Reduction of the proton radius discrepancy by 3σ

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ARTICLE INFO

ABSTRACT

We show that in previous analyses of electron–proton scattering, the uncertainties in the statistical procedure to extract the proton charge radius are underestimated. Using a fit function based on a conformal mapping, we can describe the scattering data with high precision and extract a radius value in agreement with the one obtained from muonic hydrogen.

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1. Two principally different methods are commonly used to determine the proton charge radius r_E^p. On the one hand, it enters the QED calculations of atomic energy splittings (electronic and muonic [1] hydrogen) and can thus be obtained from measurements of these. On the other hand, r_E^p can be obtained from elastic electron–proton scattering. The corresponding cross sections can be parameterized in terms of the electric and magnetic Sachs form factors $G_E(Q^2)$ and $G_M(Q^2)$, respectively, that depend on the invariant momentum transfer squared $Q^2 = -t$. Positive Q^2-values refer to the scattering process, negative to annihilation/creation. The reduced cross section, here in the one-photon approximation, describes the deviation from the scattering off a point-like particle:

$$\frac{d\sigma}{d\Omega}\bigg|_{\mathrm{red}} = \frac{\tau}{\epsilon(1+\tau)}\left[G_M^2(Q^2) + \frac{\epsilon}{\tau}G_E^2(Q^2)\right], \tag{1}$$

where $\epsilon = [1 + 2(1+\tau)\tan^2(\theta/2)]^{-1}$ is the virtual photon polarization, θ is the electron scattering angle in the laboratory frame and $\tau = -t/4m_N^2$, with m_N the nucleon mass.

Both methods refer to the same quantity, the slope of the proton form factor at the origin:

$$r_{E,M}^p = \left(\frac{-6}{G_{E,M}(0)}\frac{dG_{E,M}(Q^2)}{dQ^2}\bigg|_{Q^2=0}\right)^{1/2}. \tag{2}$$

The form factor obtained from the cross sections has to be extrapolated from the data at lowest momentum transfer to the origin. The most precise electron–proton scattering data from Ref. [2] analyzed using spline and polynomial fit functions lead to a proton

charge radius that differs by $\sim 7\sigma$ from the muonic hydrogen radius of Ref. [1], when averaged with measurements in electronic hydrogen [3].

The purpose of this Letter is to illustrate that such extrapolations lack precision in purely statistical analyses with arbitrary fit functions. For example, the fit functions quoted in the final results of Ref. [2] are polynomials and splines. In this Letter, we construct a simple function, that describes the data *equally well* and corresponds to a small radius r_E^p in agreement with the one obtained from muonic hydrogen spectroscopy. This function is based on a conformal mapping and thus obeys the analytic structure of the form factors. The following function maps the cut in the t-plane onto the unit circle in a new variable z:

$$z(t, t_{\mathrm{cut}}) = \frac{\sqrt{t_{\mathrm{cut}} - t} - \sqrt{t_{\mathrm{cut}}}}{\sqrt{t_{\mathrm{cut}} - t} + \sqrt{t_{\mathrm{cut}}}}, \tag{3}$$

where $t_{\mathrm{cut}} = 4M_\pi^2$ is the lowest singularity of the form factors with M_π the charged pion mass. The Sachs form factors can then be expanded in the new variable z:

$$G_{E/M}(z(t)) = \sum_{k=0}^{k_{\mathrm{max}}} a_k z(t)^k. \tag{4}$$

Here, the form factors are normalized to the charge and anomalous magnetic moment of the proton, respectively.

Conformal mapping techniques are a standard tool in hadron physics. So far, they have not been applied to the electron–proton scattering data by the A1 Collaboration, the data for this process with the highest quoted precision. A previous elaborate analysis of world form factor data in a similar approach was carried out by Hill and Paz [4]. In contrast to their analysis, we do not constrain the parameters a_k any further to have a most flexible fit function, which is needed for the statistical reasoning here. Moreover,

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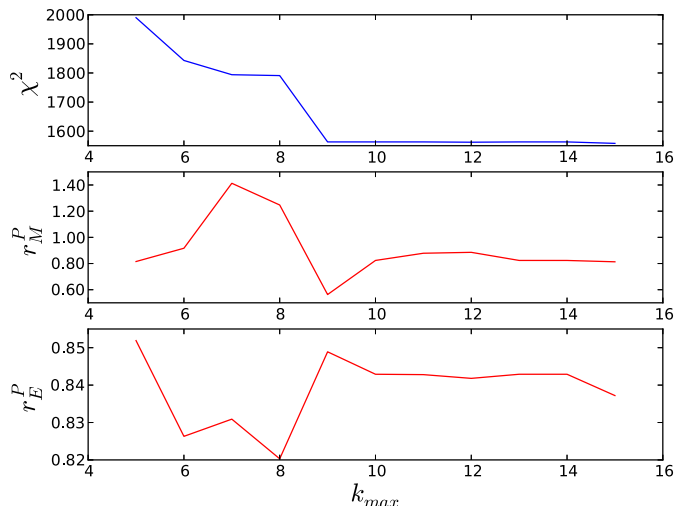meissner@hiskp.uni-bonn.de (U.-G. Meißner).

Fig. 1. Dependence of χ^2, r_E^P and r_M^P on the number of terms $k_{\max}$ in the expansion in the conformal mapping variable.

the results by Hill and Paz refer to older form factor data, that are extracted from cross sections mainly via the Rosenbluth method. We avoid the systematical uncertainties related to this procedure by directly fitting the cross sections. Also, the results by Hill and Paz show a strong ambiguity due to the included fit range. As we have shown before [5], this can be avoided in a full dispersion relation approach, since this makes use of the complete available information on the spectral function. Loose constraints on the coefficients in a z-expansion neglect the mass-related information on the spectral function.

We emphasize the mainly illustrative purposes of this work. This means that the significance of these fits lies in the comparison to the data analysis by the A1 Collaboration [2]. To allow for a direct comparison to that work, we use exactly the same data without further radiative corrections and with fixed normalization parameters (see the next section for details). Physically, the main advantage of the function used here, compared to the polynomials and splines used by the experimenters, is the correct inclusion of the lowest singularity of the form factors. Besides the basic analytic structure of the form factors, the imaginary part of the form factors can be constrained further due to unitarity, see e.g. [5], but this goes beyond the aim of this work. However, arbitrary coefficients correspond to an unconstrained spectral function. Therefore, the fits shown here have to be treated on the same footing as polynomial or spline fits.

2. To illustrate the comparison to the original analysis [2], we follow the same procedure to choose the number of parameters. This means, we increase the number of terms in the expansion (4) until the χ^2 of the fits reach a plateau and the fits stabilize. We have performed the calculations in python and checked the results with Mathematica. The fits are carried out using the lmfit package with several optimization methods as the Levenberg–Marquardt and simulated annealing algorithms [7].

The extracted radii and χ^2-values are shown in Fig. 1. For $k_{\max} = 9$, the absolute χ^2-value of 1563 is reached. This is exactly the value found in [6] for the best polynomial fit. For $k_{\max} = 10$, both electric and magnetic radii start to stabilize. The proton charge radius is found to be $r_E^P \simeq 0.84$ fm, consistent with the muonic hydrogen value of Ref. [1] and also the one obtained from a dispersion theoretical analysis of the Mainz and older data, including the ones for the neutron [5]. The level of variation in the magnetic radii is larger than in the electric case, as we expect,

since G_M is suppressed by a factor of the momentum transfer squared in the reduced cross section, cf. Eq. (1). The proton magnetic radius comes out as $r_M^P \simeq (0.85 \pm 0.04)$ fm, somewhat larger than the value found by the A1 Collaboration and within large variations compatible with the one obtained in Ref. [5]. The level of precision of the original analysis [2] is reproduced here only with unconstrained parameters. As an example, we give the parameters for the $k_{\max} = 10$ fit: $a_1 = -0.9481457$, $a_2 = -4.953483$, $a_3 = 88.55243$, $a_4 = -978.0812$, $a_5 = 6091.365$, $a_6 = -22558.25$, $a_7 = 50007.79$, $a_8 = -63978.5$, $a_9 = 42440.28$, $a_{10} = -10757.15$ for the electric form factor and $b_1 = -2.527861$, $b_2 = -12.71964$, $b_3 = 233.4448$, $b_4 = -2025.318$, $b_5 = 8129.828$, $b_6 = -15013.78$, $b_7 = 4935.569$, $b_8 = 26389.54$, $b_9 = -40617.2$, $b_{10} = 18526.76$ for the magnetic form factor. Clearly, such large coefficients generate an unphysical spectral function and thus are far from realistic, just like polynomial or spline fits. However, the crucial point is that constraints on the coefficients as suggested by Hill and Paz increase the χ^2 in our fits. This is also the case in a dispersive framework. Currently, there exist no fits in the literature with an equally good χ^2 to these data that obey all known physics constraints. In order to unambiguously distinguish between purely statistical and theoretically motivated data analyses, here we treat only statistical issues. The consideration of more physically motivated and constrained spectral functions will be covered in a later publication [8] using the statistical reasoning and unphysical fits shown here as a prerequisite. In contrast to here, the constraints from a realistic spectral function and the asymptotic behaviour naively expected from quark counting rules will be treated. The possible impact of data at larger Q^2 on the radius term is of course an additional source of uncertainty. The main point of this paper is to clearly show that the combined uncertainties of fits without physics input do not allow to distinguish between the 'small' and 'large' radius.

3. The cross sections corresponding to this fit are shown in Fig. 2 for the conformal mapping function with $k_{\max} = 10$. The six data sets for different energy settings of the incoming electron beam are separated by an offset. Note that each of these data sets contains measurements from three different spectrometers. The different experimental settings give rise to a normalization uncertainty between the individual data sets. One can take this into account by 31 floating normalization parameters, as described in the original analysis. According to Ref. [6], the data contains the normalizations determined by the spline fit. In order to show the underestimation of uncertainties in previous analyses, it is sufficient to keep the normalization parameters fixed. However, the additional normalization uncertainty covers an even larger range of radius values than given here when considered in floating normalizations. To be more precise, we exactly fit to the same data as was done by the A1 Collaboration just using an alternative fit function. Other issues like an improved treatment of radiative and two-photon corrections are not of relevance for this Letter but will be taken up in a later publication [8], as well as the rigorous inclusion of physics constraints. Here, this procedure is necessary for a proper comparison to the Mainz analysis.

4. As a further check on our fits, we now consider the form factor ratio that has been measured precisely using recoil polarization techniques. The form factor ratio from the illustrative fit with $k_{\max} = 10$ compares well to the recent measurements at Jefferson Laboratory, see Fig. 3. The displayed ratio is very similar to the one obtained in the spline fit in Ref. [6]. The 'wiggle' they found below $Q^2 = 0.2$ GeV^2 from the magnetic form factor is also reproduced. This could be interpreted as a result from overfitting, which would be consistent with the fact that the wiggle vanishes when including further physical constraints, see e.g. Ref. [5]. We
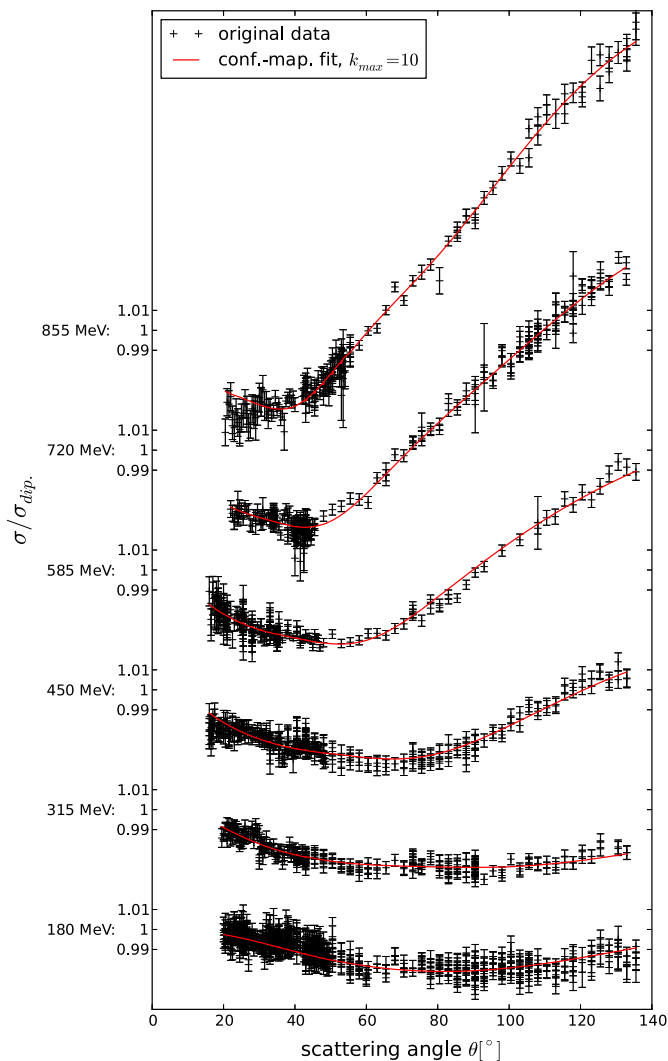
Fig. 2. Cross sections of elastic electron–proton scattering by the A1 Collaboration [2], divided by the cross section of the dipole form factors, σ_{dip}. All 1422 data points are fitted. The data measured at different energies of the incoming electron beam are shown with an offset.

Fig. 3. Prediction for the form factor ratio from the fit to cross sections. The data for the form factor ratio are from polarization measurements by Ron et al. [9] and Zhan et al. [10].

charge radius r_E^p are in perfect agreement with the values obtained via a dispersion relation approach [5] and the recent muonic hydrogen measurements. The remaining r_E^p-discrepancy is the $\sim 4\sigma$ deviation between the average of the spectroscopic measurements in electronic hydrogen and those in muonic hydrogen, see e.g. Ref. [3]. To solve this, further measurements in ordinary hydrogen are under way [3]. The planned muon–proton scattering experiment MUSE [12] might also shed further light on these issues.

Acknowledgements

do not want to enter this issue here in more detail but refer the reader to Ref. [11] for a discussion. However, under the assumption that all statistical and systematic errors are sufficiently under control, a good data description in terms of a low χ^2 is required. In this case, one has to consider polynomial, spline and unconstrained conformal mapping fits on the same footing. In principle, the latter is to be preferred due to the requirements from analyticity. Even if one were to neglect this fact, one can see from this work that a disagreement between the proton charge radius extracted from electron–proton scattering data and muonic hydrogen cannot be inferred from polynomial or spline fits, as one neglects a sizeable source of uncertainty.

5. In this Letter, we have reanalyzed the recent elastic electron–proton scattering data from Mainz with a fit function that is sufficiently flexible to describe the data with a given precision, i.e. with the same precision as achieved by the experimenters using spline and polynomial fit functions. The results for the proton
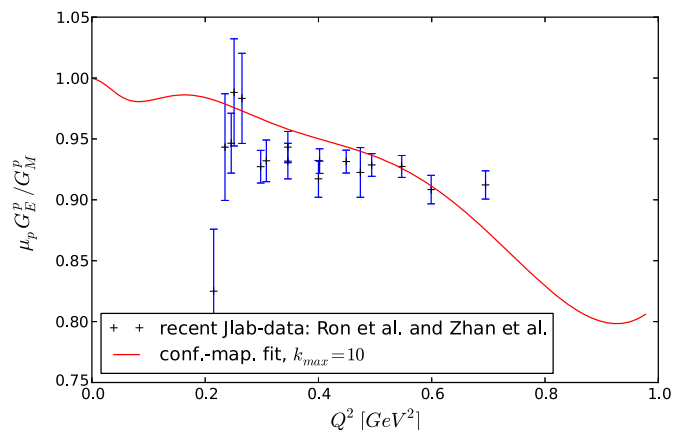
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