The Exponential Homomorphism for the Second Syntomic Cohomology

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Let $O_K$ be a complete discrete valuation ring with perfect residue field $F$ of characteristic $p \geq 3$ and $M$ a free filtered Dieudonné module. The second relative syntomic cohomology $H^2((O_K, F), \mathcal{F}(M, 2))$ with coefficients in $M$ is an object related to arithmetic geometry, such as the Albanese kernel. In this article, we study $H^2((O_K, F), \mathcal{F}(M, 2))$ by constructing the exponential homomorphism. We determine the structure of $H^2((O_K, F), \mathcal{F}(M, 2))$, under the assumption that $M$ is of Hodge–Witt type and that the absolute ramification index of $O_K$ is prime to $p$.

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1. INTRODUCTION

Let $F$ be a perfect field of characteristic $p \geq 3$, $W = W(F)$ the ring of Witt vectors of $F$, and $K_0$ be the field of fractions of $W$. Let $K/K_0$ be a totally ramified finite extension of degree $e$, and $O_K$ the ring of integers in $K$. In this article we study the relative syntomic cohomology

$$H^2((O_K, F), \mathcal{F}(M, 2))$$

with coefficients in $M$, which was defined in [14]. (We will review the definition; cf. Definition 2.3.) Here $M$ is a free filtered Dieudonné module over $W$ in the sense of Wintenberger [13] (cf. Definition 2.1; $M$ is a $W$-module endowed with a decreasing filtration $(M^i)_{i \in \mathbb{Z} \geq 0}$). Our relative syntomic cohomology with coefficients is a variant of the syntomic cohomology of Fontaine and Messing [6]; the one with coefficients has been considered in [11, 12]. It is related to objects of arithmetic geometry in many directions (e.g., Milnor $K$-groups, $p$-divisible groups, and Albanese kernel).
as explained in Examples 1.3–1.5 below. According to Bannai [1], it could be interpreted as a $p$-adic absolute Hodge cohomology.

Our study of the syntomic cohomology is based on a homomorphism

$$\exp : \frac{M}{M^e} \otimes_W \Omega_{O_K/W} \to H^2((O_K,F), \mathcal{F}(M,2)),$$

which we call the exponential homomorphism. This could be regarded as a generalization of Kurihara's exponential homomorphism [9], in which he treated the case that $M$ is the unit object. (However, he treated higher syntomic cohomology groups and did not assume the residue field $F$ to be perfect; cf. Example 1.3 for more on the relation with his work.)

By using our exponential homomorphism, we shall give a precise description of the structure of the group $H^2((O_K,F), \mathcal{F}(M,2))$, under the assumption that the extension $K/K_0$ is tamely ramified and that $M$ is of Hodge–Witt type in the sense of Kato [8] (cf. Definition 2.2). Under those assumptions, we shall introduce the group $H_M(O_K)$. We define $H_M(O_K)$ in an explicit and concrete way, depending on the choice of a prime element of $K$ and the structure of $M$. However, it is rather complicated. Here we only mention that $H_M(O_K)$ is a $p$-torsion group, and that it has a simple form when $e \leq p - 1$ (cf. Example 4.6). The reader should refer to Definition 4.5 for the precise definition.

**Theorem 1.1.** Assume that $M$ is of Hodge–Witt type. Then

(i) The homomorphism $\exp$ is surjective.

(ii) Assume further $e$ is prime to $p$. Then there exists a connected $p$-divisible group $G_M$ over $W$ (whose characterization is given in Remark 1.2 below) and an exact sequence

$$G_M(O_K/pO_K) \to H_M(O_K) \xrightarrow{\epsilon} H^2((O_K,F), \mathcal{F}(M,2)) \to 0.$$

Here, the map $\epsilon$ is defined by using the exponential homomorphism (cf. (8)).

**Remark 1.2.** Let $N$ be the level $[1, 2)$-part of $M$ (cf. Definition 2.2). The translation $N[1]$ of $N$ is a free filtered Dieudonné module of level $[0, 1)$. The connected $p$-divisible group $G_M$ over $W$ is defined to be the one corresponding to $N[1]$ via the (covariant) Dieudonné functor.

In Section 2, we review the definition of the relative syntomic cohomology. Section 3 is devoted to the definition of the exponential homomorphism and the proof of Theorem 1.1(i). In Section 4, we define the group $H_M(O_K)$ and prove Theorem 1.1(ii). In the rest of this section, we illustrate the relation of the syntomic cohomology to the arithmetic geometry.
Example 1.3. Let $M$ be the unit object, that is,

$$M = M^0 = W, \quad M^1 = 0, \quad \phi = \sigma \text{ (Frobenius endomorphism)}.$$  

Then it is an immediate consequence of the result of Kurihara [9] that $H^2((O_K, F), \mathcal{F}(M, 2))$ is isomorphic to the $p$-adic completion of the group $\ker(K_2^M(O_K) \to K_2^M(F))$, where $K_2^M$ denotes the Milnor $K_2$-functor. The composition map

$$\Omega_{O_K/W} \xrightarrow{\exp} H^2((O_K, F), \mathcal{F}(M, 2))$$

$$\cong \ker(K_2^M(O_K) \to K_2^M(F))^* \hookrightarrow K_2^M(O_K)^*$$

is nothing but Kurihara’s exponential homomorphism [9] for $\eta = p$. (Here $^*$ means the $p$-adic completion.)

In this case we have $G_M = 0$, so that Theorem 1.1 gives a complete description of the group $H^2((O_K, F), \mathcal{F}(M, 2))$ when $(e, p) = 1$; if $(e, p - 1) = 1$, the group $H_M(O_K)$ is trivial. Otherwise it is isomorphic to $\text{coker}(1 + (\pi^e/p)\phi : F \to F)$, where $\pi \in K$ is a prime element of $K$. In view of the relation with Kurihara’s result, Theorem 1.1 provides a description of the structure of $K_2^M(O_K)$, which is of course compatible with results of Kahn [7].

This method to study the structure of $K_2^M(O_K)$ is due to Nakamura [10], in which he determined the structure of higher Milnor $K$-groups under the assumption $(e, p) = 1$, without the assumption of $F$ being perfect. Our method to study $H^2((O_K, F), \mathcal{F}(M, 2))$ owes much to him, especially for the idea of using the exact sequence (3) (cf. [10, Sect. 2]).

Example 1.4. Let $G$ be a connected $p$-divisible group over $W$, and $M$ the free filtered Dieudonné module corresponding to $G$ via the (covariant) Dieudonné functor. Again we have $G_M = 0$. When $G = G_m$ (the $p$-divisible group attached to the multiplicative group), $M$ is nothing but the unit object (cf. Example 1.3).

It is shown in [14, Theorem D] that there is a canonical isomorphism

$$G(O_K) \cong H^1((O_K, F), \mathcal{F}(M, 1)).$$

The group $M/M^1$ is canonically isomorphic to the tangent space $\text{Lie}(G)$ of $G$. By the same method as in this article, one can construct a homomorphism $(\ast)$ which fits into a commutative diagram

$$\begin{array}{ccc}
M/M^1 \otimes_W O_K & \xrightarrow{(\ast)} & H^1((O_K, F), \mathcal{F}(M, 1)) \\
\cong & & \cong \\
\text{Lie}(G) \otimes_W O_K & \xrightarrow{\exp} & G(O_K),
\end{array}$$
where the map in the bottom row is the classical exponential homomorphism.

In this setting, our exponential homomorphism can be written as

$$\text{Lie}(G) \otimes_W \Omega_{O_K/W} \rightarrow H^2((O_K, F), \mathcal{F}(M, 2)).$$

Comparing with the map $(*)$ above, this could be regarded as the degree two version of the classical exponential homomorphism. The group $H^2((O_K, F), \mathcal{F}(M, 2))$ might be considered as a kind of “translated $p$-divisible group” (cf. [3]).

**Example 1.5.** Let $X$ be a smooth projective scheme over $W$ of relative dimension $d < p$. Assume that the de Rham cohomology groups $H^i_{\text{dR}}(X/W)$ are free $W$-modules for any $j$. Let $r$ be an integer such that $0 < r < p$. Set $X_{O_k} = X \otimes_W O_k$ and $X_F = X \otimes_W F$. Let $K_{oX, O_k}$ and $K_{r, X_F}$ be the (Zariski) sheaf of Milnor $K$-groups on $X_{O_k}$ and $X_F$, respectively. The cohomology group

$$(2) \quad H^d(X_{O_k}, \ker(K_{r, X_{O_k}} \rightarrow K_{r, X_F}))$$

is important in arithmetic geometry. It is related to the Chow group $\text{CH}^d(X)$ of $X$ when $r = d$, and appears in the class field theory of $X$ when $r = d + 1$. We shall explain that the group $H^2((O_K, F), \mathcal{F}(H^{d+r-2}_{\text{dR}}(X/W)[r - 2], 2))$ has something to do with a certain subgroup of this group. (Here $[r - 2]$ means the translation. There is a natural structure of a free filtered Dieudonné module on $H^j_{\text{dR}}(X/W)$; cf. [4, 8].)

It is shown in [14] that the group (2) is nicely approximated by the relative syntomic cohomology group $H^{d+r}((X_{O_k}, X_F), S(r))$. In the simplest case $X = \text{Spec}(W)$ and $r = 2$; the group $H^2((\text{Spec}(O_k), \text{Spec}(F)), \mathcal{F}(2))$ is nothing but $H^2((O_K, F), \mathcal{F}(M, 2))$ with $M$ the unit object. This group nicely approximates the Milnor $K$-group as explained in Example 1.3 above. In general, it is shown in [14] that in the infinitesimal point of view the groups (2) and $H^{d+r}((X_{O_k}, X_F), \mathcal{F}(r))$ are isomorphic modulo “essentially zero functors.”

The group $H^*((X_{O_k}, X_F), \mathcal{F}(r))$ is related to our syntomic cohomology $H^*((O_K, F), \mathcal{F}(M, r))$ by the exact sequence

$$0 \rightarrow H^2((O_K, F), \mathcal{F}(H^{d+r-2}_{\text{dR}}(X/W), r)) \rightarrow H^{d+r}((X_{O_k}, X_F), \mathcal{F}(r)) \rightarrow H^1((O_K, F), \mathcal{F}(H^{d+r-1}_{\text{dR}}(X/W), r)) \rightarrow 0.$$
Theorem C). Using the translation, we have isomorphisms (cf. Remark 2.6)
\[ H^1((O_K, F), \mathcal{F}(H^{d+r-1}_{\text{dR}}(X/W), r)) \cong H^1((O_K, F), \mathcal{F}(H^{d+r-1}_{\text{dR}}(X/W)[r - 1], 1)), \]
\[ H^2((O_K, F), \mathcal{F}(H^{d+r-2}_{\text{dR}}(X/W), r)) \cong H^2((O_K, F), \mathcal{F}(H^{d+r-2}_{\text{dR}}(X/W)[r - 2], 2)). \]

Now we assume \( H^{d+r-1}_{\text{dR}}(X/W) \) to be of Hodge–Witt type. Then, it is shown in [14, Proposition 6.12] (also cf. Example 1.4) that the former group \( H^1((O_K, F), \mathcal{F}(H^{d+r-1}_{\text{dR}}(X/W)[r - 1], 1)) \) is isomorphic to the group of \( O_K \)-rational points of some \( p \)-divisible group over \( W \). Hence the latter group \( H^2((O_K, F), \mathcal{F}(H^{d+r-2}_{\text{dR}}(X/W)[r - 2], 2)) \) is the “non-pro-representable part” of \( H^{d+r}((X_{O_K}, X_F), \mathcal{F}(r)) \). Our exponential homomorphism in this setting
\[ H^d(X, \Omega_{X/W}^{−2}) \otimes W \Omega_{O_K/W} \rightarrow H^2((O_K, F), \mathcal{F}(H^{d+r-2}_{\text{dR}}(X/W)[r - 2], 2)) \]
suggests that \( H^d(X, \Omega_{X/W}^{−2}) \otimes W \Omega_{O_K/W} \) is the “tangent space” of \( H^2((O_K, F), \mathcal{F}(H^{d+r-2}_{\text{dR}}(X/W)[r - 2], 2)) \) (cf. [2]).

When \( r = d \), as \( H^{2d}((X_{O_K}, X_F), \mathcal{F}(d)) \) approximates \( H^d(X_{O_K}, \ker(K^d_{M_{X_{O_K}} \rightarrow K^d_{M_{X_F}}})) \) (and hence \( \text{CH}^d(X) \)), the group \( H^2((O_K, F), \mathcal{F}(H^{d-2}_{\text{dR}}(X/W)[r - 2], 2)) \) should approximate the “non-pro-representable part,” that is, the Albanese kernel. It seems an interesting problem to make a precise relation between those:

**Problem 1.6.** Let \( T(X_K) \) and \( T(X_F) \) be the kernel of the Albanese maps
\[ A_0(X_K) \rightarrow \text{Alb}_{X_K}(K) \quad \text{and} \quad A_0(X_F) \rightarrow \text{Alb}_{X_F}(F), \]
respectively. (Here \( A_0(\ast) \) denotes the degree zero part of the Chow group \( \text{CH}^d(\ast) \).) Does there exist a canonical surjection
\[ H^2((O_K, F), \mathcal{F}(H^{d-2}_{\text{dR}}(X/W)[d - 2], 2)) \rightarrow \ker(T(X_K) \rightarrow T(X_F))\{p\}? \]
(Here \( \{p\} \) means the \( p \)-primary part.)

2. DEFINITION OF THE SYNTOMIC COHOMOLOGY

Let \( \sigma : W \rightarrow W \) be the Frobenius endomorphism. We recall the two definitions.

**Definition 2.1** (cf. [13]). A filtered Dieudonné module over \( W \) (of finite type) is a free \( W \)-module \( M \) of finite rank, endowed with

(i) a decreasing filtration \( (M^i)_{i \in \mathbb{Z}_{\geq 0}} \) such that \( M^0 = M, M^i = 0 \)
\( (i \gg 0) \);
(ii) a $\sigma$-linear homomorphism $\phi : M \to M$, satisfying the conditions

(iii) $M'$ is a direct summand of $M$ as a $W$-module for all $i \in \mathbb{Z}_{\geq 0}$;

(iv) $\phi(M') \subset p^i M$ for all $i \in \mathbb{Z}_{\geq 0}$ and $M = \sum_{i=0}^{\infty} (1/p^i) \phi(M')$.

**Definition 2.2.** A free filtered Dieudonné module $M$ is said to be of Hodge–Witt type if there is a (finite) sequence of subobjects of $M$ $(M_i)_{0 \leq i \leq N}$ such that

$$0 = M_0 \subset M_1 \subset \cdots \subset M_N = M$$

and such that, for each $0 \leq i \leq N - 1$, $(M_{i+1}/M_i)^i = M_{i+1}/M_i$, $(M_{i+1}/M_i)^{i+2} = 0$, and the slopes of $(M_{i+1}/M_i)$ is in the interval $[i, i+1)$. The quotient object $M_{i+1}/M_i$ is called the level $[i, i+1)$-part of $M$.

This definition coincides with Kato’s one in [8, Part II Definition 2.12], as is seen by (a slight modification of) [8, Part II Lemma 2.14].

We recall the definition of the syntomic cohomology. We shall follow the method of Kato [8], namely, we directly define a complex on the Zariski site. (We shall not use the syntomic site of Fontaine and Messing [6]; cf. [8, Remark 1.1].)

Let $M$ be a free filtered Dieudonné module. We use the notation $\phi_r = p^{-r}\phi$, which is defined on $\sum_{i=0}^\infty p^i M^{r-i} \subset M$. Let $B = W[[T]]$ be the ring of formal power series. Fix a prime element $\pi_K \in K$. Let $f(T) \in B$ be the monic minimal polynomial of $\pi_K$, so that $f(T)$ is an Eisenstein polynomial of degree $e$. We identify $B/(f(T))$ and $O_K$ by the isomorphism $B/(f(T)) \cong O_K$; $T \mapsto \pi_K$. Let $D$ be the PD-envelope of $B$ with respect to the ideal $(f(T))$, so that

$$D = B \left[ \frac{f(T)^m}{m!} \mid m \in \mathbb{Z}_{\geq 0} \right] = B \left[ \frac{T^m}{m!} \mid m \in \mathbb{Z}_{\geq 0} \right].$$

The projection map $B \to O_K$ can be naturally extended to $D \to O_K$, whose kernel is denoted by $J$. Let $I = J + pD$ be the kernel of the composite map $D \to O_K \to O_K/pO_K$. For $r \in \mathbb{Z}_{\geq 0}$, let $f^r$ and $I^r$ be the $r$th divided power of $J$ and $I$, respectively. We define $f^r = I^r = D$ for $r \leq 0$.

Let $\hat{\Omega}_{B/W}$ be the $p$-adic completion of the differential module $\Omega_{B/W}$ of $B$. This is a free $B$-module of rank one with a base $dT$. The differential map $d : B \to \hat{\Omega}_{B/W}$ can be extended to $d : D \to D \otimes_B \hat{\Omega}_{B/W}$. We shall use the (trivial) connection

$$\nabla = id \otimes d : M \otimes_W D \to M \otimes_B D \otimes \hat{\Omega}_{B/W}.$$
This connection restricts to
\[
\sum_{i=0}^{r} M'^{-i} \otimes_W I[i] \rightarrow \sum_{i=0}^{r-1} M'^{-i-1} \otimes_W I[i] \otimes_B \hat{\Omega}_{B/W},
\]
\[
\sum_{i=0}^{r} M'^{-i} \otimes_W J[i] \rightarrow \sum_{i=0}^{r-1} M'^{-i-1} \otimes_W J[i] \otimes_B \hat{\Omega}_{B/W},
\]
which are also denoted by \( \nabla \).

Let \( r \) be an integer satisfying \( 0 \leq r < p \). Let \( \phi \) be an endomorphism of \( B \) defined by \( \phi(T) = T^p \) and \( \phi(a) = \sigma(a) \) \( (a \in W) \). This endomorphism \( \phi \) can be naturally extended to an endomorphism of \( D \). Furthermore, we have \( \phi(I[i]) \subseteq pD \) by the assumption \( r < p \) (cf. [8, Part I, Lemma 1.3]). Thus we can define \( \phi_r = p^{-r} \phi : I[i] \rightarrow D \). (We need the cases \( r = 1, 2 \). This is the reason why we need to assume \( p \geq 3 \).) We define for \( 0 \leq r < p \)
\[
\phi_r : \sum_{i=0}^{r} M'^{-i} \otimes_W I[i] \rightarrow M \otimes_W D
\]
to be the unique homomorphism which coincides with \( \phi_{r-i} \otimes \phi_i \) on \( M'^{-i} \otimes I[i] \) for each \( i \). The restriction of \( \phi_r \) to \( \sum_{i=0}^{r} M'^{-i-1} \otimes J[i] \) is also denoted by \( \phi_r \).

The endomorphism \( \phi \) induces an endomorphism of \( \hat{\Omega}_{B/W} \), which is also denoted by \( \phi \). Since \( \phi(\hat{\Omega}_{B/W}) \subseteq p\hat{\Omega}_{B/W} \), we can define \( \phi_1 = p^{-1} \phi \), which is an endomorphism of \( \hat{\Omega}_{B/W} \). We define
\[
\phi_r : \left( \sum_{i=0}^{r-1} M'^{-i-1} \otimes_W I[i] \right) \otimes_B \hat{\Omega}_{B/W} \rightarrow M \otimes_W D \otimes_B \hat{\Omega}_{B/W}
\]
to be a unique homomorphism which coincides with \( \phi_{r-i-1} \otimes \phi_i \otimes \phi_1 \) on \( M'^{-i-1} \otimes I[i] \otimes \hat{\Omega}_{B/W} \) for each \( i \). The restriction of \( \phi_r \) to \( \sum_{i=0}^{r-1} M'^{-i-1} \otimes J[i] \otimes \hat{\Omega}_{B/W} \) is also denoted by \( \phi_r \).

We define the complex \( \mathcal{E}(M, r)_{O_k} \) and \( \mathcal{E}(M, r)_{O_k/pO_k} \) to be the complex of abelian groups
\[
\sum_{i=0}^{r} M'^{-i} \otimes_W J[i]^{(V, 1-\phi)} \rightarrow \left( \sum_{i=0}^{r-1} M'^{-i-1} \otimes_W J[i] \right) \otimes_B \hat{\Omega}_{B/W}
\]
\[
\oplus (M \otimes_W D)^{(1-\phi, v)} \rightarrow M \otimes_W D \otimes_B \hat{\Omega}_{B/W}
\]
and
\[
\sum_{i=0}^{r} M'^{-i} \otimes_W I[i]^{(V, 1-\phi)} \rightarrow \left( \sum_{i=0}^{r-1} M'^{-i-1} \otimes_W I[i] \right) \otimes_B \hat{\Omega}_{B/W}
\]
\[
\oplus (M \otimes_W D)^{(1-\phi, v)} \rightarrow M \otimes_W D \otimes_B \hat{\Omega}_{B/W},
\]
respectively. Since the ring \( O_K/pO_K \) is isomorphic to \( F[T]/(T^r) \), we sometimes use the notation \( \mathcal{F}(M, r)_{O_k/pO_k} = \mathcal{F}(M, r)_{F[T]/(T^r)} \).

**Definition 2.3.** Let \( M \) be a free filtered Dieudonné module and \( r \) an integer satisfying \( 0 \leq r < p \). The \( q \)th syntomic cohomology group \( H^q(O_K, \mathcal{F}(M, r)) \) (resp. \( H^q(O_K/pO_K, \mathcal{F}(M, r)) \)) of \( O_K \) (resp. \( O_k/pO_K \)) with coefficients in \( M \) is defined to be the \( q \)th cohomology group of \( \mathcal{F}(M, r)_{O_k} \) (resp. \( \mathcal{F}(M, r)_{O_k/pO_k} \)).

**Remark 2.4.** It is clear from the definition that

\[
H^q(O_K, \mathcal{F}(M, r)) = H^q(O_K/pO_K, \mathcal{F}(M, r)) = 0 \quad \text{for } q \geq 3.
\]

Next, we define the relative syntomic cohomology. Note that we already defined \( \mathcal{F}(M, r)_{W[T]/W} = \mathcal{F}(M, r)_F \) (as for the case \( K = K_0 \)). We define the complexes \( \mathcal{F}(M, r)_{O_k/pO_k, F} \) and \( \mathcal{F}(M, r)_{O_k, F} \) to be the mapping fibers of

\[
\mathcal{F}(M, r)_{O_k/pO_k} \xrightarrow{\psi} \mathcal{F}(M, r)_F \quad \text{and} \quad \mathcal{F}(M, r)_{O_k} \xrightarrow{\psi'} \mathcal{F}(M, r)_F.
\]

Here \( \psi \) is the map induced by \( O_K/pO_K = F[T]/(T^r) \rightarrow F; T \mapsto 0 \), and \( \psi' \) is the composition of the natural map \( \mathcal{F}(M, r)_{O_k} \rightarrow \mathcal{F}(M, r)_{O_k/pO_k} \) and \( \psi \). We remark that the complex \( \mathcal{F}(M, r)_F \) is quasi-isomorphic to

\[
\sum_{i=0}^{r} p^i M^{(i-1)} \rightarrow M.
\]

**Definition 2.5.** Let \( M \) and \( r \) be as in Definition 2.3. The \( q \)th relative syntomic cohomology \( H^q((O_K, F), \mathcal{F}(M, r)) \) (resp. \( H^q((O_K/pO_K, F), \mathcal{F}(M, r)) \)) of \( O_K \) (resp. \( O_k/pO_k \)) with coefficients in \( M \) is defined to be the \( q \)th cohomology group of \( \mathcal{F}(M, r)_{O_k, F} \) (resp. \( \mathcal{F}(M, r)_{O_k/pO_k, F} \)).

**Remark 2.6.**

(i) If \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) is an exact sequence of filtered Dieudonné modules, then the sequence of complexes

\[
0 \rightarrow \mathcal{F}(L, r)_* \rightarrow \mathcal{F}(M, r)_* \rightarrow \mathcal{F}(N, r)_* \rightarrow 0
\]

is exact for \( * = O_K, O_K/pO_K, (O_K, F) \) and \((O_K/pO_K, F)\).

(ii) If \( 0 \leq q \leq r < p \) and \( M \) is a free filtered Dieudonné module such that \( M^q = M \), then

\[
\mathcal{F}(M, r)_* = \mathcal{F}(M[q], r-q)_*,
\]

for \( * = O_K, O_K/pO_K, (O_K, F) \) and \((O_K/pO_K, F)\). (Here \( M[q] \) is the translation of \( M \), which is defined by \( M[q]^i = M^{i+q} \) and \( p^{-q} \).

The following lemma will be used later.
Lemma 2.7. Let $M$ be a free filtered Dieudonné module and $0 \leq r < p$. Then, we have

\[
H^q(F, S(M, r)) = 0 \quad \text{for any } q \geq 2
\]
\[
H^q(F, F, S(M, r)) = 0 \quad \text{for any } q \geq 0
\]
\[
H^q(O_K, S(M, r)) = H^q(O_K/pO_K, S(M, r)) = 0
\]
\[
\quad \text{for any } q \geq \max(r + 1, 2)
\]
\[
H^q((O_K, F, S(M, r)) = H^q((O_K/pO_K, F), S(M, r)) = 0
\]
\[
\quad \text{for any } q \geq r + 1.
\]

Proof. The second equation is clear from the definition. Other equations are proved in [14]. \[\square\]

3. Definition of the Exponential Homomorphism

For a free filtered Dieudonné module $M$ and $0 \leq r < p$, we define the complex $S(M, r)$ by the exact sequence

\[
0 \to S(M, r)_{O_K} \to S(M, r)_{O_K/pO_K} \to S(M, r) \to 0.
\]

Then, we also have an exact sequence

\[
0 \to S(M, r)_{O_K/F} \to S(M, r)_{O_K/pO_K/F} \to S(M, r) \to 0.
\]

Since $S(M, 2)$ is acyclic outside $[0, 1]$, we have an exact sequence

\[
H^1(S(M, 2)) \to H^2((O_K, F), S(M, 2)) \to 0.
\]

Lemma 3.1. There is an isomorphism

\[
\beta : H^1(S(M, 2)) \simeq \frac{M}{M^1} \otimes_W \frac{\Omega_{O_K/W}}{p\Omega_{O_K}}.
\]

Proof (cf. Nakamura [10]). By definition, we have

\[
H^1(S(M, 2)) = \text{coker}\left(\frac{\sum_{i=0}^{2} M^{2-i} \otimes I[i]}{\sum_{i=0}^{1} M^{1-i} \otimes J[i]} \otimes \Omega_{B/W}\right).
\]

Via the isomorphism $M \otimes D/(M \otimes J + M^1 \otimes D) \simeq M/M^1 \otimes O_K$, the subgroup $(M \otimes I + M^1 \otimes D)/(M \otimes J + M^1 \otimes D)$ maps to $M/M^1 \otimes pO_K$. Thus we have

\[
H^1(S(M, 2)) \simeq M/M^1 \otimes_W \text{coker}(I[2] \xrightarrow{d} pO_K \otimes \hat{\Omega}_{B/W}).
\]
Using the exact sequence

\[(f(T))/(f(T)^2) \to O_K \otimes \hat{\Omega}_{B/W} \to \Omega_{O_K/W} \to 0,\]

we have

\[H^1(\epsilon(M, 2)) \cong \frac{M}{M^1} \otimes \frac{pO_K \otimes \hat{\Omega}_{B/W}}{d(p^2B + pf(T)B)} \]

\[\cong \frac{M}{M^1} \otimes \frac{O_K \otimes \hat{\Omega}_{B/W}}{d(pB + f(T)B)} \cong \frac{M}{M^1} \otimes \frac{\Omega_{O_K/W}}{d(pO_K)}.\]

(Note that \(O_K \otimes \hat{\Omega}_{B/W}\) is a free \(W\)-module.) This completes the proof. □

**Definition 3.2.** We define the exponential homomorphism \(\exp\) by the composition map

\[\begin{align*}
M/M^1 \otimes \Omega_{O_K/W} &\xrightarrow{\text{proj.}} M/M^1 \otimes \frac{\Omega_{O_K/W}}{pdO_K} \\
&\xrightarrow{\beta^{-1}} H^1(\epsilon(M, 2)) \\
&\xrightarrow{\alpha} H^2((O_K, F), \mathcal{F}(M, 2)).
\end{align*}\]

**Remark 3.3.** When \(F\) is a finite field, the group \(H^2((O_K, F), \mathcal{F}(M, 2))\) is finite (without assumption that \(M\) be of Hodge–Witt type nor \((e, p) = 1\). This fact can be shown as follows. Since the group \(\Omega_{O_K/W}\) is a finite group and by the exact sequence (3), we have only to show the finiteness of \(H^2((O_K/pO_K, F), \mathcal{F}(M, 2)) = H^2((F[T]/(T^e), F), \mathcal{F}(M, 2))\). By using the method of [14], one can show that there exists an exact sequence

\[\begin{align*}
M/M^1 \otimes_W \ker(F[T]/(T^e)) &\to F[T]/(T^{e-1}) \otimes_B \hat{\Omega}_{B/W} \\
&\to H^2((F[T]/(T^e), F), \mathcal{F}(M, 2)) \\
&\to H^2((F[T]/(T^{e-1}), F), \mathcal{F}(M, 2)) \to 0.
\end{align*}\]

(cf. [14, Proposition 6.2] and its proof). Since \(H^2((F, F), \mathcal{F}(M, 2)) = 0\) (cf. Lemma 2.7), this exact sequence deduces the finiteness by the induction on \(e\). A similar fact also holds for the non-relative cohomology \(H^2(O_K, \mathcal{F}(M, 2))\).

**Proof of Theorem 1.1(i).** Assume \(M\) to be of Hodge–Witt type. In view of the exact sequence (3), we need to show that the group \(H^2((O_K/pO_K, F), \mathcal{F}(M, 2))\) is trivial. Let \(N\) be the level \((0, 1)\)-part of \(M\). There exists an exact sequence

\[H^2((O_K/pO_K, F), \mathcal{F}(N, 2)) \to H^2((O_K/pO_K, F), \mathcal{F}(M, 2)) \to H^2((O_K/pO_K, F), \mathcal{F}(M/N, 2)).\]

The vanishing of the third term is a consequence of Remark 2.6(ii) and Lemma 2.7(iv). The following lemma shows the vanishing of the first term.
Lemma 3.4. If $M$ is a free filtered Dieudonné module satisfying $M^2 = 0$, then we have

(i) $H^1(F, \mathcal{F}(M, 2)) = 0,$

(ii) $H^2((O_K/pO_K, \mathcal{F}(M, 2)) = 0.$

Proof. The complex $\mathcal{F}(M, 2)_F$ is quasi-isomorphic to $p^2M + pM^1 \xrightarrow{1-\phi_2} M.$

Since $M^2 = 0$, the map $p^2M + pM^1 \xrightarrow{1-\phi_2} M$ is an isomorphism (cf. Definition 2.1(iv)). Hence this complex is exact, and assertion (i) follows.

Remember the group $H^2((O_K/pO_K, \mathcal{F}(M, 2))$ is defined to be the cokernel of the map $(1-\phi_2, \nabla) : ((M \otimes_W I + M^1 \otimes_W D) \otimes_B \hat{\Omega}_{B/W}) \\
\oplus (M \otimes_W D) \to M \otimes_W D \otimes_B \hat{\Omega}_{B/W}.$

Take $v \in M$ and $m \in \mathbb{Z}_{>0}$. To finish the proof, we need to show that $w = v \otimes (T^{-1}dT/[(m-1)/e])$ is in the image of $(1-\phi_2, \nabla)$.

If $m > e$, then $w \in M \otimes I \otimes \hat{\Omega}_{B/W}$, so that $w_1 = \phi_2(w)$ is defined and again in $M \otimes I \otimes \hat{\Omega}_{B/W}$. We can define inductively $w_{i+1} = \phi_2(w_i)$. Then the series

$$w + w_1 + w_2 + \cdots$$

is convergent ($p$-adically) in $M \otimes I \otimes \hat{\Omega}_{B/W}$. If we write this element by $y$, then

$$(1-\phi_2, \nabla)(y, 0) = w.$$

Now assume $m \leq e$. In this case, $[(m-1)/e]! = 1$, so that $w = v \otimes T^{-1}dT.$ Write $m = m_0p^i$ with $(m_0, p) = 1$. We use the induction on $i$. If $i = 0$, then $m$ is invertible, and we have

$$(1-\phi_2, \nabla)(0, \frac{1}{m}v \otimes T^m) = w.$$ 

Assume $i > 0$. Since $M^2 = 0$, we can write $v = \phi(v_0) + \phi_1(v_1)$ with $v_0 \in M$, $v_1 \in M^1$. Then we have

$$(1-\phi_2, \nabla)((pv_0 + v_1) \otimes T^{m_0p^{i-1}}dT, 0)$$

$$= w + (pv_0 + v_1) \otimes T^{m_0p^{i-1}}dT.$$

By the inductive hypothesis, the second term of the right hand side is also in the image of $(1-\phi_2, \nabla)$. This completes the proof. $\blacksquare$
4. THE KERNEL OF THE EXPONENTIAL HOMOMORPHISM

In this section, we assume that \( e \) is prime to \( p \). Then, we can take a prime element \( \pi \) of \( K \) which satisfies \( a = \pi^e/p \in W^* \), so that \( f(T) = T^e - pa \).

In what follows, we fix such a \( \pi \). We first define the group \( H_M(O_K) \).

Let \( M \) be a free filtered Dieudonné module such that \( M^2 = 0 \). Then we have a \( \sigma^{-1} \)-linear map \( V : M \to M \) which is characterized by \( V(\phi(x) + \phi_1(y)) = px + y \) (\( x \in M, y \in M^1 \)) (cf. [5, Sect. 9]). The image of \( V \) is \( pM + M^1 \) and \( V \) is the inverse map of \( \phi_1 : pM + M^1 \to M \). Set \( \bar{M} = M/(pM + M^1) \). We define two maps:

\[
\begin{align*}
\rho : M & \to \bar{M}; \quad \text{the projection,} \\
\tau : pM + M^1 & \to \bar{M}; \quad \tau(px + y) = \rho(x) \quad (x \in M, y \in M^1).
\end{align*}
\]

Let \( n \) be a positive integer. We consider a series of positive integers \( L = (i_j)_{j \in \mathbb{Z}/n \mathbb{Z}} \). (We also use the notation \( L = (i_0, \ldots, i_{n-1}) \). We call \( n \) the length of \( L \).) For \( L = (i_j)_{j \in \mathbb{Z}/n \mathbb{Z}} \), we define the group \( H_M(O_K, L) \) to be the quotient group of \( \bar{M} \) by the subgroup

\[
\left\{ a\rho(u_0) + \tau V^{i_{j-1}}(u_{j-1}) \mid \text{there exists } u_0, \ldots, u_{n-1} \in M \text{ such that } \tau V^{i_j}(u_j) + a\rho(u_{j+1}) = 0 \right\}
\]

\((V^i = V \circ \cdots \circ V \text{ (} i \text{ times)})\). For given \( M \) and \( L \), it is a problem of linear algebra to calculate \( H_M(O_K, L) \).

**Example 4.1.** When \( L = (1) \) (length 1), we have

\[
H_M(O_K, (1)) \cong \frac{\bar{M}}{(1 + a\phi)(M) + \rho(\phi_1(M^1))}.
\]

**Example 4.2.** Assume \( M^1 = 0 \) (e.g., the unit object (cf. Example 1.3)). For \( L = (1, 1, \ldots, 1) \) of length \( n \), we have

\[
H_M(O_K, L) \cong \frac{\bar{M}}{(1 + a^{1+p+\cdots+p^{n-1}}\phi^n)(\bar{M})}.
\]

If \( L \) is not of the form \((1, 1, \ldots, 1)\), then the group \( H_M(O_K, L) \) is trivial.

Let \( S = \{ m \in \mathbb{Z} \mid 1 \leq m \leq e/(p - 1) \} \). For \( k \in S \), we define a series \( \{m_j^{(k)}\}_{j \in \mathbb{Z}/n \mathbb{Z}} \) of integers inductively as follows: Set \( m_0^{(k)} = k \). Assume we have defined \( m_j^{(k)} \). If \( (m_j^{(k)} + e, p) = 1 \) or if \( m_j^{(k)} = 0 \), then we define \( m_j^{(k)} = 0 \). Otherwise, writing \( m_j^{(k)} + e = np^s \) with \( (n, p) = 1 \), we define \( m_{j+1}^{(k)} = n \). Then \( 0 \leq m_j^{(k)} \leq e/(p - 1) \) for all \( j \). Let \( S_0 \) be the set of all \( k \) such that \( m_j^{(k)} = 0 \) for some \( j \). For \( k \in S - S_0 \), let \( S_k = \{m_j^{(k)} \mid j = 0, 1, 2, \ldots\} \) and \( n^{(k)} = \min\{j \mid m_j^{(k)} = k, j > 0\} \) (= Card\((S_k)\)). Define \( \Gamma = \{k \in S - S_0 \mid k = \min(S_k)\} \).
There exists a decomposition

\[ S = S_0 \cup \left( \bigcup_{k \in \Gamma} S_k \right) \]  

(disjoint union).

**Example 4.3.** If \( e \leq p - 2 \), then \( S = S_0 \). If \( e = p - 1 \), the decomposition (4) can be written as

\[ S = S_0 \cup \{1\}. \]

**Example 4.4.** When \( p = 3 \), the decomposition (4) in some cases is as follows:

- If \( e = 8 \), then \( S = S_0 \cup \{1\} \cup \{4\} \).
- If \( e = 13 \), then \( S = S_0 \cup \{2, 5\} \).
- If \( e = 80 \), then \( S = S_0 \cup \{1\} \cup \{4, 28\} \cup \{10\} \cup \{13, 31, 37\} \).

For \( k \in S - S_0 \), we define a series of integers of length \( n^{(k)} \)

\[ L^{(k)} = (i_0^{(k)}, i_1^{(k)}, \ldots, i_{n-1}^{(k)}) \]

by \( i_j^{(k)} = \min \{ i \mid e < m_j^{(k)} p^i \} \). The numbers \( i_j^{(k)} \) and \( m_j^{(k)} \) depend only on \( j \) modulo \( n^{(k)} \). We remark that the series \( L^{(k)} \) can recover \( \{m_j^{(k)}\} \):

\[ m_j^{(k)} = \frac{p^{\sum_{r=1}^{\infty} i_{j+r}} + p^{\sum_{r=1}^{\infty} i_{j+r} + 1} + \cdots + p^{j+1} + 1}{p^{\sum_{r=1}^{\infty} i_{j} - 1}}. \]

**Definition 4.5.** Let \( M \) be a free filtered Dieudonné module of Hodge–Witt type. Let \( N \) be the level \([0, 1]\)-part of \( M \). (Hence \( \overline{N} \cong \overline{M} \).) We define the group \( H_M(O_K) \) to be

\[ \bigoplus_{k \in \Gamma} H_N(O_K, L^{(k)}). \]

**Example 4.6.** If \( e \leq p - 2 \), \( H_M(O_K) = 0 \). If \( e = p - 1 \), writing by \( N \) the level \([0, 1]\)-part of \( M \), we have

\[ H_M(O_K) \cong \frac{\overline{N}}{(1 + a\phi)(\overline{N}) + \rho(\phi_1(\overline{N}^1))}. \]

The following lemma is the main step of the proof of Theorem 1.1(ii).

**Lemma 4.7.** Let \( M \) be a free filtered Dieudonné module satisfying \( M^2 = 0 \). Then the group \( H^2(O_K, \mathcal{J}(M, 2)) \) is isomorphic to \( H_M(O_K) \).
Proof. Since $e$ is prime to $p$, the group $\Omega_{O_K/W}$ is isomorphic to (as a $W$-module)

$$\bigoplus_{m=1}^{e-1} F\pi^{m-1} d\pi.$$ 

In particular, $pdO_K$ vanishes in $\Omega_{O_K/W}$. By the definition of $\mathcal{C}(M, 2)$, together with Lemmas 3.4 and 3.1, we have an exact sequence

$$H^1(O_K/pO_K, \mathcal{J}(M, 2)) \xrightarrow{\delta} M \otimes W \Omega_{O_K/W} \xrightarrow{\exp} H^2(O_K, \mathcal{J}(M, 2)) \to 0.$$ 

We shall calculate the image of $\delta$.

**Lemma 4.8.** The image of $\delta$ is generated by all elements of the forms

\begin{align*}
(5) \quad & \rho(v) \otimes \pi^{m-1} d\pi, \quad v \in M, \, 1 \leq m \leq e - 1, \, (m + e, p) = 1, \\
(6) \quad & (\tau V^{(m)}(v) \otimes \pi^{m-1} + a\rho(v) \otimes \pi^{mp^{i(m)} - e - 1}) d\pi, \\
& v \in M, \, 1 \leq m \leq e - 1, \, (m, p) = 1,
\end{align*}

where $i(m)$ is the least integer which satisfies $e < mp^{i(m)}$.

We first complete the proof of Lemma 4.7, admitting Lemma 4.8. We show that if $k > e/(p - 1)$ then $w = v \otimes \pi^{k-1} d\pi \in \text{Im}(\delta)$ for any $v \in \bar{M}$. To show this, we show the following assertion by the induction on $n$: If $k > p^{-n}(p^{n-1} + p^{n-2} + \cdots + p + 2)e$, then $w \in \text{Im}(\delta)$. Note that if $k$ is divisible by $p$, then $w$ is of the form (5), so the assertion holds. Thus we may assume $k$ is prime to $p$. Then $w$ may appear as the first term of (6) (note $i(k) = 1$ since $k > e/(p - 1)$). When $n = 1$, the assumption shows $kp - e > e$. Hence the second term of (6) is trivial, so the assertion holds. Assume $n > 1$. Then $kp - e > p^{-n-1}(p^{n-2} + p^{n-2} + \cdots + p + 2)e$, so the second term of (6) is in $\text{Im}(\delta)$ by the inductive hypothesis. The assertion is proved.

Next, we consider $w = v \otimes \pi^{k-1} d\pi$ for $k \in S = \{m \mid 1 \leq m \leq e/(p - 1)\}$. (We use the notations in the beginning of this section.) If $k \in S_0$, we see that $w$ is in the image of $\delta$ by (6) and (5). Let $k \in S - S_0$. We see by (6) that

$$\left(\bigoplus_{j=0}^{n^{(k)}-1} M \otimes \pi^{m^{(k)}-1} d\pi \right) \text{ modulo } \text{Im}(\delta) \cong H_M(O_K, L^{(k)}).$$

Now the decomposition (4) completes the proof of Lemma 4.7

**Proof of Lemma 4.8.** Remember that $H^1(O_K/pO_K, \mathcal{J}(M, 2))$ is a quotient of

$$\ker\left(\left( (M \otimes I + M^1 \otimes D) \otimes \hat{\Omega}_{B/W} \right) \oplus (M \otimes D)^{(1-\phi_H,v)} \to M \otimes D \otimes \hat{\Omega}_{B/W} \right).$$


The map

\[
1 - \phi_2 : (M \otimes I + M^1 \otimes D) \otimes \hat{\Omega}_{B/W} \rightarrow M \otimes D \otimes \hat{\Omega}_{B/W}
\]

is injective, as is seen by simple calculation. Thus, any element of \(H^1(O_K/pO_K, \mathcal{F}(M, 2))\) is represented by an element \(w\) of \((M \otimes I + M^1 \otimes D) \otimes \hat{\Omega}_{B/W}\) such that \((1 - \phi_2)(w) \in \nabla(M \otimes D)\).

Any element of \(M \otimes D \otimes \hat{\Omega}_{B/W}\) can be written as

\[
w' = \sum_{m=0}^{\infty} \nu_m' \otimes T^{m-1} dT, \quad \nu_m' \in ((m - 1)!)^{-1} M.
\]

Assuming \(w'\) is in \((M \otimes I + M^1 \otimes D) \otimes \hat{\Omega}_{B/W}\), the condition \((1 - \phi_2)(w') \in \text{Im}(\nabla)\) is equivalent to

\[
(1 - \phi_2) \left( \sum_{i=0}^{\infty} \nu_{m_i}' \otimes T^{m_{i-1}} dT \right) \in \text{Im}(\nabla)
\]

for any \(m_0 \in \mathbb{Z}_{\geq 0}\) such that \((p, m_0) = 1\).

Fixing \(m_0 \in \mathbb{Z}_{\geq 0}\) such that \((p, m_0) = 1\), we consider an element

\[
w = \sum_{i=0}^{\infty} \nu_i \otimes T^{m_{i-1}} dT.
\]

Considering the case \(m_0 > e\), a straightforward calculation shows that the element of the form (5) is in \(\text{Im}(\delta)\). We consider the case \(m_0 \leq e\). Let \(i(m_0)\) be the least integer which satisfies \(e < m_0 p^{i(m_0)}\). The condition \(w \in (M \otimes I + M^1 \otimes D) \otimes \hat{\Omega}_{B/W}\) is equivalent to

\[
v_i \in pM + M^1 \quad \text{for } 0 \leq i < i(m_0).
\]

On the other hand, we have

\[
(1 - \phi_2)(w) = \left( \nu_0 \otimes T^{m_0-1} + \sum_{i=1}^{\infty} (\nu_i - \phi_1(v_{i-1})) \otimes T^{m_{i-1}} dT \right).
\]

(Even if \(i > i(m_0)\), \(\phi_1(v_{i-1})\) is well-defined in \(((m_0p^{i(m_0)} - 1)/e)\). Thus the condition \((1 - \phi_2)(w) \in \text{Im}(\nabla)\) is equivalent to

\[
v_i = \phi_1(v_{i-1}) \bmod \frac{p^i}{((m_0p^{i(m_0)} - 1)/e)!} M \quad \text{for any } i \geq 1.
\]

This condition implies that \(\tau(v_i) = \tau V(v_{i+1})\) for \(0 \leq i < i(m_0)\). Setting \(v = v_{i(m_0)}\), we have

\[
\delta(u) = \left( \sum_{j=0}^{i(m_0)-1} \tau V^{i(m_0)-j}(v) \otimes \pi^{m_0 p^{i(m_0)} - 1} + ap(v) \otimes \pi^{m_0 p^{i(m_0)} - e - 1} \right) d\pi.
\]
(Here we used the fact \( f(T) = T^2 - pa \).) Conversely, for any \( v \in M \), there exists an element \( w \) satisfying this equation. By using elements of the form (5), we can replace (7) by (6). This completes the proof of the lemma.

**Proof of Theorem 1.1(ii).** Assume that \( M \) is a free filtered Dieudonné module of Hodge–Witt type and that \( e \) is prime to \( p \). Let \( N \) be the level \([0, 1)\)-part of \( M \). We have a commutative diagram whose rows and columns are exact

\[
\begin{array}{ccc}
H^0(\mathcal{E}(M/N, 2)) & \xrightarrow{\gamma} & H^1(\mathcal{E}(M, 2)) \\
\downarrow & & \downarrow \\
H^1(\mathcal{O}_K, F, \mathcal{F}(M/N, 2)) & \xrightarrow{\gamma} & H^2(\mathcal{O}_K, F, \mathcal{F}(M, 2)) \\
\downarrow & & \downarrow \\
H^1(\mathcal{O}_K \otimes \mathcal{O}_K, F, \mathcal{F}(M, 2)) & \xrightarrow{\gamma} & H^2(\mathcal{O}_K, F, \mathcal{F}(N, 2)) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

By Lemma 3.1, we have \( H^1(\mathcal{E}(M, 2)) \cong M/M^1 \otimes \Omega_{O_K/W}/pdO_K \) and \( H^1(\mathcal{E}(N, 2)) \cong N/N^1 \otimes \Omega_{O_K/W}/pdO_K \). They are isomorphic by the definition of \( N \). Thus \( \gamma \) is the zero map. The map \( \gamma' \) is surjective by Remark 2.6 and Lemma 2.7. Now we obtain an exact sequence

\[
H^1(\mathcal{O}_K \otimes \mathcal{O}_K, F, \mathcal{F}(M, 2)) \rightarrow H^2(\mathcal{O}_K, F, \mathcal{F}(N, 2)) \rightarrow 0.
\]

By Remark 2.6 and [14, Proposition 6.12] (cf. also Example 1.4), we have an isomorphism

\[
G_M(\mathcal{O}_K \otimes \mathcal{O}_K) \cong H^1(\mathcal{O}_K \otimes \mathcal{O}_K, F, \mathcal{F}(M, N, 2)),
\]

where \( G_M \) is a connected \( p \)-divisible group defined in Remark 1.2. By Lemmas 2.7, 3.4, and 4.7, we see

\[
H^2(\mathcal{O}_K, F, \mathcal{F}(N, 2)) \cong H^2(\mathcal{O}_K, \mathcal{F}(N, 2)) \cong H_M(\mathcal{O}_K).
\]

This completes the proof. By definition, the map \( \epsilon: H_M(\mathcal{O}_K) \rightarrow H^2(\mathcal{O}_K, \mathcal{F}(M, 2)) \) could be written as

\[
(8) \quad \epsilon \left( \sum_{k \in \Gamma} u_k \right) = \sum_{k \in \Gamma} \exp(\tilde{u}_k \otimes \pi^{k-1} d\pi),
\]

where \( u_k \in H_N(\mathcal{O}_K, L^{(k)}) \) and \( \tilde{u}_k \in M/M^1 \) is a lift of \( u_k \).
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REFERENCES