Discrete Mathematics 312 (2012) 3373–3383

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Critically paintable, choosable or colorable graphs

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ARTICLE INFO

Article history:
Received 31 March 2012
Received in revised form 19 July 2012
Accepted 26 July 2012
Available online 23 August 2012

Keywords:
Critical graph
Graph coloring
List coloring
Paintability
Graphs on surfaces

ABSTRACT

We extend results about critically k-colorable graphs to choosability and paintability (list colorability and on-line list colorability). Using a strong version of Brooks' Theorem, we generalize Gallai's Theorem about the structure of the low-degree subgraph of critically k-colorable graphs, and introduce a more adequate lowest-degree subgraph. We prove lower bounds for the edge density of critical graphs, and generalize Heawood's Map-Coloring Theorem about graphs on higher surfaces to paintability. We also show that on a fixed given surface, there are only finitely many critically k-paintable/k-choosable/k-colorable graphs, if $k \geq 6$. In this situation, we can determine in polynomial time k-paintability, k-choosability and k-colorability, by giving a polynomial time coloring strategy for "Mrs. Correct". Our generalizations of k-choosability theorems also concern the treatment of non-constant list sizes (non-constant k). Finally, we use a Ramsey-type lemma to deduce all 2-paintable, 2-choosable, critically 3-paintable and critically 3-choosable graphs, with respect to vertex deletion and to edge deletion.

1. Introduction

About 1950 G.A. Dirac introduced the concept of *criticality* (Definition 2.1) as a methodological approach in the study of graph colorability (see Proposition 2.2). This approach was then further developed in over 70 papers. In Kostochka's survey [11] it is discussed which of the results also hold for choosability. A main difficulty in finding criticality results for *choosability* and *paintability* (Definition 2.1, [17]) is that most construction methods for critically colorable graphs do not work for choosability and paintability. Even Dirac's very basic construction of $(k_1 + k_2)$-critical graphs G out of k_1-critical and k_2-critical graphs G_1 and G_2, by just taking the complete join, does not work for choosability and paintability. If, e.g., $G_1 = G_2 = \overline{K_n}$, with $n \geq 3$, then the complete join $G = G_1 \vee G_2 = K_{n,n}$ has a choice number bigger than the sum of the choice numbers of its components G_1 and G_2.

In Section 2, after the initial introduction of *almost ℓ-paintability/choosability* (with a slight notational difference to criticality), we briefly will discuss one special construction method for graphs that are at the same time critical for colorability, choosability and paintability. However, the main content of this section is Gallai's Theorem [6]. His theorem about the structure of the *low-degree subgraph* of a graph (our Theorem 2.5) is a cornerstone in the whole theory, see also [11,16]. It is a close relative to Brooks' Theorem, and it is frequently mentioned that Gallai's Theorem implies Brooks' Theorem. We show that the converse is also true. Gallai's Theorem follows easily from Brooks' Theorem, but one has to use Erdős, Rubin and Taylor's strengthened version of Brooks' result for degree-choosability (Definition 2.1) from [5]. Even if we want to prove Gallai's Theorem just for colorability, we have to use the choosability version of Brooks' Theorem.

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0012-365X/$ – see front matter
doi:10.1016/j.disc.2012.07.035

We will prove Gallai's Theorem even for paintability, and have to use a further strengthening of Brooks' Theorem for paintability [9, Theorem 5]. The choosability version of Gallai's Theorem, for a fixed system of color lists $L = (L_v)$ (we call it *almost L-colorability*), was already proven before, by Thomassen in [20].

In Section 3, we will use Gallai's Theorem to deduce the usual lower bounds for the edge density of critical graphs (almost ℓ-paintable/choosable/colorable graphs). Since we will generalize these bounds to the case of non-constant ℓ, we will need to work here with the *lowest-degree subgraph*; a variant of the low-degree subgraph, which we introduced in the section before.

Section 4, is concerned with graphs on surfaces. We show that Heawood's well known Map-Coloring Theorem [8], the generalization of the Four Color Theorem to higher surfaces, holds for paintability as well. We also show that there are only finitely many critically ℓ-paintable graphs on a fixed surface, if $\min(\ell) = \min_v(\ell_v) \geq 6$. Afterwards, we use this to deduce polynomial running time results for *L*-coloration and the Paint-Correct Game on surfaces.

Finally, in Section 5, we provide all 2-paintable/choosable graphs, and all almost 2-paintable/choosable (critically 3-paintable/choosable) graphs. The 2-choosable graphs were already classified by Erdős et al. in [5], and Zhu extended this to 2-paintability in [23]. The almost 2-choosable graphs, with respect to edge deletion, were classified by Voigt in [22]. The almost 2-paintable graphs, again with respect to edge deletion, are presently also studied in the still unpublished manuscript [2]. So, only the vertex critical results, i.e., the classification of almost 2-choosable and almost 2-paintable graphs, with respect to vertex deletion, are new. Vertex criticality is here a little bit more difficult to study. We also observed that all results should follow from just one Ramsey-type result (Lemma 5.1). After proving this central lemma, we split the remaining section into two quite similar subsections, one about almost 2-choosable graphs, the other about almost 2-paintable graphs. We put some effort into optimizing our approach, as we had the hope that it could be generalized from graphs to hypergraphs or even matrices (as a generalization of incidence matrices). This would be of relevance to problems like the Three Flow Conjecture and the Four Color Problem, but seems to be difficult.

2. Criticality and the low-vertex subgraph

We start with some basic definitions, including a small notational trick regarding paintability and the allowed extension $\mathcal{S} \supseteq V$ of sets of indices:

Definition 2.1 (*Choosability, Paintability*). Let $G = (V, E)$ be a graph and $\mathcal{S} \supseteq V$. For a system $L = (L_v)_{v \in \mathcal{S}}$ of *color lists* (sets) L_v, for $\ell = (\ell_v)_{v \in \mathcal{S}} \in \mathbb{Z}^{\mathcal{S}}$ and $k \in \mathbb{N}$ we define:

(i) G is *L-choosable* if its vertices can be colored such that every vertex $v \in V$ receives a color from its list L_v and adjacent vertices receive different colors.
(ii) G is *ℓ-colorable* if it is L-choosable for $L := (\{1, 2, \ldots, \ell_v\})_{v \in V}$.
(iii) G is *ℓ-choosable* if it is L-choosable for any *ℓ-list system* $L = (L_v)_{v \in V}$ of G, i.e. $|L_v| = \ell_v$ for all $v \in V$. If $\ell = d_G$ (i.e. $\ell_v = d_G(v)$ for all $v \in V$) then G is called *degree-choosable*. If $\ell_v = k$ for all $v \in V$ then G is *k-choosable*.
(iv) G is *ℓ-paintable* if either $G = \emptyset$ or if $\ell_v - 1 \geq 0$ for each $v \in V$ and each non-empty subset $V_P \subseteq V$ of vertices contains a *good* subset V_C. A good subset is an independent subset $V_C \subseteq V_P$ with the additional property that $G \setminus V_C$ is $(\ell - 1_{V_P})$-paintable (where $1_U(v)$ equals 1 if $v \in U$ and 0 else).
"*Mrs. Correct has to be able to trim Mr. Paint's continuing suggestions $V_P \subseteq V$ for the next forthcoming color, using at most $\ell_v - 1$ erasers for any vertex $v \in V$*".
(v) G is *almost* ℓ-paintable if, for any $v \in V$, $G \setminus v$ is ℓ-paintable, but G is it not.
(vi) The terms *almost L-choosable, almost ℓ-choosable, almost ℓ-colorable, degree-paintable*, *k-paintable* and so forth are defined similarly.

We emphasis that – with respect to colorability, choosability and paintability – the prefix "*almost k*-" means the same as "*critically* $(k+1)$-". It implies "$(k+1)$-", by Lemma 2.4, i.e., "almost k-something" implies "$(k+1)$-something". In literature, the *critically terminology* is used usually (and the term *k-critical* stands for critically k-colorable, but mostly with respect to edge deletion). However, since we do not see a good way to define criticality with respect to non-constant tuples ℓ or list systems L, we work in this paper with the *almost terminology* (some authors use the term "critically non-L-" instead of "almost L-"). The idea behind this definition is that any graph with a "non-" property must contain an induced subgraph which is "almost" as borderline case, since small enough subgraphs have the property. Therefore, the absence of such a subgraph implies the property:

Proposition 2.2. *If a graph does not contain an* almost *ℓ-paintable/choosable/colorable or almost L-choosable induced subgraph then it is (*entirely*) ℓ-paintable/choosable/colorable or L-choosable.*

Note also that almost ℓ-choosability implies almost L-choosability for at least one ℓ-list system L, but not the other way around. It is also not hard to see that ℓ-paintability implies ℓ-choosability (as described in [17]), and there is a whole series of papers that show that very many choosability theorems remain true for paintability (e.g. [2,7,9,10,17,19,18,21,23]). Nevertheless, there is no direct implication between almost ℓ-paintability and almost ℓ-choosability. Proposition 2.2 only ensures that there is an almost ℓ-paintable subgraph in any almost ℓ-choosable graph (as induced subgraph in our vertex critical setting). In [15,2] it is shown, that if we attach to a graph G a big enough complete graph K_s, by taking the complete join, then the *chromatic number* χ, the *choosability number* χ_{ch} and the *paintability number* χ_{P} will become the same. This observation can be used to construct graphs that are at the same time almost k-paintable, almost k-choosable and almost k-colorable. One just has to start with an almost k-colorable graph G, if $\chi_{\text{P}}(G \vee K_s) = \chi(G \vee K_s)$ $[= k+1+s]$ then $G \vee K_s$ is almost $(k+s)$-paintable, almost $(k+s)$-choosable and almost $(k+s)$-colorable at the same time. However, it might be more honest to say that we attach a small critical graph G to K_s, since s might have to be big. (One referee mentioned that $s := \alpha(G)\deg(G)$ with the independence number $\alpha(G)$ and the degeneracy $\deg(G)$ would suffice.)

The main tool of this section is the following strong version of Brooks' Theorem from [9]. It highlights an exceptional property of *Gallai Trees*. A Gallai Tree is a connected *Gallai Forest*, which is a graph whose *blocks* (i.e. 2-connected components) are all complete graphs or odd cycles (see [3] for general block forests):

Theorem 2.3. *Any connected graph is degree-paintable, the only exceptions are Gallai Trees (which are not even degree-choosable).*

We also will need the following simple lemma, which can also be applied to single vertices (the case $|U| = 1$ or $|W| = 1$ in the partition $V = U \uplus W$ of the vertex set):

Lemma 2.4 (*Cut Lemma*)**.** *Let $L = (L_v)_{v\in V}$ be an ℓ-list system of a graph G with vertex set $V = U \uplus W$, and denote the number of neighbors of $u \in U$ inside W by $\eta_u := |N(u) \cap W|$. Then the following holds:*

(i) *If $G[W]$ is L-choosable and $G[U]$ is $(\ell - \eta)$-choosable then G is L-choosable.*
(ii) *If $G[W]$ is ℓ-paintable and $G[U]$ is $(\ell - \eta)$-paintable then G is ℓ-paintable.*

Proof. The choosability part of this lemma is pretty obvious. Just color $G[W]$ first. Then remove from the color list L_u of any vertex $u \in U$ the at most η_u colors used for its neighbors in W, and color $G[U]$ from the remaining lists. The paintability version of the lemma is just a little bit more technical to prove, see [17, Lemma 2.3]. □

From this lemma follows that G is ℓ-paintable if $G \setminus v$ is ℓ-paintable and $\ell_v > d(v)$. Hence, any Gallai Tree T is almost degree-paintable/choosable. With the map $d_T : V(T) \longrightarrow \mathbb{N}$, $v \longmapsto d(v)$, this means that $T \setminus v$ is d_T-paintable/choosable (but not necessarily degree-paintable/choosable, i.e. $d_{T\setminus v}$-paintable/choosable), for any $v \in V$. This can, e.g., be proven by repeated application of Lemma 2.4 to single vertices. It also follows that, in an almost ℓ-paintable graph G, the degrees of vertices are high (relative to ℓ),

$$d(v) \geq \ell_v \quad \text{for all } v \in V. \tag{1}$$

The same holds for almost ℓ-choosability/colorability and almost L-choosability with $\ell_v := |L_v|$. Vertices v for which equality hold are the *low-degree* or *ℓ-degree* vertices. The graph induced by these vertices is the low-degree or ℓ-degree subgraph. Similarly, we introduce *lowest-degree* or $\min(\ell)$*-degree* vertices as vertices of degree

$$\min(\ell) := \min_{v\in V}(\ell_v) \leq \delta(G). \tag{2}$$

In other papers only the low-degree subgraph is defined, but we observed that we get more general results using the lowest-degree subgraph. Of course, for constant ℓ both notions coincide. The low- and the lowest-degree subgraph have a special structure:

Theorem 2.5. *Let G be an almost ℓ-paintable, almost ℓ-choosable, almost L-choosable or almost ℓ-colorable graph. Then the low- and the lowest-degree subgraphs of G are Gallai Forests (possibly empty).*

Proof. Since any induced subgraph of a Gallai Forest is a Gallai Forest, it suffices to prove that the low-degree subgraph is a Gallai Forest. We prove this with respect to almost L-choosability. The proof for almost ℓ-paintability/choosability/colorability works the same:

Assume that a block B of the low-degree subgraph of an almost L-choosable graph G is neither an odd cycle nor a complete graph. Then, on one hand, by Theorem 2.3, B is degree-choosable (even degree-paintable). On the other hand, $G \setminus B$ is L-choosable, by the almost-property of G. Hence, if we apply Lemma 2.4, the L-choosability of G follows, a contradiction. □

3. A lower bound for the edge density

In this section, we will provide a lower bounds for the edge density of critical graphs. Already Inequality (1) includes a lower bound for the average vertex degree, and may be viewed as such a bound. However, we will need a slightly better one. As Gallai Trees $G \neq K_n$ have highly nonconstant degree sequences, it is plausible that many vertices v in our $\min(\ell)$-degree subgraph must have a degree much smaller than the constant $\min(\ell)$. Hence, many of their edges must lead to sufficiently many vertices outside of this lowest-degree subgraph. As these outside vertices make a higher contribution to the averaged degree, the Gallai Forests structure of the lowest-degree subgraph should lead to a better lower bound for the edge density of the graph. More precisely, we can use the following upper bound for the edge density of Gallai Forests, which was already proven in [6, Lemma 4.5] (see also [14, Lemma 8.3.5]):

Lemma 3.1. *Let $G = (V, E)$ be a Gallai Forest of maximal degree $\Delta \geq 3$ different from $K_{\Delta+1}$. Then*

$$\frac{|E|}{|V|} < \frac{\Delta - 1}{2} + \frac{1}{\Delta}.$$

This bound is best possible. If we connect t complete graphs K_Δ with $t - 1$ edges, e.g. in a linear chain "$K_\Delta - K_\Delta - K_\Delta - \cdots - K_\Delta$", then

$$\frac{|E|}{|V|} = \frac{\Delta - 1}{2} + \frac{1}{\Delta} - \frac{1}{t\Delta}. \tag{3}$$

Only with more information about the structure of the Gallai Forests would better bounds be available (see [13]). We use the lemma, as in [6], to deduce the following lower bound:

Lemma 3.2. *Let $G = (V, E)$ be a connected graph of minimal degree $\delta \geq 3$ different from $K_{\delta+1}$. If the δ-degree subgraph of G is a Gallai Forest, then*

$$2\frac{|E|}{|V|} > \delta + \frac{\delta - 2}{\delta^2 + 2\delta - 2}.$$

Proof. Let L be the δ-degree subgraph of G and $H := G \setminus L$. Then

$$\emptyset \neq L \neq K_{\delta+1}, \tag{4}$$

as $G \neq K_{\delta+1}$ is connected, and applying Lemma 3.1 yields

$$\begin{aligned} |E| &\geq |E| - |E(H)| = \delta|V(L)| - |E(L)| \\ &> \delta|V(L)| - \left(\frac{\Delta(L) - 1}{2} + \frac{1}{\Delta(L)}\right)|V(L)| \\ &\geq \left(\delta - \frac{\delta - 1}{2} - \frac{1}{\delta}\right)|V(L)| \\ &= \frac{\delta^2 + \delta - 2}{2\delta}|V|. \end{aligned} \tag{5}$$

So, the number of δ-degree vertices is bounded, and we must have more vertices of higher degree, a high average degree and many edges:

$$2|E| \geq (\delta + 1)|V(H)| + \delta|V(L)| = (\delta + 1)|V| - |V(L)| > (\delta + 1)|V| - \frac{2\delta}{\delta^2 + \delta - 2}|E|, \tag{6}$$

$$2\big((\delta^2 + \delta - 2) + \delta\big)|E| > (\delta + 1)(\delta^2 + \delta - 2)|V|, \tag{7}$$

and

$$2\frac{|E|}{|V|} > \frac{(\delta + 1)(\delta^2 + \delta - 2)}{\delta^2 + 2\delta - 2} = \delta + \frac{\delta - 2}{\delta^2 + 2\delta - 2}. \quad \square \tag{8}$$

With this lemma, we can generalize Gallai's Inequality [6],

$$2\,\frac{|E|}{|V|} \geq \kappa - 1 + \frac{\kappa - 3}{\kappa^2 - 5}, \tag{9}$$

for κ-critical graphs (edge-critically κ-colorable graphs with $\kappa \geq 4$). Gallai stated his bound with "$\geq$", but the inequality is actually strict. Our k and his κ differ by 1, so that the right term looks different:

Theorem 3.3. *Let L be an ℓ-list system of $G \neq K_{k+1}$ with $k := \min(\ell) \geq 3$. If G is almost ℓ-paintable, almost ℓ-choosable, almost L-choosable or almost ℓ-colorable, then:*

$$2\,\frac{|E|}{|V|} > k + \frac{k - 2}{k^2 + 2k - 2}.$$

Proof. If $\delta(G) > k$ then

$$2\frac{|E|}{|V|} \geq k+1 > k + \frac{k-2}{k^2+2k-2}. \tag{10}$$

Otherwise, $\delta(G) = k$, and the statement follows from Theorem 2.5 and Lemma 3.2 as critical graphs are always connected. □

4. Graphs on surfaces

In this section, we study simple graphs G that are drawn on a surface without crossing edges. The sphere with $g \geq 0$ handles is denoted by S_g, and the sphere with $g \geq 1$ crosscaps is denoted by N_g. For example, N_1 can be obtained from the sphere with one blanked out hole by gluing a Möbius Band along its boundary cycle to the boundary cycle of the hole. The Klein Bottle N_2 requires 2 Möbius Bands. We use the Euler genus $\varepsilon \in \mathbb{N}$ instead of the Euler characteristic $\chi = 2 - \varepsilon$, and the Heawood number $H(\varepsilon)$,

$$\varepsilon(S_g) = 2g, \qquad \varepsilon(N_g) = g \quad \text{and} \quad H(\varepsilon) := \left\lfloor \frac{7+\sqrt{1+24\varepsilon}}{2} \right\rfloor. \tag{11}$$

Euler's Formula, and the fact that any face of G is adjacent to at least three edges, imply

$$2 - \varepsilon = |V| - |E| + |F| \leq |V| - |E| + \frac{2}{3}|E| = |V| - \frac{1}{3}|E|, \tag{12}$$

where F is the set of faces of the surface embedding of $G = (V, E)$. Now, we look at the minimal degree $\delta = \delta(G)$. Obviously,

$$6(2-\varepsilon) \leq 6|V| - 2|E| \leq (6-\delta)|V|, \tag{13}$$

so that, for $\delta \geq 6$,

$$0 \geq 6(2-\varepsilon) + (\delta-6)|V| \geq 6(2-\varepsilon) + (\delta-6)(\delta+1) = \delta^2 - 5\delta - 6\varepsilon + 6 \tag{14}$$

(as G is simple) and

$$\delta \leq \frac{5+\sqrt{1+24\varepsilon}}{2} < H(\varepsilon). \tag{15}$$

The resulting inequality,

$$\delta < H(\varepsilon), \tag{16}$$

even remains true for $\delta \ngeq 6$, if we additionally assume $\varepsilon \geq 1$, as, in this case,

$$\delta \leq 5 < H(1) \leq H(\varepsilon). \tag{17}$$

Hence, if $\varepsilon \geq 1$ then G is $(H(\varepsilon) - 1)$-degenerate and $H(\varepsilon)$-paintable by Lemma 2.4. This follows also from Proposition 2.2, as almost k-paintable graphs G for $k := H(\varepsilon) > \delta(G)$ do not exist. It proves the biggest part of the following paintability generalization of Heawood's Map-Coloring Theorem [8]:

Theorem 4.1. *Let G be a graph on a surface with Euler genus $\varepsilon > 0$. Then G is $H(\varepsilon)$-paintable, $H(\varepsilon)$-choosable and $H(\varepsilon)$-colorable. $H(\varepsilon)$ is the best possible general bound, except for graphs on the Klein Bottle N_2, for which $6 = H(2) - 1$ is the best bound. The best bound for the sphere S_0 is $5 = H(0) + 1$ with respect to paintability and choosability, and $4 = H(0)$ with respect to colorability.*

The reason for the best possible statement is that the complete graph with $H(\varepsilon)$ vertices can be embedded in any surface of Eulerian genus ε, except in the Klein Bottle (if $\varepsilon = 2$), see [14, Theorems 4.4.4 and 4.4.6]. In Corollary 4.5, we will obtain the tight bound for the Klein Bottle. The 5-paintability of planar graphs was proven in [17] following the ideas of Thomassen's 5-choosability proof. See [22, Theorem 5.10] for the non-4-choosability. The 4-colorability on S_0 is the very most difficult part and the content of the famous Four Color Theorem, see [14].

We think that the upper bound $H(\varepsilon)$ in this theorem can be improved by one if the graph does not contain $K_{H(\varepsilon)}$. This was proven for choosability in the quite long papers [1,12]. In [14, Theorem 8.3.7] the original coloring version of this so called Dirac Map-Coloring Theorem can be found. Thomassen's book also contains Dirac's proof for all $\varepsilon > 0$ different from 1 and 3. This proof works for paintability as well, so only the cases $\varepsilon = 1$ and $\varepsilon = 3$ are open. Dirac's proof is based on criticality examinations similar to the ones that we will use next.

We want to study the size of critical graphs G on surfaces of Eulerian genus ε. Inequality (13) implies for $\delta(G) - 6 > 0$ the following upper bound:

Theorem 4.2. *If $\delta(G) \geq 7$ then*

$$|V| \leq \frac{6(\varepsilon-2)}{\delta(G)-6}.$$

Corollary 4.3. *If* $\min(\ell) \geq 7$ *and G is almost* ℓ*-paintable/choosable/colorable, or almost L-choosable for an* ℓ*-list system L, then*

$$|V| \leq \frac{6(\varepsilon-2)}{\min(\ell)-6}.$$

This corollary does not hold if $\min(\ell) \leq 6$, but, at least for $\min(\ell) = 6$, and with the additional assumption $G \neq K_{\min(\ell)+1} = K_7$, we know that

$$6(\varepsilon-2) \overset{(12)}{=} 2|E| - 6|V| \overset{3.3}{>} \left(6 + \frac{6-2}{6^2+2\cdot 6-2} - 6\right)|V| \geq \frac{2}{23}|V|. \tag{18}$$

Collecting all cases with $\min(\ell) \geq 6$, we have:

Theorem 4.4. *If* $G \neq K_7$ *is almost* ℓ*-paintable/choosable/colorable or almost L-choosable for an* ℓ*-list system L, and if* $\min(\ell) \geq 6$*, (or, more generally, if* $G \neq K_7$ *is connected,* $\delta(G) \geq 6$*, and its* 6*-degree subgraph is a Gallai Forest) then:*

$$|V(G)| < 69(\varepsilon - 2).$$

For surfaces with Euler genus $\varepsilon \leq 2$, the right side of the inequality in this theorem is not positive. This shows that there are no "almost-ℓ graphs" with $\min(\ell) \geq 6$ on such surfaces, except possibly K_7. As only K_6, but not K_7, can be embedded in the Klein Bottle [14, Theorem 4.4.6], we obtain as corollary:

Corollary 4.5. *Any graph on the Klein Bottle is* 6*-paintable,* 6*-choosable and* 6*-colorable; and* 6 *is the best possible general bound.*

Another important consequence of the last results is:

Theorem 4.6. *On a fixed given surface, there are, up to canonically defined isomorphy, only finitely many pairs* (G, ℓ) *of embedded graphs G and tuples* $\ell \in \mathbb{N}^{V(G)}$ *with* $\min(\ell) \geq 6$ *such that G is almost* ℓ*-paintable/choosable/colorable. The same holds for pairs* (G, L) *with respect to almost L-choosability, provided* $\ell_v := |L_v| \geq 6$ *for all* $v \in V$*.*

Assuming that we know the finitely many pairs (G, ℓ) or (G, L) in this theorem (which is a constant time problem), it is easy to decide in polynomial time whether a graph G on the particular surface is ℓ-paintable/choosable/colorable or L-choosable (provided $\min(\ell) \geq 6$). Due to Proposition 2.2, we just have to check if any induced subgraph is in the finite list of almost "something" graphs.

If $\min(\ell) \geq 7$, we even can find an L-coloring in polynomial time, if it exists. The idea is to successively remove vertices v of degree less than ℓ_v until no such vertex exists anymore. This takes polynomial time (and is, in a similar way, also possible for usual 6-coloring), see [14, Remark after Corollary 8.4.2] and [4]. The remaining graph $H \leq G$ belongs to a finite set of graphs (by Theorem 4.2), which can be colored (or verified as uncolorable) in constant time. The coloring of H can then be extended (in linear time) to G.

Similar procedures apply to paintability, only Lemma 2.4(ii) (in the case $|U| = 1$) has to be checked for polynomial running time. We have:

Theorem 4.7. *To any fixed given surface S, there exists a polynomial running time algorithm* $\mathcal{A}_S$ *that does the following: To any graph G on S, and to any* ℓ*-list system L of G with* $\min(\ell) \geq 7$*,* $\mathcal{A}_S$ *determines if an L-coloring exists, and finds an L-coloring of G if it does exist.*

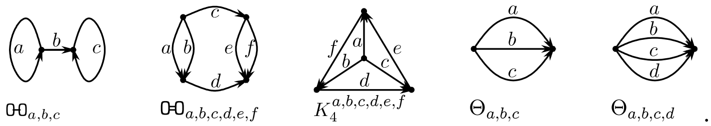
There also exists a polynomial running time implementation $\mathcal{C}_S$ *of Mrs. Correct in the Paint–Correct Game for S-embedded graphs with at least* 6 *erasers at each vertex* v $(\ell_v - 1 \geq 6)$*. More precisely,* $\mathcal{C}_S$ *will tell in polynomial time if there is a winning strategy for Mrs. Correct to any given "mounted graph"* (G, ℓ) *on S. If so, then, during the whole game,* $\mathcal{C}_S$ *will always find a good subset* V_C *in any suggestion* V_P *of Mr. Paint, in polynomial time.*

On surfaces with even Eulerian genus, this result also holds under the weaker assumption $\ell \geq 6$. For choosability, this was proven by Thomassen in [20, Theorems 4.4 and 4.5], and the proof there works for paintability as well. As mentioned there, it is not known if the 6 in this restriction can be relaxed to 5, see [14, Corollary 8.4.13] for a negative result in this direction.

5. (Almost) 2-paintable and 2-choosable graphs

In this section, whenever we speak about a set of graphs, we identify two graphs if they are isomorphic. We denote the set of all such graphs as $\mathcal{G}$. To ensure that $\mathcal{G}$ actually is a set with proper elements, we may imagine that vertices are always taken from a fixed countable set $\{v_1, v_2, v_3, \ldots\}$. In connection with paintability and choosability vertices v of degree 1 are quite uninteresting, since, if $\ell_v \geq 2$, then the graph G is ℓ-paintable/choosable if and only if $G \setminus v$ is so. Hence, with respect to 2-paintability/choosability, we always can replace G with $\mathrm{core}(G)$, the graph that is obtained from G by successively pruning away vertices of degree 1. We also can examine the connected components of G separately. Therefore, we work here with the subset $\mathcal{G}_2 := \{G \in \mathcal{G} \mid \delta(G) \geq 2$ and G is connected$\}$ of graphs. Important will be the following graphs; where $P_1 := v_{1,0}v_{1,1}\cdots v_{1,a}$, $P_2 := v_{2,0}v_{2,1}\cdots v_{2,b}$ and so forth are pairwise vertex disjoint paths of lengths a, b, c and so forth, and where the symbol = stands for identification of vertices:

$$\begin{aligned}
&C_a := P_1/v_{1,0} = v_{1,a} \quad (\text{“the” cycle of length } a),\\
&\text{O–O}_{a,b,c} := P_1 \cup P_2 \cup P_3/v_{1,0} = v_{1,a} = v_{2,0}/v_{2,b} = v_{3,0} = v_{3,c},\\
&\text{O=O}_{a,\dots,f} := P_1 \cup \cdots \cup P_6/v_{1,0} = v_{2,0} = v_{3,0}/v_{1,a} = v_{2,b} = v_{4,0}/v_{3,c} = v_{5,0} = v_{6,0}/v_{4,d} = v_{5,e} = v_{6,f},\\
&K_4^{a,\dots,f} := P_1 \cup \cdots \cup P_6/v_{1,0} = v_{2,0} = v_{3,0}/v_{1,a} = v_{5,e} = v_{6,0}/v_{2,b} = v_{6,f} = v_{4,0}/v_{3,c} = v_{4,d} = v_{5,0},\\
&\Theta_{a,b,c} := P_1 \cup P_2 \cup P_3/v_{1,0} = v_{2,0} = v_{3,0}/v_{1,a} = v_{2,b} = v_{3,c},\\
&\Theta_{a,b,c,d} := \text{O=O}_{a,b,0,0,c,d}.
\end{aligned} \tag{19}$$



In this definition, we do not allow paths of length 0, except for the path b in $\text{O–O}_{a,b,c}$, and the paths c and d in $\text{O=O}_{a,b,c,d,e,f}$. We further want to avoid multiple edges and loops, so that, e.g., we do not allow $\Theta_{1,1,3}$. Parameters $a, b, c, \ldots \in \mathbb{N} = \{0, 1, \ldots\}$ and graphs that follow these rules may be called *proper*, but we usually assume the rules without mentioning them. For example, the second parameter in $\Theta_{3,0,3}$ is not proper, but, in this special case, we may view the graph as proper as it is equal to $\text{O–O}_{3,0,3}$. We also use abbreviations like

$$\text{O–O}_{\{1,7\},*,2*} := \{\text{O–O}_{a,b,2c} \mid a, b, c \in \mathbb{N} \text{ and } a, b, 2c \text{ proper and } a \in \{1, 7\}\}, \tag{20}$$

and will focus on the following sets of graphs:

$$\begin{aligned}
&\mathcal{T} := C_* \cup \Theta_{*,*,*},\\
&\mathcal{T}_{\mathrm{ch}} := C_{2*} \cup \Theta_{2,2,2*},\\
&\mathcal{T}_{\mathrm{P}} := C_{2*} \cup \{\Theta_{2,2,2}\} = C_{2*} \cup \{K_{2,3}\}.
\end{aligned} \tag{21}$$

We want to recognize and understand that the elements of $\mathcal{T}_{\mathrm{P}}$ (resp. $\mathcal{T}_{\mathrm{ch}}$) are exactly the 2-paintable (resp. 2-choosable) graphs (as already proven in [5,23]). That these elements actually are 2-paintable (resp. 2-choosable) is relatively easy to check, so that we focus on proving that there are no others:

We have to find the minimal elements of $\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}}$ (resp. $\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{ch}}$) with respect to vertex minors ($\preceq$). A *vertex minor* of a graph G is a graph H that can be obtained from G by taking first a subgraph and then contracting vertices. To *contract* a vertex v, $G \mapsto G/v$, means to contract all edges incident with v and, afterwards, to replace multiple edges with single edges and to discard loops. This operation has the property of preserving 2-paintability (resp. 2-choosability), as we will see. Therefore, it is enough to prove that the $\preceq$-minimal elements of $\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}}$ (resp. $\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{ch}}$) are not 2-paintable (resp. 2-choosable).

Once we have established that the elements of $\mathcal{T}_{\mathrm{P}}$ (resp. $\mathcal{T}_{\mathrm{ch}}$) are exactly the 2-paintable (resp. 2-choosable) graphs, it is obvious that the minimal elements of $\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}}$ (resp. $\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{ch}}$) with respect to induced subgraphs ($\leq$) are precisely the almost 2-paintable (resp. almost 2-choosable) graphs in $\mathcal{G}_2$ with respect to vertex deletion. The minimal elements with respect to subgraphs ($\subseteq$), are the almost 2-paintable (resp. almost 2-choosable) graphs with respect to edge deletion. We denote the sets of these subgraphs by $\min_{\leq}(\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}})$, $\min_{\subseteq}(\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}})$ and so forth. However, we start with the structure of the elements of $\mathcal{G}_2 \setminus \mathcal{T}$:

Lemma 5.1. *Any $G \in \mathcal{G}_2 \setminus \mathcal{T}$ contains one of the following graphs as induced subgraph:*

$C_3, K_{3,3}$, *an* O–O, *an* O=O *or a subdivided* K_4.

Proof. Assume the lemma is false, and let $G \in \mathcal{G}_2 \setminus \mathcal{T}$ be a smallest counterexample. Then, for any $x \in V$, any component H of $\mathrm{core}(G \setminus x)$ is either a K_1, a C_a, or a $\Theta_{a,b,c}$. (That is because, if $K_1 \neq H \notin \mathcal{T}$ then $H \in \mathcal{G}_2$ must contain one of the graphs in our list by the minimality of G. But then G would contain the same graph from this list, and the theorem yet would hold for G, a contradiction.) However, this observation leads to one of the following contradictions:

Case 1, *G contains a vertex x of degree* 2: Then x lies inside a path $P \leq G$ of length $t \geq 2$, such that G is obtained from $\mathrm{core}(G \setminus x)$ by connecting two vertices u and w of $\mathrm{core}(G \setminus x)$ with the path P, where $u = w$ is possible.

Case 1.1, $\mathrm{core}(G \setminus x)$ *is disconnected*: Then P must connect all components, as G is connected. It follows that $G \in \mathcal{G}_2$ contains an $\text{O–O}_{*,t,*}$-graph from our list.

Case 1.2, $\mathrm{core}(G \setminus x) = K_1$: Then $u = w$, and our $G \in \mathcal{G}_2 \setminus \mathcal{T}$ is a cycle and lies in $\mathcal{T}$.

Case 1.3, $\mathrm{core}(G \setminus x)$ is a C_a:

Case 1.3.1, $u = w$: Then $G = C_a \cup P$ contains $\text{O–O}_{a,0,t}$ from our list.

Case 1.3.2, $u \neq w$: Then G is a $\Theta_{*,t,*}$-graph and lies in $\mathcal{T}$.

Case 1.4, $\mathrm{core}(G \setminus x)$ is a $\Theta_{a,b,c} = P_1 \cup P_2 \cup P_3/\cdots$:

Case 1.4.1, $\{u, w\} \not\subseteq V(P_i)$ *for* $i = 1, 2, 3$: Then G contains a $K_4^{a,\dots,f}$.
Case 1.4.2, $\{u, w\} \subseteq V(P_i)$: Then G contains an O=O - or an $\text{O-O}_{*,0,t}$-graph.
Case 2, *G contains no vertex of degree* 2: Then $\operatorname{core}(G \setminus x) = G \setminus x$ for any vertex $x \in V$, which simplifies the remaining studies.
Case 2.1, *G contains a vertex* v of *degree* $d(v) > 3$:
Case 2.1.1, $V \neq N(v) \cup \{v\}$: Let $x \in V \setminus (N(v) \cup \{v\})$, and let $H \in \mathcal{G}_2$ be the component of $G \setminus x$ containing v. Then $d_H(v) = d_G(v) > 3$ implies $H \notin \mathcal{T}$, and, by the minimality of G, $H < G$ contains a graph from the list.
Case 2.1.2, $V = N(v) \cup \{v\}$: Then $G \geq C_3$ as $G \in \mathcal{G}_2$ is not a star.
Case 2.2, *G is* 3-*regular*: For $x \in V$, $G \setminus x$ contains exactly 3 vertices of degree 2.
Case 2.2.1, $G \setminus x = \Theta_{2,2,2}$: Then $G \geq K_{3,3}$.
Case 2.2.2, $G \setminus x = \Theta_{1,2,3}$: Then $G \geq G \setminus x = \Theta_{1,2,3} \geq C_3$.
Case 2.2.3, $G \setminus x = C_3$: Then $G \geq G \setminus x = C_3$. □

5.1. 2-choosable and almost 2-choosable graphs

In this subsection, we examine $\mathcal{G}_2 \setminus \mathcal{T}_{\text{ch}} \supseteq \mathcal{G}_2 \setminus \mathcal{T}$, and its minimal elements:

Lemma 5.2. *With respect to induced subgraphs, $\mathcal{G}_2 \setminus \mathcal{T}_{\text{ch}}$ has the following set of minimal elements:*

$$\min_{\leq}(\mathcal{G}_2 \setminus \mathcal{T}_{\text{ch}}) = \{K_{3,3}\} \cup C_{2*+1} \cup \Theta_{2*+1,2*+1,2*+1} \cup \Theta_{2*,2*+4,2*+4} \cup \Theta_{2,2,2,2*} \cup \text{O-O}_{2*,*,2*} \cup \{\text{O=O}_{2,2,1,1,2,2}\} \cup K_4^{2*,1,1,2*,1,1}.$$

Proof. Obviously, if we add the elements of

$$\min_{\leq}(\mathcal{T} \setminus \mathcal{T}_{\text{ch}}) = C_{2*+1} \cup \Theta_{2*+1,2*+1,2*+1} \cup \Theta_{2*,2*+4,2*+4} \tag{22}$$

to the list in Lemma 5.1, we obtain

$$\min_{\leq}(\mathcal{G}_2 \setminus \mathcal{T}_{\text{ch}}) \subseteq \{K_{3,3}\} \cup C_{2*+1} \cup \Theta_{2*+1,2*+1,2*+1} \cup \Theta_{2*,2*+4,2*+4} \cup \text{O-O}_* \cup \text{O=O}_* \cup K_4^*, \tag{23}$$

where $\text{O-O}_* := \text{O-O}_{*,*,*}$ and so forth. From the right side of this relation we can remove some of the graphs because other elements on the right side are smaller. For example, any non-bipartite graph contains an odd cycle as induced subgraph. Hence, it is either identic with this odd cycle or *redundant*, as we say. Next, assume that $a > 1$ in an non-redundant and necessarily bipartite $K_4^{a,b,c,d,e,f}$. Then $\Theta_{d,b+c,e+f}$ occurs as induced subgraph, which forces d to be even, as otherwise $\Theta_{d,b+c,e+f}$ already appears in $\Theta_{2*+1,2*+1,2*+1}$. So, $a > 1$ implies that d is even and, in particular, $d > 1$; but then, in turn, a must be even. Hence, paths of length greater than 1 have even length and occur pairwise. They are located on opposite (disjoint) edges of the tetrahedra K_4. Now, it cannot be that all three pairs, i.e. all six parameters a, b, c, d, e, f are even, as then $K_4^{a,b,c,d,e,f} \geq \Theta_{d,b+c,e+f} \in \Theta_{2*,2*+4,2*+4}$. It also cannot be that no pair or exactly two pairs are even, because of odd cycles. The only cases that are left have exactly one even pair and belong to $K_4^{2*,1,1,2*,1,1}$.

Most O=O-graphs are redundant as well. Assume that $\text{O=O}_{a,b,c,d,e,f}$ is not, and that without loss of generality $a, f > 1$, so that we can remove P_1 and P_6 by deleting their interior vertices. Then $c, d \leq 1$ is needed to avoid an induced O-O-subgraph. If $c = 1$ and $d = 0$, or $c = 0$ and $d = 1$, then removal of P_6 results in a $\Theta_{a,b,1+e} \notin \Theta_{2*+1,2*+1,2*+1}$, so that e must be odd; but then $\text{O=O}_{a,b,c,d,e,f} \geq \Theta_{b+1,e,f} \in \Theta_{2*+1,2*+1,2*+1}$. If $c = d = 1$, then removal of P_6 results in $\Theta_{a,b,1+1+e}$, so that $a = b = 2$ is needed to avoid redundancy. Similarly, $e = f = 2$ is needed here, and we obtain $\text{O=O}_{2,2,1,1,2,2}$ as sole survivor in this case. Finally, if $c = d = 0$ then necessarily $\Theta_{a,b,e}, \Theta_{b,e,f} \in \Theta_{2,2,2*}$. Hence, a, b, e, f are even, P_2 and P_5 are removable, and the induced subgraphs $\Theta_{a,b,f}, \Theta_{a,e,f}$ must lie in $\Theta_{2,2,2*}$ too. It follows that only one of the variables a, b, e, f might be different from 2, and we are left with a $\Theta_{2,2,2,2*}$-graph. Summarizing,

$$\min_{\leq}(\mathcal{G}_2 \setminus \mathcal{T}_{\text{ch}}) \subseteq \{K_{3,3}\} \cup C_{2*+1} \cup \Theta_{2*+1,2*+1,2*+1} \cup \Theta_{2*,2*+4,2*+4} \cup \Theta_{2,2,2,2*} \cup \text{O-O}_{2*,*,2*} \cup \{\text{O=O}_{2,2,1,1,2,2}\} \cup K_4^{2*,1,1,2*,1,1}, \tag{24}$$

with equality, as none of the elements on the right side is redundant. □

Based on this lemma, we may now go a step further and look at the $\subseteq$-minimal elements. We observe that

$$K_{3,3} \supseteq \Theta_{1,3,3}, \qquad \text{O=O}_{2,2,1,1,2,2} \supseteq \text{O-O}_{4,1,4} \quad \text{and} \quad K_4^{2a,1,1,2d,1,1} \supseteq \Theta_{1,2a+1,2d+1}, \tag{25}$$

so that we obtain Voigt's result [22]:

Lemma 5.3. *With respect to usual subgraphs, $\mathcal{G}_2 \setminus \mathcal{T}_{\text{ch}}$ has the following set of minimal elements:*

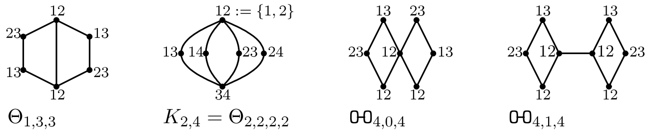
$$\min_{\subseteq}(\mathcal{G}_2 \setminus \mathcal{T}_{\text{ch}}) = C_{2*+1} \cup \Theta_{2*+1,2*+1,2*+1} \cup \Theta_{2*,2*+4,2*+4} \cup \Theta_{2,2,2,2*} \cup \text{O-O}_{2*,*,2*}.$$

Finally, if we continue by taking vertex minors then, e.g., $\Theta_{2,4,6} \succeq \Theta_{2,4,4} \succeq \Theta_{1,3,3}$ and $\Theta_{2,2,2,4} \succeq \Theta_{2,2,2,2} = K_{2,4}$, and we obtain:

Lemma 5.4. *With respect to vertex minors, $\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{ch}}$ has the following set of minimal elements:*

$$\min_{\preceq}(\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{ch}}) = \{C_3,\, \Theta_{1,3,3},\, K_{2,4},\, \text{0-0}_{4,0,4},\, \text{0-0}_{4,1,4}\}.$$

These five graphs are all not 2-choosable, as one easily can check in the following diagrams with obstructive 2-list systems (C_3 is not even 2-colorable):



.

Now, on one hand, all elements in $\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{ch}}$ are non-2-choosable since vertex contraction preserves 2-choosability. Because if, for a vertex v of a 2-choosable graph G, the 2-lists L_x of all vertices $x \in N(v) \cup \{v\}$ are the same, then an L-coloring would require equal colors for all $x \in N(v)$. Hence, the coloring would give rise to a coloring of G/v from the inherited lists. As any 2-list system of G/v can be obtained as an inherited system, this proves the preservation of 2-choosability under vertex contraction. On the other hand, the elements of the complementary set $\mathcal{T}_{\mathrm{ch}}$ are 2-choosable. Hence, we have found all 2-choosable graphs in $\mathcal{G}_2$, and, with the minimal sets above, also all almost 2-choosable graphs in $\mathcal{G}_2$. We obtain the following theorem, whose first two parts are already proven in [5,22]:

Theorem 5.5. *A connected graph G is 2-choosable if and only if*

$$\mathrm{core}(G) \in \{K_1\} \cup \mathcal{T}_{\mathrm{ch}} := \{K_1\} \cup C_{2*} \cup \Theta_{2,2,2*}.$$

The set of almost 2-choosable graphs with respect to edge deletion is:

$$\min_{\subseteq}(\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{ch}}) = C_{2*+1} \cup \Theta_{2*+1,2*+1,2*+1} \cup \Theta_{2*,2*+4,2*+4} \cup \Theta_{2,2,2,2*} \cup \text{0-0}_{2*,*,2*}.$$

The set of almost 2-choosable graphs with respect to vertex deletion is:

$$\min_{\le}(\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{ch}}) = \min_{\subseteq}(\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{ch}}) \cup \{K_{3,3}\} \cup \{\text{0-0}_{2,2,1,1,2,2}\} \cup K_4^{2*,1,1,2*,1,1}.$$

5.2. 2-paintable and almost 2-paintable graphs

In this subsection, we examine $\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}} \supseteq \mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{ch}}$, and its minimal elements:

Lemma 5.6. *With respect to induced subgraphs, $\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}}$ has the following set of minimal elements:*

$$\min_{\le}(\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}}) = \{K_{3,3}\} \cup C_{2*+1} \cup \Theta_{2*+1,2*+1,2*+1} \cup \Theta_{2*,2*,2*+4} \cup \{K_{2,4}\} \\ \cup\ \text{0-0}_{2*,*,2*} \cup \{K_4^{2,1,1,2,1,1}\}.$$

Proof. Obviously, if we add the elements of

$$\mathcal{T}_{\mathrm{ch}} \setminus \mathcal{T}_{\mathrm{P}} = \Theta_{2,2,2*+4} \tag{26}$$

to the right side of the equation in Lemma 5.2, we obtain

$$\min_{\le}(\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}}) \subseteq \{K_{3,3}\} \cup C_{2*+1} \cup \Theta_{2*+1,2*+1,2*+1} \cup \Theta_{2*,2*,2*+4} \cup \Theta_{2,2,2,2*} \\ \cup\ \text{0-0}_{2*,*,2*} \cup \{\text{0-0}_{2,2,1,1,2,2}\} \cup K_4^{2*,1,1,2*,1,1}, \tag{27}$$

and

$$\min_{\le}(\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}}) = \{K_{3,3}\} \cup C_{2*+1} \cup \Theta_{2*+1,2*+1,2*+1} \cup \Theta_{2*,2*,2*+4} \cup \{K_{2,4}\} \\ \cup\ \text{0-0}_{2*,*,2*} \cup \{K_4^{2,1,1,2,1,1}\}, \tag{28}$$

as $\text{0-0}_{2,2,1,1,2,2}$ and almost all graphs in $K_4^{2*,1,1,2*,1,1}$ and $\Theta_{2,2,2,2*}$ became redundant (and $\Theta_{2,2,2,2} = K_{2,4}$). □

As before, we may go a step further and look at the $\subseteq$-minimal element in the last obtained classification:

Lemma 5.7. *With respect to usual subgraphs, $\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}}$ has the following set of minimal elements:*

$$\min_{\subseteq}(\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}}) = C_{2*+1} \cup \Theta_{2*+1,2*+1,2*+1} \cup \Theta_{2*,2*,2*+4} \cup \{K_{2,4}\} \cup \text{O-O}_{2*,*,2*}.$$

From this lemma we easily can deduce:

Lemma 5.8. *With respect to vertex minors, $\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}}$ has the following set of minimal elements:*

$$\min_{\preceq}(\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}}) = \{C_3,\ \Theta_{1,3,3},\ \Theta_{2,2,4},\ K_{2,4},\ \text{O-O}_{4,0,4},\ \text{O-O}_{4,1,4}\}.$$

All the elements in this list are not 2-paintable and, aside from the new element $\Theta_{2,2,4}$, not even 2-choosable, as we have seen. The non-2-paintability of $\Theta_{2,2,4}$ is left as an exercise, see also [23]. It follows that all elements of $\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}}$ are non-2-paintable, as vertex contraction preserves 2-paintability:

Lemma 5.9. *Let G be a graph and $v \in V(G)$ a vertex. If G is 2-paintable then G/v is also 2-paintable.*

Proof. Let $\check{v}$ be the contracted vertex in G/v and $\hat{v} := N(v) \cup \{v\}$. Let $\check{V}_P \subseteq V(G/v)$ be Mr. Paint's move in G/v. Then

$$\hat{V}_P := \begin{cases} \check{V}_P & \text{if } \check{v} \notin V_P, \\ (\check{V}_P \setminus \check{v}) \cup \hat{v} & \text{if } \check{v} \in V_P, \end{cases} \tag{29}$$

is a possible painting move in G, i.e. $\hat{V}_P \subseteq V(G)$. Assuming the 2-paintability of G, there must be a good subset $\hat{V}_C \subseteq \hat{V}_P$. As here $\ell \equiv 2$, neither $\hat{V}_C$ nor $\hat{V}_P \setminus \hat{V}_C$ may contain two neighbored vertices. Hence, we have either

$$N(v) \subseteq \hat{V}_C \quad (\text{and } v \notin V_C) \quad \text{or} \quad N(v) \cap \hat{V}_C = \emptyset \quad (\text{and } v \in V_C). \tag{30}$$

In the first case $\check{V}_C := (\hat{V}_C \setminus \hat{v}) \cup \{\check{v}\}$ would be a good subset of $\check{V}_P$, in the second case $\check{V}_C := \hat{V}_C \setminus \hat{v}$. $\square$

Instead of using this lemma, one may also avoid this argument and examine all elements of

$$\mathcal{T}_{\mathrm{ch}} \setminus \mathcal{T}_{\mathrm{P}} = \Theta_{2,2,2*+4} \tag{31}$$

for non-paintability, which is not more difficult than the case $\Theta_{2,2,4}$. Exactly as in the last subsection, this yields the following theorem, whose first two parts already appear in [23,2]:

Theorem 5.10. *A connected graph G is 2-paintable if and only if*

$$\mathrm{core}(G) \in \{K_1\} \cup \mathcal{T}_{\mathrm{P}} := \{K_1\} \cup C_{2*} \cup \{K_{2,3}\}.$$

The set of almost 2-paintable graphs with respect to edge deletion is:

$$\min_{\subseteq}(\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}}) = C_{2*+1} \cup \Theta_{2*+1,2*+1,2*+1} \cup \Theta_{2*,2*,2*+4} \cup \{K_{2,4}\} \cup \text{O-O}_{2*,*,2*}.$$

The set of almost 2-paintable graphs with respect to vertex deletion is:

$$\min_{\preceq}(\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}}) = \min_{\subseteq}(\mathcal{G}_2 \setminus \mathcal{T}_{\mathrm{P}}) \cup \{K_{3,3}\} \cup \{K_4^{2,1,1,2,1,1}\}.$$

Acknowledgments

We want to thank the Abdus Salam School of Mathematical Sciences for providing a good research environment. The second author also gratefully acknowledges the support provided by the King Fahd University of Petroleum and Minerals under the project numbers FT090010.

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