((\Omega, \mathcal{F}, (\mathcal{F}(s,t))_{(s,t)\in[0,1]^2}, \mathbb{P})\). Until the end of this paper we assume that \((M_{(s,t)})_{(s,t)\in[0,1]^2}\) is a continuous and \((\mathcal{F}(s,t))_{(s,t)\in[0,1]^2}\)-adapted process. The following assumption will be used in the results presented below.

Hypothesis \((H_0)\):

\[\sigma : \mathbb{R} \rightarrow \mathbb{R}\] is in \(C^2(\mathbb{R})\) with,

\[\sup_{(s,t)\in[0,1]^2} \mathbb{E}\left[ |\sigma^{(a)}(W(s,t))|^2 \right] < \infty, \quad p \in (0, 4q), \quad a = 0, 1, 2,\] with the convention \(\sigma^{(0)} := \sigma\).

For \(n \geq 1\), let

\[Y_n^{(s,t)} := n \left( V_{(s,t)} - C_{(s,t)} \right) = n \left( \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} |\Delta_{i,j}^n W|^{2} - \int_{[0,s] \times [0,t]} \sigma^2(W_{(u,v)}) \, du \, dv \right). \quad (4.2)\]

Once again in this section \(C\) denotes a constant which can differ from one line to another.

**Proposition 4.1.** Under hypothesis \((H_1)\) the estimator \(V^n\) defined in (1.4) of the quadratic variation \(C(1.5)\) is consistent. That is, for every \((s, t)\) in \([0, 1]^2\),

\[V_n^{(s,t)} \xrightarrow{n \to \infty} C_{(s,t)}.\]

**Proof.** Fix \((s, t)\) in \([0, 1]^2\). We show \((V_n^{(s,t)})_n\) converges to \(C_{(s,t)}\) in \(L^2(\Omega, \mathcal{F}, \mathbb{P})\). It is enough to show that

\[\sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} \sigma^2 \left( W_{(\lfloor i-1/n \rfloor, \lfloor j-1/n \rfloor)} \right) \left( |\Delta_{i,j} W|^2 - \frac{1}{n^2} \right)\]

tends to zero in \(L^2(\Omega, \mathcal{F}, \mathbb{P})\) as \(n\) goes to infinity. We have,

\[
\mathbb{E}\left[ \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} \sigma^2 \left( W_{(\lfloor i-1/n \rfloor, \lfloor j-1/n \rfloor)} \right) \left( |\Delta_{i,j} W|^2 - \frac{1}{n^2} \right) \right]^2
\]

\[
= \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} \mathbb{E}\left[ \sigma^4 \left( W_{(\lfloor i-1/n \rfloor, \lfloor j-1/n \rfloor)} \right) \mathbb{E}\left[ \left| \Delta_{i,j} W \right|^2 - \frac{1}{n^2} \right]^2 \right]
\]

\[
= \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} \frac{1}{n^4} \mathbb{E}\left[ \sigma^4 \left( W_{(\lfloor i-1/n \rfloor, \lfloor j-1/n \rfloor)} \right) \mathbb{E}\left[ \left| n \Delta_{i,j} W \right|^2 - 1 \right]^2 \right]
\]

\[
\leq \frac{C}{n^2}, \quad \text{by using hypothesis \((H_1)\).} \quad \square
\]

In order to prove the asymptotic normality of \(V^n\) in Corollary 4.5 we need the following theorem.
Theorem 4.2. Let $X$ be the process defined by (3.2) where $f$ in (3.2) is replaced by $\sigma^2$. Under assumption $(H_1)$, $(Y^n)_{n \geq 1}$ defined by (4.2) converges $\mathcal{F}$-stably in law in the Skorohod space $(\mathcal{D}([0, 1]^2), d, \mathcal{L}_2)$ to the non-Gaussian continuous process $X$ defined on the extension $\tilde{\mathcal{B}}$ described in the proof of Theorem 3.1.

Remark 4.3. Note that $X$ above is not a Gaussian process but it is conditionally Gaussian given the filtration generated by $W$.

Proof. Using a localization argument the finite variation part of $Y$ has no contribution in the limit. So we can assume that $M = 0$.

From Theorem 3.1 with $f = \sigma^2$, the process $(X^n)_n$ converges $\mathcal{F}$-stably in law to $X$ with

$$X^n_{(s, t)} = n \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} \sigma^2 \left( W\left(\frac{i-1}{n}, \frac{j-1}{n}\right) \right) \left( |\Delta_{i,j} W|^2 - \frac{1}{n^2} \right), \quad (s, t) \in [0, 1]^2.$$

To conclude the proof we show that $Y^n$ is equal to $X^n$ plus a term $r_n$ which becomes negligible when $n$ goes to infinity. More precisely using the notations

$$\eta_{i,j} := \left( \int_{\Delta_{i,j}} \sigma (W_u) \, dW_u \right)^2 - \sigma^2 \left( W\left(\frac{i-1}{n}, \frac{j-1}{n}\right) \right) |\Delta_{i,j} W|^2$$

$$\eta'_{i,j} := - \int_{\Delta_{i,j}} \sigma^2 (W_\rho) - \sigma^2 \left( W\left(\frac{i-1}{n}, \frac{j-1}{n}\right) \right) \, d\rho \quad 1 \leq i, j \leq n, \, n \geq 1,$$

$r_n$ can be decomposed as

$$r_n(s, t) := r_n^{(1)}(s, t) + r_n^{(2)}(s, t)$$

where

$$r_n^{(1)}(s, t) := \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} \eta_{i,j} + \eta'_{i,j},$$

$$r_n^{(2)}(s, t) := - \int_{[ns]/n}^{s} \int_{[nt]/n}^{t} \sigma^2 (W_\rho) \, d\rho.$$

Using a standard argument of the form [11, Theorem 3.1] and Theorem 3.1 the proof is finished if we show that $n \sup_{(s, t) \in [0, 1]^2} |r_n(s, t)|$ converges in probability to zero. We use the decomposition (4.3) and we show

$$n \sup_{(s, t) \in [0, 1]^2} |r_n^{(1)}(s, t)| \overset{p}{\to} 0,$$

$$n \sup_{(s, t) \in [0, 1]^2} |r_n^{(2)}(s, t)| \overset{p}{\to} 0.$$

Proof of (4.4). Using Burkholder’s inequality for two-parameter martingales (see Remark 2 of [29]) it is enough to show that

$$\mathbb{E} \left[ \left| \eta_{i,j} + \eta'_{i,j} \right|^2 \right] \leq \frac{C}{n^5},$$

where $C$ is a constant.
The tool used here is the Malliavin calculus (see Appendix A.2) and especially the Malliavin integration by parts formula (A.3). The main problem comes from the computation of $\mathbb{E} \left[ |\eta_{i,j}|^2 \right]$. First we express $\eta_{i,j}$ as,

$$
\eta_{i,j} = \left( \int_{\Delta_{i,j}} \sigma \left( W_{(i/n, i-1/n)} \right) - \sigma \left( W_{(i-1/n, i/n)} \right) \, dW_{\rho} \right) \left( \int_{\Delta_{i,j}} \sigma \left( W_{(i/n, i-1/n)} \right) + \sigma \left( W_{(i-1/n, i/n)} \right) \, dW_{\rho} \right)
$$

where

$$
\begin{align*}
\begin{cases}
    u_{(s,t)} := 1_{\Delta_{i,j}}(s, t) \left( \sigma \left( W_{(s,t)} \right) - \sigma \left( W_{(s-1/n, t)} \right) \right) & \text{by (A.3)}, \\
    v_{(s,t)} := 1_{\Delta_{i,j}}(s, t) \left( \sigma \left( W_{(s,t)} \right) + \sigma \left( W_{(s-1/n, t)} \right) \right), & (s, t) \in [0, 1]^2.
\end{cases}
\end{align*}
$$

$\delta(u)$ denotes the Skorohod integral of the process $(u_{(s,t)}, (s,t) \in [0,1]^2$ which coincides since $u$ is adapted with the Itô stochastic integral. Consequently,

$$
\begin{align*}
\mathbb{E} \left[ |\eta_{i,j}|^2 \right] &= \mathbb{E} \left[ \delta(u) \left( \delta(u) \delta(v)^2 \right) \right] \\
&= \mathbb{E} \left[ (u, D(\delta(u)\delta(v)^2))_{L^2([0,1]^2)} \right], \\
&= \int_{\Delta_{i,j}} \mathbb{E} \left[ u_{(s,t)} D_{(s,t)} \delta(u) \delta(v)^2 \right] \, ds \, dt \\
&= \int_{\Delta_{i,j}} \mathbb{E} \left[ u_{(s,t)} \delta(v)^2 D_{(s,t)} \delta(u) \right] \, ds \, dt \\
&\quad + 2 \int_{\Delta_{i,j}} \mathbb{E} \left[ u_{(s,t)} \delta(u) \delta(v) D_{(s,t)} \delta(v) \right] \, ds \, dt, \\
&= \int_{\Delta_{i,j}} \mathbb{E} \left[ u_{(s,t)}^2 \delta(v)^2 \right] \, ds \, dt + \int_{\Delta_{i,j}} \mathbb{E} \left[ \delta(D_{(s,t)} u) \delta(v)^2 \right] \, ds \, dt \\
&\quad + 2 \int_{\Delta_{i,j}} \mathbb{E} \left[ u_{(s,t)} v_{(s,t)} \delta(u) \delta(v) \right] \, ds \, dt \\
&\quad + 2 \int_{\Delta_{i,j}} \mathbb{E} \left[ u_{(s,t)} \delta(u) \delta(v) \delta(D_{(s,t)} v) \right] \, ds \, dt \tag{4.7}
\end{align*}
$$

where the last equality is deduced from the chain rule formula (A.5) and from the “Heisenberg commutativity relationship” (A.6). We compute the first term of (4.7) since the estimates for the other terms can be derived from mimicking the following computations. Using Malliavin calculus arguments already mentioned above we obtain that,

$$
\begin{align*}
\int_{\Delta_{i,j}} \mathbb{E} \left[ u_{(s,t)}^2 \delta(v)^2 \right] \, ds \, dt &= \int_{\Delta_{i,j}} \int_{\Delta_{i,j}} \mathbb{E} \left[ v_{(a,b)} D_{(a,b)} (u_{(s,t)}^2 \delta(v)) \right] \, da \, db \, ds \, dt \\
&= 2 \int_{\Delta_{i,j}} \int_{\Delta_{i,j}} \mathbb{E} \left[ v_{(a,b)} u_{(s,t)} \delta(v) D_{(a,b)} (u_{(s,t)}) \right] \, da \, db \, ds \, dt \\
&\quad + \int_{\Delta_{i,j}} \int_{\Delta_{i,j}} \mathbb{E} \left[ v_{(a,b)}^2 u_{(s,t)}^2 \right] \, da \, db \, ds \, dt \\
&\quad + \int_{\Delta_{i,j}} \int_{\Delta_{i,j}} \mathbb{E} \left[ v_{(a,b)} u_{(s,t)}^2 \delta(D_{(a,b)} v) \right] \, da \, db \, ds \, dt. \tag{4.8}
\end{align*}
$$
Using the Cauchy–Schwarz inequality we have that,
\[
\mathbb{E}\left[ v(a,b)u(s,t)\delta(v)D(a,b)(u(s,t)) \right] \leq \mathbb{E}\left[ |v(a,b)u(s,t)D(a,b)(u(s,t))|^2 \right]^{1/2} \mathbb{E}\left[ \delta(v)^2 \right]^{1/2}
\]
which leads to
\[
\left| \int_{\Delta_{i,j}} \int_{\Delta_{i,j}} \mathbb{E}\left[ v(a,b)u(s,t)\delta(v)D(a,b)(u(s,t)) \right] \, da \, db \, ds \, dt \right| \leq \frac{C}{n^5},
\]
by Itô isometry (A.4) and by assumption (H₂). The same methods are valid for the two remaining terms in (4.8). Thus we obtain
\[
\mathbb{E}\left[ |\eta_{i,j}|^2 \right] \leq \frac{C}{n^5}.
\]

**Proof of (4.5).** Recall that \(|s - [ns]| \leq 1/n\) and use assumption (H₂). \(\square\)

**Remark 4.4.** We have chosen to use the Malliavin calculus which leads to a short proof due to the specific form (4.1) of the process \(Y\). We could also have used the two-parameter stochastic calculus techniques developed in [17,29,30,35] which are valid for more general processes however with a longer argument.

We state and prove that the estimator \(V^n\) of \(C\) is asymptotically normal.

**Corollary 4.5 (Asymptotic Normality).** For \((s, t)\) in \([0, 1]^2\), let \(S^n_{(s,t)}\) be
\[
S^n_{(s,t)} := n^2 \sum_{i=1}^{[ns]} \sum_{j=1}^{[nt]} |\Delta_i, j Y|^4, \quad n \geq 1.
\]
Let \((s, t)\) be fixed in \((0, 1)^2\) such that \(S^n_{(s,t)}\) and \(S_{(s,t)}\) do not vanish. Then, under hypothesis (H₂) we have,
\[
\left( S^n_{(s,t)} \right)^{-\frac{1}{2}} n \left( V^n_{(s,t)} - C_{(s,t)} \right) \xrightarrow{\text{law}} n \rightarrow \infty \sqrt{\frac{2}{3}} N, \quad N \sim N(0, 1).
\]

**Proof.** Using a localization argument, only the stochastic integral part of \(Y\) gives a contribution to the limit so we assume \(M = 0\) in (4.1). The main argument of the proof is the convergence in law of \((S^n, Y^n)_n\) to \((S, X)\) where \(S\) is defined by
\[
S_{(s,t)} := 3 \int_{[0,s] \times [0,t]} \sigma^4(W_\rho) \, d\rho
\]
and \(X\) is defined by (3.2) with \(f\) replaced by \(\sigma^2\). Actually assume this convergence holds. Since \((x, y) \mapsto x^{-\frac{1}{2}} y\) is continuous on \(\mathbb{R}_+^* \times \mathbb{R}\) we have for every \((s, t)\) in \((0, 1]^2\),
\[
\left( S^n_{(s,t)} \right)^{-\frac{1}{2}} Y^n_{(s,t)} \xrightarrow{\text{law}} n \rightarrow \infty \sqrt{\frac{2}{3}} \int_{[0,s] \times [0,t]} \sigma^2(W_\rho) \, dB_\rho \left( \int_{[0,s] \times [0,t]} \sigma^4(W_\rho) \, d\rho \right)^{\frac{1}{2}}.
\]
Computing the characteristic function with respect to the probability measure \( \tilde{\mathbb{P}} \) we can show that

\[
\sqrt{2 \over 3} \int_{[0,s] \times [0,t]} \sigma^2 \left( W_{\rho} \right) dB_{\rho} \overset{law}{=} \sqrt{2 \over 3} N, \quad N \sim \mathcal{N}(0, 1).
\]

Now we have to show that \( (S^n, Y^n)_n \) converges in law to \( (S, X) \). The key point is the \( \mathcal{F} \)-stable convergence in law of \( (Y^n)_n \) to \( X \) obtained in Theorem 4.2 stated and proved at the end of this section. Using a result of Aldous and Eagleson (presented in [3]) concerning stable convergence in law if \( (S^n)_n \) converges in \( \mathbb{P} \)-probability to \( S \) then \( (S^n, Y^n)_n \) converges in law to \( (S, X) \) (and the convergence is even \( \mathcal{F} \)-stable convergence in law).

Let us finally show that \( (S^n)_n \) converges in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) to \( S \).

First we show that

\[
\mathbb{E} \left[ n^2 \sum_{i=1}^{\lfloor ns \rfloor} \sum_{j=1}^{\lfloor nt \rfloor} \sigma^4 \left( W_{i-1 \over n}, j-1 \over n \right) \left( |\Delta_{i,j} W| - 3 \over n^4 \right) \right] \to 0.
\]

Actually for every \( (s, t) \) in \([0, 1]^2\),

\[
\mathbb{E} \left[ n^2 \sum_{i=1}^{\lfloor ns \rfloor} \sum_{j=1}^{\lfloor nt \rfloor} \sigma^4 \left( W_{i-1 \over n}, j-1 \over n \right) \left( |\Delta_{i,j} W| - 3 \over n^4 \right) \right] \\
= n^4 \sum_{i=1}^{\lfloor ns \rfloor} \sum_{j=1}^{\lfloor nt \rfloor} \mathbb{E} \left[ \sigma^8 \left( W_{i-1 \over n}, j-1 \over n \right) \left| \Delta_{i,j} W\right|^4 \right] \left( -3 \over n^4 \right)^2 \\
\leq n^4 \sup_{(a,b) \in [0,1]^2} \mathbb{E} \left[ \sigma^8 (W_{a,b}) \right] \sum_{i=1}^{\lfloor ns \rfloor} \sum_{j=1}^{\lfloor nt \rfloor} \mathbb{E} \left[ \left| \Delta_{i,j} W\right|^4 \right] \left( -3 \over n^4 \right)^2 \\
\leq C \over n^2, \quad \text{by assumption (H$_2$).}
\]

Using a Riemann approximation for integrals we have that

\[
3n^2 \sum_{i=1}^{\lfloor ns \rfloor} \sum_{j=1}^{\lfloor nt \rfloor} \frac{\sigma^4 \left( W_{i-1 \over n}, j-1 \over n \right)}{n^4} \overset{L^2}{\to} 3 \int_{[0,s] \times [0,t]} \sigma^4 \left( W_{\rho} \right) d\rho, \quad \forall (s, t) \in [0, 1]^2.
\]

The proof is finished if we can show the estimate (4.9).

\[
\mathbb{E} \left[ n^2 \sum_{i=1}^{\lfloor ns \rfloor} \sum_{j=1}^{\lfloor nt \rfloor} \left( |\Delta_{i,j} Y| - \frac{\sigma^4 \left( W_{i-1 \over n}, j-1 \over n \right)}{n^4} \right) \right] \to 0.
\]

(4.9)
This goal is achieved by adapting arguments of Theorem 4.2. Since we have to mimic some estimates done in Theorem 4.2 with $\sigma^2$ replaced with $\sigma^4$ we have to assume that $\sigma$ fulfills hypothesis (H$_2$) and not only assumption (H$_1$).

Using classical techniques the asymptotic normality property of $V^n$ enables construction of confidence interval for $C_{(s,t)}$ for every $(s, t)$ in $[0, 1]^2$.

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Appendix

In this section we present some definitions and results used in Sections 3 and 4. First we provide some background on set-indexed processes. Then we briefly present the Malliavin calculus for two-parameter Brownian motion, including the Malliavin integration by parts formula which has used to obtain the estimates of the previous section.

A.1. Set-indexed processes

In the following definition it will be convenient to think of two-parameter processes on $[0, 1]^2$ as set-indexed processes on $\mathcal{A} := \{[0, z], z \in [0, 1]^2\}$ where $Y_{[0, z]} := Y_z$ and $\mathcal{F}_{[0, z]} := \mathcal{F}_z$, $z \in [0, 1]^2$. We will use indifferently one of these two points of view. Let $\mathcal{A}(u)$ be the set of all finite unions of sets from $\mathcal{A}$. A simple flow (see [19, Definition 1.4.5]) $\varphi$ is an application $\varphi : [0, 1] \to \mathcal{A}(u)$ which is increasing, continuous with $\varphi(0) = \emptyset$ such that for some $k$ in $\mathbb{N}$ there exist some increasing functions $\varphi_i : [0, 1] \to \mathcal{A}$, $i = 1, \ldots, k$ such that

$$\varphi(s) = \varphi\left(\frac{i - 1}{k}\right) \cup \varphi_i(s), \quad s \in \left[\frac{i - 1}{k}, \frac{i}{k}\right], 1 \leq i \leq k.$$ 

Let $\mathcal{C}$ denote the set of elements of the form $A \backslash B$ with $A$ in $\mathcal{A}$ and $B$ in $\mathcal{A}(u)$ (the set of all finite unions of elements of $\mathcal{A}$).

A process $(X_z)_{z \in [0, 1]^2}$ identified with $(X_A)_{A \in \mathcal{A}}$ is called additive if for every $C, C_1, C_2$ in $\mathcal{C}$ with $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$ we have,

$$X_C = X_{C_1} + X_{C_2}.$$ 

We recall a particular case of [19, Lemma 5.1.2].

Lemma A.1. Let $X$ be a strong martingale and let $\varphi$ be an $\mathcal{A}(u)$-valued simple flow defined on $[0, 1]$. Then the one-parameter process $(X_{\varphi(s)})_{s \in [0, 1]}$ is a martingale with respect to the filtration $(\mathcal{F}_{\varphi(s)})_{s \in [0, 1]}$.

A.2. Malliavin calculus for two-parameter Brownian motion

Let $(W_{(s,t)})_{(s,t) \in [0, 1]^2}$ be a two-parameter Brownian motion defined on a probability basis $(\Omega, \mathcal{F}, (\mathcal{F}_z)_{z \in [0, 1]^2}, \mathbb{P})$. This process is a centered Gaussian process whose covariance given by

$$\mathbb{E}\left[W_{(s,t)}W_{(s',t')}\right] = (s \land s')(t \land t'), \quad (s, t) \in [0, 1]^2, (s', t') \in [0, 1]^2.$$
The Malliavin calculus for general Gaussian processes has been described in [31] and the reader can refer to it for a complete explanation about this topic. Here we give the definition of the Malliavin derivative and we present the integration by parts formula which is hardly used in Section 3.

**Definition A.2.** Let \( S \) be the space of random variable \( F \) of the form

\[
F = f(W(h_1), \ldots, W(h_n)),
\]

where \( h_i \) is an element of \( L^2([0, 1]^2, dz) \) and \( W(h_i) \) denotes the stochastic integral \( W(h_i) := \int_{[0,1]^2} h_i(z) \, dz \) for \( i = 1, \ldots, n \) and \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is infinitely continuously differentiable.

For \( F \) of the form (A.1) we define the Malliavin derivative \( DF \) of \( F \) as the following \( L^2([0, 1]^2, dz) \)-valued random variable

\[
DF := \sum_{i=1}^n \partial_i f(W(h_i)) \, h_i. \tag{A.2}
\]

Here we give the Malliavin integration by parts formula [31, Lemma 1.2.1].

**Lemma A.3.** Let \( F \) be in \( S \) and \((u(s,t))_{(s,t) \in [0,1]^2}\) in the domain of the divergence operator \( \delta \). We have

\[
\mathbb{E}[F \delta(u)] = \mathbb{E} \left[ (DF, \delta(u))_{L^2([0,1]^2,dz)} \right]. \tag{A.3}
\]

\( \delta \) is called the divergence operator (or the Skorohod integral of \( u \)) and it extends the Itô integral since when \( u \) is adapted to the filtration generated by the two-parameter Brownian motion \( W \),

\[
\delta(u) = \int_{[0,1]^2} u(s,t) \, dW(s,t).
\]

Moreover we recall the Itô isometry

\[
\mathbb{E}[\delta(u)^2] = \|u\|_{L^2(\Omega \times [0,1])}^2. \tag{A.4}
\]

Note also that the Malliavin derivative \( D \) is a closable operator (see [31, Proposition 1.2.1]) and we denote by \( \text{Dom}(D) \) its domain. Furthermore \( D \) satisfies a chain rule property that is, for each of the \( F \) and \( G \) elements of \( \text{Dom}(D) \) such that \( FG \) belongs to \( \text{Dom}(D) \) we have

\[
D(FG) = FDG + DFG. \tag{A.5}
\]

We end this section with the “Heisenberg commutativity relationship” which enables the computation of the gradient of a Itô stochastic integral, more precisely we have for a process \( u \) such that the right-hand side of (A.6) is well-defined (more details about the assumption on \( u \) can be found in [31, (1.46)]),

\[
D_{(s,t)} \delta(u) = u_{(s,t)} + \delta(D_{(s,t)}u), \quad (s, t) \in [0, 1]^2. \tag{A.6}
\]

**References**
