ADVANCES IN MATHEMATICS 106, 151–168 (1994)

Toric Structures on the Moduli Space of Flat Connections on a Riemann Surface: Volumes and the Moment Map

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In earlier papers we constructed a Hamiltonian torus action on an open dense set in the moduli space of flat $SU(2)$ connections on a compact Riemann surface, where the dimension of the torus is half the dimension of the moduli space. This torus action shows that this set can be viewed symplectically as a (noncompact) toric variety. The number of integral points of the moment map for the torus action turns out to be identical to the Verlinde dimension $D(g, k)$. As an application, we furnish a new proof of the relation between the large-k limit of $D(g, k)$ and the volume of the moduli space. From our point of view, this relation follows from the equality between the symplectic volume of a toric variety and the Euclidean volume of the image of the moment map. Similar considerations are shown to give rise to the volumes of moduli spaces of parabolic bundles on a Riemann surface. Knowledge of these volumes has been shown to allow a proof of the Verlinde formula for the dimension of the space of holomorphic sections of line bundles on this space.

1. INTRODUCTION

In this paper we continue our study of the symplectic geometry of the moduli space $\bar{\mathscr{S}}_g$ of flat $SU(2)$ connections on a Riemann surface Σ^g. Our previous results were stated in the language of geometric quantization: in [13] it was shown that the space $\bar{\mathscr{S}}_g$ possessed a real polarization, which in [6] was shown to yield a quantization whose dimension coincided with

* Supported in part by a grant in aid from The J. Seward Johnson Charitable Trust.
† Supported in part by NSF Mathematical Sciences Postdoctoral Research Fellowship DMS 88-07291.

0001-8708/94 $9.00

the Verlinde dimension of the holomorphic quantization of $\bar{\mathscr{S}}_g$. However, the existence of the real polarization studied in [6, 13] can be seen as a manifestation of the structure of $\bar{\mathscr{S}}_g$ as a symplectic manifold, quite apart from any applications to quantization. The purpose of this paper is to clarify the meaning of our previous results in terms of symplectic geometry, and to show how they can be applied to the study of the geometry and topology of $\bar{\mathscr{S}}_g$.

The basic geometric fact underlying the constructions of [6, 13] is the existence on (an open set U_g in) $\bar{\mathscr{S}}_g$ of $3g-3$ commuting Hamiltonian S^1 actions. These flows, which are the analogs of the twist flows on Teichmüller space, were studied by Goldman [5], and the study of their orbits was the main topic of [6, 13]. The dimension of $\bar{\mathscr{S}}_g$ is $6g-6$ which is twice the number of flows; thus these flows give the space U_g, *considered as a symplectic manifold*, the structure of a (noncompact) toric variety. This is an open dense set in the toric variety constructed from the convex polyhedron given by the closure of the image of the moment map of the torus action.

The basic application we have for this structure is the computation of the symplectic volume of the space $\bar{\mathscr{S}}_g$. By the Duistermaat–Heckman theorem, this volume is equal to the volume of the image of the moment map. Now the convex polyhedron corresponding to this image is equipped with a lattice, corresponding to the integral points of the moment map; we may thus compute the volume by counting the number of $1/k$-integral points in the polyhedron, dividing by $k^{(\dim \bar{\mathscr{S}}_g)/2}$, and taking the limit as $k \to \infty$. Thus we may reduce the computation of the volume to the calculation of the number of integral points of multiples of the moment map.

This calculation was the main result of [6], where it was shown that the number of integral points coincided precisely with the combinatorial formula $D(g, k)$ (see Definition 4.7) first proposed by Verlinde in the context of conformal quantum field theory, and which has been recently shown (see, e.g., [2]) to compute the dimension of the space of holomorphic sections of the kth power of a line bundle on $\bar{\mathscr{S}}_g$. A rigorous proof of the relation between the large k limit of $D(g, k)$ and the volume of the moduli space was given by Witten [14], using Reidemeister torsion methods. The novelty of our approach is in the relation of this limit to the toric structure on $\bar{\mathscr{S}}_g$ and to the corresponding moment map.

The next natural question is whether the combinatorics of the Verlinde dimension formula itself can be understood in terms of the toric structure we are working with. In fact, the dimension of the space of holomorphic sections of the appropriate line bundle on the natural compactification of U_g (as a toric variety) is given precisely by the number of integral points of the moment map; this is just the classical result on the relation between combinatorics and algebraic geometry [9]. But it is not too hard to see that the complex structure on this compactification does not concord with the structure coming from the moduli space. Hence such a direct interpretation of the combinatorial nature of the Verlinde dimension, as an algebraic manifestation of the toric structure, cannot be sustained.

There is, however, a more indirect method of relating the dimension of this space to symplectic geometry of moduli spaces, since in [3, 11] it is shown that knowledge of the intersection pairings in the cohomology ring of a related moduli space $\mathscr{S}_g(1)$ can be used to compute the dimension of the space of holomorphic sections of a line bundle.[1] Now an argument given in [3] shows that these intersection pairings may be obtained from the volumes of moduli spaces of parabolic bundles on the surface Σ^g.[2] As it turns out, the moduli spaces of parabolic bundles can be given a toric structure in a manner wholly analogous to that applied to $\mathscr{S}_g$ itself. Hence the results of [6, 13] have a natural extension to the spaces of parabolic bundles, whose volumes can then be computed by counting the lattice points of the corresponding polyhedra, just as in the case of $\mathscr{S}_g$.

This paper is structured as follows. In Section 2, we recall some basic facts about the moduli spaces in question, followed in Section 3 by a quick review of the results of [6, 13]; this serves both to make this paper reasonably self-contained, and to extend these results to moduli spaces of parabolic bundles. In this section we describe the S^1 actions which underlie the work of [6, 13], and which we use to compute the volumes. In Section 4 we recall how symplectic volumes are related by the Duistermaat–Heckman theorem to integral points of the moment map, and apply this method to compute the volumes of our moduli spaces. Finally we recall in Section 5 how these volumes allow the calculation of the intersection pairings in the cohomology ring of $\mathscr{S}_g(1)$, and of the dimension of the space of holomorphic sections of line bundles on $\mathscr{S}_g$ and $\mathscr{S}_g(1)$—the Verlinde dimension formula. We emphasize that this last section is a summary of other work in the literature and is not original.

2. Moduli Spaces Associated to Riemann Surfaces

In this section we recall the construction of the moduli spaces we study, and the symplectic structures on them. This material is standard; we refer the reader to, e.g., [1] for more details.

[1] More precisely, in [3] this dimension is computed for line bundles over $\mathscr{S}_g(1)$; in order to compute the dimension for line bundles on $\mathscr{S}_g$ in terms of the intersection pairings on $\mathscr{S}_g(1)$ we combine the results of [3] with those of [2]. See Section 5.

[2] These moduli spaces are described in subsection 2.2.

2.1. *The Moduli Space of Flat Connections on a Closed Surface*

Let G denote $SU(2)$, and $\mathfrak{g}$ its Lie algebra. Also, denote by T the maximal torus $U(1)$. We treat the moduli space $\bar{\mathscr{S}}_g$ of gauge equivalence classes of flat G connections on a compact Riemann surface Σ^g of genus g. More precisely, we have the space of all G connections, $\mathscr{A} = \Omega^1(\Sigma^g, \mathfrak{g})$, and $\mathscr{A}_F = \{A \in \mathscr{A} : F_A = dA + A \wedge A = 0\}$. The gauge group $\mathscr{G} = \mathrm{Map}(\Sigma^g, G)$ acts on $\mathscr{A}_F$, with $g \in \mathscr{G}$ taking $A \in \mathscr{A}_F$ to $A^g = g^{-1}Ag + g^{-1}\,dg$. Then $\bar{\mathscr{S}}_g = \mathscr{A}_F/\mathscr{G}$.

We may alternatively identify $\bar{\mathscr{S}}_g = \mathrm{Hom}(\pi_1(\Sigma^g), G)/G$, the space of conjugacy classes of representations of $\pi_1(\Sigma^g)$ into G. This is a stratified symplectic space, with an open dense set $\mathscr{S}_g$ consisting of conjugacy classes of *irreducible* representations of $\pi_1(\Sigma^g)$ into G. The space $\mathscr{S}_g$ is a smooth manifold of dimension $6g-6$.

On $\mathscr{S}_g$, there is a natural symplectic form ω, which descends from the following symplectic form $\tilde{\omega}$ on $\mathscr{A}$: if $a, b \in \Omega^1(\Sigma^g, \mathfrak{g})$, then we define

$$\tilde{\omega}(a, b) = \frac{1}{4\pi^2} \int \mathrm{Tr}(a \wedge b). \tag{2.1}$$

The symplectic form $\tilde{\omega}$ is invariant under the action of $\mathscr{G}$, and descends to give a symplectic form ω on $\mathscr{S}_g$.

2.2. *The Moduli Space of Flat Connections on a Surface with Boundary*

Let Σ_d^g be the surface of genus g with d boundary components, and let $t_a \in [0, 1]$ for $a = 1, ..., d$.

Let $\tilde{C}_a$, $a = 1, ..., d$, denote the ath boundary component, and $[\tilde{C}_a]$ the corresponding element of $\pi_1(\Sigma_d^g)$. Then we make the following

DEFINITION 2.1. The moduli space of flat connections on Σ_d^g with weights $t_1, ..., t_d$ is

$$\bar{\mathscr{S}}_g(\underline{t}) = \{\rho \in \mathrm{Hom}(\pi_1(\Sigma_d^g), G) \mid \mathrm{Tr}\, \rho([\tilde{C}_a)] = 2\cos \pi t_a,\ a = 1, ..., d\}/G.$$

Remark. We denote by $\mathscr{S}_g(\underline{t})$ the space of (conjugacy classes of) irreducible representations of $\pi_1(\Sigma_d^g)$ lying in $\bar{\mathscr{S}}_g(\underline{t})$.

We can also interpret this space as a moduli space of flat connections on a noncompact surface. Let $\tilde{\Sigma}_d^g = \Sigma_d^g \cup \coprod_{a=1}^d (S^1 \times \mathbb{R}^+)$ be the noncompact surface consisting of Σ_d^g extended by a half-infinite cylinder along each boundary component. We denote by (ζ_a, s_a) the coordinates on the ath copy of $S^1 \times \mathbb{R}^+$. Then we define $\bar{\mathscr{S}}_g(\underline{t}) = \mathscr{A}_F(\tilde{\Sigma}_d^g)/\mathscr{G}_0(\tilde{\Sigma}_d^g)$, where

$$\begin{aligned} \mathscr{A}_F(\tilde{\Sigma}_d^g) = \{A \in \Omega^1(\tilde{\Sigma}_d^g, \mathfrak{g}) \mid F_A = 0, \text{ and } \exists s_a^0 \in \mathbb{R}^+ \text{ s.t.} \\ A|_{(\zeta_a, s_a)} = \pi t_a \operatorname{diag}(i, -i)\, d\zeta_a \text{ for } s_a \geqslant s_a^0\}, \end{aligned} \tag{2.2}$$

and

$$\mathscr{G}_0(\tilde{\Sigma}^g_d) = \{g : \tilde{\Sigma}^g_d \to G \mid \exists s^0_a \in \mathbb{R}^+ \text{ and } h_a \in \text{Stab}(\text{diag}(e^{i\pi t_a}, e^{-i\pi t_a})) \\ \text{s.t.} \quad g(\zeta_a, s_a) = h_a \text{ for } s_a \geq s^0_a\}. \tag{2.3}$$

Remark. If $t_a \in (0, 1)$, the stabilizer $\text{Stab}(\text{diag}(e^{i\pi t_a}, e^{-i\pi t_a}))$ is just the maximal torus T.

Then for $A \in \mathscr{A}_F(\tilde{\Sigma}^g_d)$, the tangent space to $\mathscr{A}_F(\tilde{\Sigma}^g_d)/\mathscr{G}_0(\tilde{\Sigma}^g_d)$ at $[A]$ is identified with $Z^1_A/d_A C^0_A$, where

$$Z^1_A = \{a \in \Omega^1(\tilde{\Sigma}^g_d, \mathfrak{g}) \mid d_A a = 0 \text{ and } a \text{ has compact support}\}, \tag{2.4}$$

and

$$C^0_A = \{\phi \in \Omega^0(\tilde{\Sigma}^g_d, \mathfrak{g}) \mid \exists \eta_a \in \text{Lie}(\text{Stab}(\text{diag}(e^{i\pi t_a}, e^{-i\pi t_a}))) \\ \text{and } s^0_a \in \mathbb{R}^+ \text{ s.t. } \phi(s_a, \zeta_a) = \eta_a \text{ for } s_a \geq s^0_a\}. \tag{2.5}$$

The symplectic form on $T_{[A]}\mathscr{S}_g(\underline{t})$ descends from an antisymmetric pairing on Z^1_A: for $a, b \in Z^1_A$, we define

$$\omega(a, b) = \frac{1}{4\pi^2} \int_{\tilde{\Sigma}^g_d} \text{Tr}(a \wedge b). \tag{2.6}$$

Using Stokes' Theorem, we see that this definition depends only on the equivalence classes $[a]$, $[b]$.

Now $\bar{\mathscr{S}}_g(\underline{t})$ has dimension $6g - 6 + 2d'$, where d' is the number of t_a that are in $(0, 1)$. A particularly important special case is $\bar{\mathscr{S}}_g(1)$, which is a compact *smooth* manifold (see [1]) of dimension $6g - 6$.

3. The Toric Structure on the Moduli Spaces

In this section we recall the construction of the toric structure on $\bar{\mathscr{S}}_g$ from [6, 13], and describe how this structure may be constructed also on the moduli spaces $\bar{\mathscr{S}}_g(\underline{t})$ of parabolic bundles on Σ^g. We use, as in [6], two methods to describe the Hamiltonian flows giving rise to this structure; the first, that of [13], makes the symplectic nature of the flows clear but masks the topology of the orbits of the flows. The second method, related to the work of [14], can then be used to understand these orbits, and show that they in fact correspond to a torus action.

3.1. *Poisson Commuting Functions on the Moduli Space*

The *trinion* or *pair of pants* P is the space $S^2 - (D_1 \cup D_2 \cup D_3)$, where the D_i are three disjoint disks. Recall that we may choose a *trinion* *decomposition* ξ of a closed surface Σ^g: this means Σ^g is decomposed as the union of $2g-2$ trinions P_γ ($\gamma = 1, ..., 2g-2$). The boundary components of the trinions determine $3g-3$ disjoint simple closed curves C_j ($j = 1, ..., 3g-3$) in Σ^g.

We use these curves C_j to give a family of Poisson commuting Hamiltonian functions on $\bar{\mathscr{S}}_g$. Now a simple closed curve C in Σ^g determines a function $f^C: \bar{\mathscr{S}}_g \to \mathbb{R}$ as follows. By choosing a base point $*$ and an arc joining $*$ to some point in C, we obtain an element $[C] \in \pi_1(\Sigma^g)$. Thus, if $\rho \in \mathrm{Hom}(\pi_1(\Sigma^g), G)$, we may define a function $f^C: \mathrm{Hom}(\pi_1(\Sigma^g), G)/G \to \mathbb{R}$ by

$$f^C([\rho]) = \mathrm{Tr}\, \rho([C]). \tag{3.1}$$

We have the following

PROPOSITION 3.1 [5, 13]. *If C, C' are disjoint simple closed curves in Σ^g, then the Poisson bracket $\{f^C, f^{C'}\}$ is* 0.

Let us also define the function $h^C: \bar{\mathscr{S}}_g \to [0, 1]$ by $f^C([\rho]) = 2\cos \pi h^C([\rho])$. The function h^C is continuous on $\bar{\mathscr{S}}_g$, though not differentiable when $h^C = 0$ or $h^C = 1$. It follows from the explicit description of the Hamiltonian flows given in [5, Theorems 4.5, 4.7] that

PROPOSITION 3.2. *Suppose $x \in \bar{\mathscr{S}}_g$ and $h^C(x) \in (0, 1)$. Then the Hamiltonian flow of h^C through x has period* 1 *if C is a nonseparating loop, and period* 1/2 *if C is a separating loop.*

A trinion decomposition ξ determines $3g-3$ disjoint simple closed curves C_j: thus we obtain $3g-3$ Poisson commuting functions $f_j = f^{C_j}$ on $\bar{\mathscr{S}}_g$. The Hamiltonian flows of the functions h_j are defined on the open dense set $U_g = \bigcap_{j=1}^{3g-3} h_j^{-1}((0, 1)) \subset \bar{\mathscr{S}}_g$, and these functions Poisson commute on U_g. We define a map $\mu = (h_1, ..., h_{3g-3}): U_g \to \mathbb{R}^{3g-3}$.

We want to identify the obits of the Hamiltonian flows of the functions h_j. We denote by $\Phi_t^h(x)$ the Hamiltonian flow of the function h at time t, starting at the point $x \in \bar{\mathscr{S}}_g$. An action of $\mathbb{R}^{3g-3}$ on U_g is then given by

$$((\lambda_1, ..., \lambda_3), x) \mapsto \Phi_{(t=1)}^{\sum_j \lambda_j h_j}(x). \tag{3.2}$$

The *kernel* of this action (the subgroup of $\mathbb{R}^{3g-3}$ that acts trivially for all x) is then a lattice Λ in $\mathbb{R}^{3g-3}$: thus we obtain an action of a *torus* $K = \mathbb{R}^{3g-3}/\Lambda$ on U_g. A generic orbit is identified with K.

In order to identify the lattice Λ, it will be convenient to have an alternative description of the Hamiltonian flows, which takes up the next subsection.

3.2. *Alternative Description of Hamiltonian Flows on the Moduli Space*

In order to understand the geometry of the orbits of the torus actions, it will be useful to use another construction of these actions. Suppose Σ is a (not necessarily connected) surface with two distinguished boundary components S, S'. We may form a surface $\hat{\Sigma}$ with two fewer boundary components, by gluing S to S'. We denote by $\mathscr{S}(\Sigma)$ and $\mathscr{S}(\hat{\Sigma})$ the corresponding moduli spaces of gauge equivalence classes of flat connections. There is thus a map $\alpha:\mathscr{S}(\hat{\Sigma})\to\mathscr{S}(\Sigma)$.

The fibre of α above a point $[A]\in\mathscr{S}(\Sigma)$ for which $A|_S$ is gauge equivalent to $A|_{S'}$ is given as follows. We pick a representative connection A in the class $[A]$ such that $A|_S=A|_{S'}$. Denote by $\mathscr{G}_0(\Sigma)$ the subgroup of the gauge group $\mathscr{G}(\Sigma)$ consisting of gauge transformations which restrict on S, S' to elements of $H=\mathrm{Stab}(A|_S)$, the stabilizer of $A|_S$ under the gauge group action. We denote also by J the stabilizer of A under the action of $\mathscr{G}(\Sigma)$. There is a surjective map $\phi:\mathscr{G}_0(\Sigma)/J\to\alpha^{-1}([A])$ given by $\phi([g])=[A^g]$, the gauge equivalence class on $\hat{\Sigma}$ of A^g viewed as a connection on $\hat{\Sigma}$.

The kernel of ϕ is then identified as

$$\mathrm{Ker}(\phi)=\{g\in\mathscr{G}_0(\Sigma)\mid j|_S g|_S=j|_{S'}g|_{S'} \text{ for some } j\in J\}. \tag{3.3}$$

Thus $\mathrm{Im}(\phi)=\mathscr{G}_0(\Sigma)/\mathrm{Ker}(\phi)$, and we have

LEMMA 3.3. $\mathrm{Im}(\phi)\cong H/J$, *where J acts on H by*

$$j: h\mapsto j|_S h j|_{S'}^{-1}. \tag{3.4}$$

Proof. We have a map $\delta:\mathscr{G}_0(\Sigma)\to H/J$ given by $\delta(g)=[g|_S g|_{S'}^{-1}]$, where the latter is the equivalence class under the action of J. Furthermore, $\mathrm{Ker}(\delta)=\mathrm{Ker}(\phi)$.

Thus finally we have

THEOREM 3.4. *The fibre* $\alpha^{-1}([A])$ *is diffeomorphic to* $\mathrm{Stab}(A|_S)/\mathrm{Stab}(A)$, *where the action of* $\mathrm{Stab}(A|_S)$ *on* $\mathrm{Stab}(A)$ *is given by* (3.4).

We want to use this identification of the fibre to relate flat connections on Σ^g to their restrictions to individual trinions P_γ in the trinion decomposition ξ. First we quote

PROPOSITION 3.5 [6, Proposition 3.1]. *The moduli space $\mathscr{S}(P)$ of gauge equivalence classes of flat connections on a trinion P is in bijective correspondence with the set*

$$\{(t_1,t_2,t_3)\in[0,1]^3\mid |t_1-t_2|\leqslant t_3\leqslant t_1+t_2,\ t_1+t_2+t_3\leqslant 2\}. \tag{3.5}$$

The correspondence is induced by the map $x \to (h_1(x), h_2(x), h_3(x))$, where the h_i are the functions on $\bar{\mathscr{S}}(P)$ defined above, corresponding to the three boundary circles of P.

This proposition shows that the fibre of the map $\mu\colon U_g \to \mathbb{R}^{3g-3}$ coincides on U_g with the fibre of the restriction map $\bar{\mathscr{S}}_g \to \prod_{\gamma=1}^{2g-2} \bar{\mathscr{S}}(P_\gamma)$.

PROPOSITION 3.6. *The closure $\overline{\mu(U_g)} \subset \mathbb{R}^{3g-3}$ of the image of U_g under μ is a convex polyhedron.*

Proof. The polyhedron is determined by imposing the inequalities (3.5) for every trinion P_γ.

Applying Theorem 3.4 repeatedly to a surface Σ^g formed by gluing together successive boundary circles of trinions, we obtain

PROPOSITION 3.7. *Suppose A is a flat connection on Σ^g. Then the fibre of the map $\mu\colon \bar{\mathscr{S}}_g \to \mathbb{R}^{3g-3}$ containing $[A]$ is diffeomorphic to*

$$\mu^{-1}(x) = \prod_{j=1}^{3g-3} \mathrm{Stab}(A|_{C_j}) \Big/ \prod_{\gamma=1}^{2g-2} \mathrm{Stab}(A|_{P_\gamma}). \tag{3.6}$$

Let $U_g^{gen} \subset U_g$ denote $\{x \in \bar{\mathscr{S}}_g \mid$ there is a flat connection A in the gauge equivalence class of x such that $[A|_{P_\gamma}] \in \mathrm{Interior}(\bar{\mathscr{S}}(P_\gamma))$ for every trinion $P_\gamma\}$. If $[A] \in U_g^{gen}$, then the stabilizers $\mathrm{Stab}(A|_{C_j})$ are just copies of $U(1)$, while $\mathrm{Stab}(A|_{P_\gamma})$ is a copy of $\mathbb{Z}_2 = Z(G)$.

Furthermore, let us represent the copy of $U(1)$ corresponding to the jth boundary circle by $\{e^{2\pi i\phi_j},\ 0 \leqslant \phi_j \leqslant 1\}$. By Proposition 3.7, we have the following identification:

PROPOSITION 3.8. *The generic fibre of μ (i.e., the fibre through a point of U_g^{gen}) may be identified with*

$$\mu^{-1}(x) = U(1)^{3g-3}/\mathbb{Z}_2^{2g-2}, \tag{3.7}$$

where the action of $(\varepsilon(1), ..., \varepsilon(2g-2)) \in \mathbb{Z}_2^{2g-2}$ is given by $e^{2\pi i\phi_j} \to \varepsilon(\gamma)\,\varepsilon(\gamma')\,e^{2\pi i\phi_j}$, where P_γ, $P_{\gamma'}$ are the two trinions bounding the curve C_j.

We have a vector field $\partial/\partial\phi_j$ on $h_j^{-1}((0,1)) \subset \bar{\mathscr{S}}_g$. There is the following identification:

THEOREM 3.9 [6, Proposition 5.4]. *The vector field $\partial/\partial\phi_j$ is equal to the Hamiltonian vector field X_{h_j} corresponding to the function h_j.*

Thus we have identified $\{e^{2\pi i \phi_j}: 0 \leqslant \phi_j \leqslant 1\}$ with the copy of $U(1)$ arising from the Hamiltonian flow of the function h_j. Now Proposition 3.8 permits us to identify the lattice Λ that is the kernel of the Hamiltonian flow 3.2. Summarizing, we have shown the following:

PROPOSITION 3.10. *The trinion decomposition* ξ *determines a* $3g-3$-*dimensional torus* $K = \mathbb{R}^{3g-3}/\Lambda$ *which acts on* U_g *preserving the symplectic form, and acts effectively (in other words, the only element of* K *acting trivially everywhere is* 1). *The action of* K *is given by* (3.2). *The lattice* $\Lambda \subset \mathbb{R}^{3g-3}$ *has rank* $3g-3$, *and is spanned by* $\hat{e}_j = (0, ..., 1, ..., 0)$ $(j = 1, ..., 3g-3)$ *and* $\tilde{e}_\gamma$ $(\gamma = 1, ..., 2g-2)$, *where*

$$\tilde{e}_\gamma = \tfrac{1}{2}(\hat{e}_{j_1(\gamma)} + \hat{e}_{j_2(\gamma)} + \hat{e}_{j_3(\gamma)}), \tag{3.8}$$

for $C_{j_1(\gamma)}, C_{j_2(\gamma)}, C_{j_3(\gamma)}$ *the three boundary circles of the trinion* P_γ.

The map μ is the *moment map* for the action of the torus K.

3.3. *Surfaces with Boundary*

In this section we return to the notation of Subsection 2.2. We equip Σ^g_d with a trinion decomposition ξ, which has $2g-2+d$ trinions and $3g-3+d$ boundary circles in the *interior* of Σ^g_d. The methods of [13, Lemma 3.3 and Corollary 3.4] show that we have Poisson commuting functions h_j $(j = 1, ..., 3g-3+d')$ on $\bar{\mathscr{S}}_g(\underline{t})$, defined as in Subsection 3.1. Here, d' was defined in Subsection 2.2 as the number of boundary circles $\tilde{C}_a$ for which $t_a \neq 0, 1$. Observe that if one of the t_a is 0 then the functions $h_{j_1(a)}$, $h_{j_2(a)}$ corresponding to the other two boundary circles $C_{j_1(a)}$, $C_{j_2(a)}$ of the trinion containing $\tilde{C}_a$ satisfy $h_{j_1(a)} = h_{j_2(a)}$, while if $t_a = 1$ they satisfy $h_{j_1(a)} = 1 - h_{j_2(a)}$. Hence the number of linearly independent Poisson commuting Hamiltonian flows is $3g-3+d'$ rather than $3g-3+d$.

Denote by $U_g(\underline{t})$ the subset $\bigcap_{j=1}^{3g-3+d'} h_j^{-1}((0, 1))$. The analog of Theorem 3.9 also extends to this setting, so that on $U_g(\underline{t})$, if $\partial/\partial\phi_j$ is the vector field at the point $[A] \in U_g(\underline{t})$ corresponding to $\mathrm{Stab}(A|_{C_j}) = U(1)$, we have

THEOREM 3.11. *The vector field* $\partial/\partial\phi_j$ *is equal to the Hamiltonian vector field* X_{h_j} *corresponding to the function* h_j.

Proof. The proof given in Proposition 5.4 of [6] does not generalize directly to this case. However, one may give an alternative proof by finding a vector field on the subspace of flat connections $\mathscr{A}_F \subset \Omega^1(\Sigma^g_d, \mathfrak{g})$, which represents the Hamiltonian vector field X_{h_j} corresponding to the function h_j: this was done in [13, Lemma 3.3]. The vector field $\partial/\partial\phi_j$ may likewise be explicitly determined as a vector field on $\mathscr{A}_F$: this was done in the proof of Theorem 2.1 in [7]. One may then explicitly check that these two vector fields agree.

Remark. If all the weights t_a are in $(0, 1)$, an alternative proof of the existence of Poisson commuting functions on $\bar{\mathscr{S}}_g(\underline{t})$ and of Theorem 3.11 may be obtained as follows. One observes that $\bar{\mathscr{S}}_g(\underline{t})$ is a *symplectic quotient* of a moduli space $\bar{\mathscr{S}}_{g+d}$ corresponding to a surface Σ^{g+d}, which is formed by attaching one-holed tori N_a, $a = 1, ..., d$, to the boundary circles $\tilde{C}_a$. Let us assume also that N_a is equipped with a distinguished nonseparating simple closed curve C'_a, so that $N_a - C'_a$ is a trinion. Thus $\{C'_a\}$ and ξ together yield a trinion decomposition $\tilde{\xi}$ of Σ^{g+d}. Let us label the boundary circles C_j of ξ in such a way that $C_{3g-3+d+1}, ..., C_{3g-3+2d}$ are the boundary components $\tilde{C}_a = \partial N_a$, while $C_{3g-3+2d+1}, ..., C_{3g-3+3d}$ correspond to the distinguished curves C'_a in N_a. Thus the Hamiltonian flows of the functions h_j ($j = 3g - 3 + d + 1, ..., 3g - 3 + 3d$) are defined on $U = \bigcap_{j \geq 3g-3+d+1} h_j^{-1}((0, 1)) \subset \bar{\mathscr{S}}_{g+d}$. These flows define an action of $U(1)^{2d}$ on U, with moment map μ'.

Then $\bar{\mathscr{S}}_g(\underline{t})$ may be identified with the symplectic quotient $(\mu')^{-1}(x)/U(1)^{2d}$ for a suitable $x \in \mathbb{R}^{2d}$, with $(x_1, ..., x_d) = (t_1, ..., t_d)$. The symplectic form (2.6) on $\bar{\mathscr{S}}_g(\underline{t})$ is that obtained from the symplectic reduction. The functions h_j ($j \leqslant 3g - 3 + d$) are invariant under the action of $U(1)^{2d}$, and hence descend to give Poisson commuting Hamiltonian functions on $\bar{\mathscr{S}}_g(\underline{t})$.

PROPOSITION 3.12. *For $t_a \in (0, 1)$, the symplectic volume of $\bar{\mathscr{S}}_g(\underline{t})$ is a piecewise continuous function of $\underline{t}$.*

Proof. This follows immediately from the identification of $\bar{\mathscr{S}}_g(\underline{t})$ as a symplectic quotient, and from the Duistermaat–Heckman theorem [4].

If any of the t_a are equal to 1, it is no longer possible to identify $\bar{\mathscr{S}}_g(\underline{t})$ as a symplectic quotient, but one may nonetheless embed an open dense subset of $\bar{\mathscr{S}}_g(\underline{t})$ symplectically as a submanifold of $\bar{\mathscr{S}}_{g+d}$, and hence likewise obtain an alternative proof of Theorem 3.11 in this case.

As in the case of closed surfaces, one here obtains a torus acton on an open dense subset $U_g(\underline{t}) = \bigcap_{j=1}^{3g-3+d'} h_j^{-1}((0, 1))$ of $\bar{\mathscr{S}}_g(\underline{t})$: we define the moment map $\mu = (h_1, ..., h_{3g-3+d'})\colon U_g(\underline{t}) \to \mathbb{R}^{3g-3+d'}$. We see by extending the proof of Proposition 3.10 that

PROPOSITION 3.13. *The trinion decomposition ξ of Σ^g_d determines a torus $K = \mathbb{R}^{3g-3+d}/\Lambda$ which acts effectively on $U_g(\underline{t})$, preserving the symplectic form. The kernel Λ is spanned by the elements $\hat{e}_j = (0, ..., 1, ..., 0)$ $(j = 1, ..., 3g - 3 + d')$ and $\tilde{e}_\gamma$, where*

$$\tilde{e}_\gamma = (1/2) \sum_{j(\gamma)} \hat{e}_{j(\gamma)}: \tag{3.9}$$

here we sum over $j(\gamma)$ corresponding to those boundary circles $C_{j(\gamma)}$ of P_γ which are not components of $\partial\Sigma_d^g$.

As before, μ is the moment map for the action of K. We have also

PROPOSITION 3.14. *The closure $\overline{\mu(U_g(\underline{t}))}$ of the image of $U_g(\underline{t})$ under μ is a convex polyhedron.*

Proof. The polyhedron is determined by the inequalities

$$|h_{j_1(\gamma)}(x) - h_{j_2(\gamma)}(x)| \leqslant h_{j_3(\gamma)}(x) \leqslant h_{j_1(\gamma)}(x) + h_{j_2(\gamma)}(x), \tag{3.10}$$
$$h_{j_1(\gamma)}(x) + h_{j_2(\gamma)}(x) + h_{j_3(\gamma)}(x) \leqslant 2,$$

for trinions P_γ not intersecting $\partial\Sigma_d^g$. If P_γ intersects $\partial\Sigma_d^g$ in boundary components $\tilde{C}_a$, then in (3.10) the coordinates $h_a(x)$ must be replaced by the weights t_a.

4. INTEGRAL POINTS AND THE SYMPLECTIC VOLUME

Let (M^{2n}, ω) be a symplectic manifold, and let K^n be a rank n abelian Lie group with a Hamiltonian action on M. Suppose the action of K on M is effective. The action may be described as an action of the Lie algebra $\mathfrak{k}$, for which the integer lattice Λ is the subgroup of $\mathfrak{k}$ that acts trivially. The *moment map* for the action of K is then a map $\mu: M \to \mathfrak{k}^*$.

Choose a basis (e_j) $(1 \leqslant j \leqslant n)$ for Λ and the corresponding dual basis (f^j) for Λ^*: we may then write the moment map as $\mu = \sum_j \mu_j f^j$. Then we may take coordinates (x^j) $(j = 1, ..., n;\ x^j \in \mathbb{R}/\mathbb{Z})$ on K, corresponding to $\sum_j x^j e_j$ on $\mathfrak{k}$.

Suppose $U \subset \mathrm{Im}(\mu) \subset \mathfrak{k}^*$ is a subset for which there is a Lagrangian submanifold L in $\mu^{-1}(U)$ transverse to the fibres of the K action, and mapped diffeomorphically onto U by μ. Suppose also that the action of K on $\mu^{-1}(U)$ is free. Then by setting $x^j = 0$ on L, the x^j, μ_j become coordinates on $\mu^{-1}(U)$ which identify it with $U \times K$. We thus have

PROPOSITION 4.1. *The symplectic form is given on $\mu^{-1}(U)$ by $\omega = \sum_j d\mu_j\, dx^j$, and the volume form is $\omega^n/n! = d\mu_1 \cdots d\mu_n\, dx^1 \cdots dx^n$.*

COROLLARY 4.2. *The symplectic volume satisfies* $\mathrm{vol}(\mu^{-1}(U)) = \mathrm{vol}(U)$, *where the Euclidean volume on $\mathfrak{k}^*$ is chosen to assign volume* 1 *to $\mathfrak{k}^*/\Lambda^*$.*

Remark. This corollary is the simplest case of the Duistermaat–Heckman theorem [4, Corollary 3.3].

If $B \subset \mathfrak{t}^*$, let $i(B)$ denote the number of points in $B \cap \Lambda^*$. For $k \in \mathbb{R}^+$, denote by kB the dilation of B by a factor k: thus $\mathrm{vol}(kB) = k^n \mathrm{vol}(B)$. The following is immediate:

LEMMA 4.3. *Suppose B is a submanifold of $\mathfrak{t}^*$ with piecewise smooth boundary. Then* $\mathrm{vol}(B) = \lim_{k \to \infty} i(kB)/k^n$.

COROLLARY 4.4. *Let B be as in Lemma 4.3. Then* $\mathrm{vol}(\mu^{-1}(B)) = \lim_{k \to \infty} i(kB)/k^n$.

We can apply Lemma 4.3 to compute the volume of the symplectic manifolds $\mathcal{S}_g$ and $\mathcal{S}_g(\underline{t})$. In the above notation, we have the following

PROPOSITION 4.5. *The volumes of $\mathcal{S}_g$ and $\mathcal{S}_g(\underline{t})$ are given by*

$$\mathrm{vol}(\mathcal{S}_g) = \lim_{k \to \infty} \frac{i(k\overline{\mu(U_g)})}{k^{3g-3}},$$

$$\mathrm{vol}(\mathcal{S}_g(\underline{t})) = \lim_{k \to \infty} \frac{i(k\overline{\mu(U_g(\underline{t}))})}{k^{3g-3+d}}.$$

It remains to identify the lattice Λ^*. Let us examine the condition $k\mu(x) \in \Lambda^*$. Recalling (3.8), (3.9), if we identify $\mathfrak{t}^*$ with $\mathbb{R}^{3g-3+d}$ via a basis $\hat{e}_j$, then the lattice Λ is generated by $\hat{e}_j$ and $\tilde{e}_\gamma = (1/2) \sum_{j(\gamma)} \hat{e}_{j(\gamma)}$. The moment map μ is given in terms of the dual basis $\{\hat{f}^j\}$ to $\{\hat{e}_j\}$ by $\mu = \sum_j h_j \hat{f}^j$: thus the condition $k\mu(x) \in \Lambda^*$ is just

$$kh_j(x) \in \mathbb{Z}, \tag{4.1}$$

$$kg_\gamma(x) \in \mathbb{Z}, \tag{4.2}$$

where $g_\gamma = (1/2) \sum_{j(\gamma)} h_{j(\gamma)}$ and $j(\gamma)$ denote those boundary components of P_γ that do not intersect $\partial \Sigma^g_d$.

Let Σ^g be a closed surface equipped with a trinion decomposition ξ, and let $k \in \mathbb{Z}^+$. We consider labellings of the boundary circles C_j of the trinion decomposition by integers l_j, $0 \leqslant l_j \leqslant k$.

DEFINITION 4.6. A labelling (l_j) of the trinion decomposition ξ is admissible if for every trinion P_γ, the labels $l_{j_1(\gamma)}$, $l_{j_2(\gamma)}$, $l_{j_3(\gamma)}$ of the three boundary circles $j_1(\gamma)$, $j_2(\gamma)$, $j_3(\gamma)$ satisfy the quantum Clebsch–Gordan condition

$$\begin{aligned} &\text{(a)} \quad l_{j_1(\gamma)} + l_{j_2(\gamma)} + l_{j_3(\gamma)} \in 2\mathbb{Z} \\ &\text{(b)} \quad |l_{j_1(\gamma)} - l_{j_2(\gamma)}| \leqslant l_{j_3(\gamma)} \leqslant l_{j_1(\gamma)} + l_{j_2(\gamma)} \\ &\text{(c)} \quad l_{j_1(\gamma)} + l_{j_2(\gamma)} + l_{j_3(\gamma)} \leqslant 2k. \end{aligned} \tag{4.3}$$

DEFINITION 4.7. Let $\mathcal{D}(\Sigma^g, \xi, k)$ be the set of admissible labellings of ξ, and let $D(g, \xi, k)$ be the number of admissible labellings.

Remark. In fact, $D(g, \xi, k)$ is independent of the choice of trinion decomposition ξ [8], so we denote it by $D(g, k)$.

Let Σ_d^g be a genus g surface with d boundary components: we assume all weights $t_a \in \mathbb{Q}$. Consider $k \in \mathbb{Z}$ such that $kt_a = n_a \in 2\mathbb{Z}$ for all a.

Assume Σ_d^g is equipped with a trinion decomposition ξ. We consider labellings of the boundary circles C_j ($j = 1, ..., 3g-3+d$) in the interior of Σ_d^g by integers l_j, $0 \leqslant l_j \leqslant k$. We obtain a labelling of *all* boundary circles of the trinion decomposition by endowing the boundary component $\tilde{C}_a$ with the label $l_{3g-3+d+a} = n_a$.

DEFINITION 4.8. A labelling (l_j, $j = 1, ..., 3g-3+2d$) of the trinion decomposition ξ of Σ_d^g is *admissible* if for every trinion D_γ, the labels $l_{j_1(\gamma)}$, $l_{j_2(\gamma)}$, $l_{j_3(\gamma)}$ of the three boundary circles satisfy the conditions (4.3).

(Notice that here the labels on the boundary circles $\tilde{C}_a$ are now fixed as $n_a = kt_a$.)

DEFINITION 4.9. Let $\mathscr{D}(\Sigma_d^g, \xi, \underline{t}, k)$ be the set of admissible labellings of ξ, and let $D(g, d, \xi, \underline{t}, k)$ be the number of admissible labellings.

As in the case $d = 0$, this number is independent of the choice of ξ, so we may denote it by $D(g, d, \underline{t}, k)$.

We now observe

THEOREM 4.10. (a) *The set* $\mathscr{D}(\Sigma^g, \xi, k)$ *is in bijective correspondence with* $k\overline{\mu(U_g)} \cap \Lambda^*$.

(b) *The set* $\mathscr{D}(\Sigma_d^g, \xi, \underline{t}, k)$ *is in bijective correspondence with* $k\overline{\mu(U_g(\underline{t}))} \cap \Lambda^*$.

Proof. The labels l_j correspond to the integer values of the functions h_j. Equation (4.3)(a) arises from the condition that the kg_γ take integer values (recalling that the labels n_a corresponding to the boundary components $\tilde{C}_a$ are *even* integers). Equations (4.3)(b), (c) arise from the inequalities (3.5) specifying the moduli space $\mathscr{S}(P)$ of gauge equivalence classes of flat connections on one trinion P.

Our main result is an immediate corollary of Theorem 4.10 and Lemma 4.3:

THEOREM 4.11. *The volumes of the moduli spaces* $\mathscr{S}_g$ *and* $\mathscr{S}_g(\underline{t})$ *are given by*

$$\text{(a)} \qquad \text{vol}(\mathscr{S}_g) = \lim_{k \to \infty} \frac{D(g, k)}{k^{3g-3}},$$

$$\text{(b)} \quad \text{vol}(\mathscr{S}_g(\underline{t})) = \lim_{k \to \infty} \frac{D(g, d, \underline{t}, k)}{k^{3g-3+d}}.$$

(In case (a), *we take the limit over* $k \in \mathbb{Z}^+$. *In case* (b), *we take the limit over those* $k \in \mathbb{Z}^+$ *for which* $kt_a \in 2\mathbb{Z}$ *for all a.)*

In Section 5 we need alternative formulas for $D(g, k)$. By considering particular trinion decompositions, it is possible to rewrite $D(g, k)$ and $D(g, d, \underline{t}, k)$ as trigonometric sums. For $0 \leqslant m, n \leqslant k \in \mathbb{Z}$, define

$$S_{m,n} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi(m+1)(n+1)}{k+2}.$$

In terms of this quantity, we have

PROPOSITION 4.12. *The quantities* $D(g, k)$ *and* $D(g, d, \underline{t}, k)$ *are given by the trigonometric series* (a)

$$D(g, k) = \sum_{j=0}^{k} \frac{1}{(S_{0,j})^{2g-2}}.$$

(b) *If all* $t_a \in (0, 1)$, *and* $n_a = kt_a \in \mathbb{Z}$, *then*

$$D(g, d, \underline{t}, k) = \sum_{j=0}^{k} \frac{1}{(S_{0,j})^{2g-2+d}} \prod_{a=1}^{d} S_{j,n_a}.$$

Proof. These formulas result from a set of identities due to Verlinde [12]. For (a), see [11, (6)–(12)]. Formula (b) is given in [14 (3.16)], and is proved by a straightforward extension of the calculation in [11].

5. Review of Topological Consequences

We close with an expository section describing some of the consequences of Theorem 4.11 identifying the volumes of moduli spaces. These consequences have been worked out by Donaldson [3], Thaddeus [11], and Witten [14]. The volumes yield information about the ring structure of the cohomology ring $H^*(\bar{\mathcal{S}}_g(1))$ of the smooth moduli space $\bar{\mathcal{S}}_g(1)$, and also the dimension of the space of holomorphic sections of powers of a distinguished line bundle $\mathcal{L}_1$ over $\bar{\mathcal{S}}_g(1)$. We emphasize that this section summarizes results found in the literature and is not original.

Via results of Bertram and Szenes using the *Hecke correspondence* [2], one may also obtain the dimension of the space of holomorphic sections of powers of a distinguished line bundle $\mathcal{L}_0$ over the non-smooth moduli space $\bar{\mathcal{S}}_g$.

5.1. *The Cohomology Ring of the Smooth Moduli Space*

The cohomology ring $H^*(\bar{\mathcal{S}}_g(1))$ is generated by classes which are obtained from the characteristic classes of a universal rank 2 bundle $\mathcal{U}$ over $\bar{\mathcal{S}}_g(1) \times \Sigma^g$. There is a distinguished line bundle $\mathcal{L}_1$ over $\bar{\mathcal{S}}_g(1)$ such that $c_1(\mathcal{L}_1) = 2[\omega_1]$ (where $[\omega_1]$ is the cohomology class determined by the symplectic form ω_1 (2.6) on $\bar{\mathcal{S}}_g(1)$). Further, $\mathcal{U}$ has the property that $\det(\mathcal{U}_p) = \mathcal{L}_1$, where $\mathcal{U}_p$ denotes the restriction of $\mathcal{U}$ to $\bar{\mathcal{S}}_g(1) \times \{p\}$ for a point $p \in \Sigma^g$. In terms of the *adjoint* bundle ad $\mathcal{U}$ with fibre sl(2, $\mathbb{C}$), we have classes

$$\alpha = -(1/2)\, p_1(\text{ad}\,\mathcal{U})/[\Sigma^g] \in H^2(\bar{\mathcal{S}}_g(1)), \tag{5.1}$$

$$\beta = p_1(\text{ad}\,\mathcal{U})/[p] \in H^4(\bar{\mathcal{S}}_g(1)). \tag{5.2}$$

The slant product of $p_1(\text{ad}\,\mathcal{U})$ with all the cycles in $H_*(\Sigma^g, \mathbb{Z})$ gives a set of generators for $H^*(\bar{\mathcal{S}}_g(1))$; however, we shall only be concerned with the classes α and β. Here, $\alpha = 2[\omega_1]$.

In order to extract information about the cohomology ring, it is necessary to recast the result (Theorem 4.11) for $\text{vol}(\mathcal{S}_g(\underline{t}))$ as a polynomial in t. When all $t_i \in (0, 1)$, one may extract the leading term in k in Proposition 5.8(b): one has [14, (3.17)]

$$\lim_{k \to \infty} \frac{1}{k^{3g-3+d}} D(g, d, \underline{t}, k) = 2 \frac{1}{2^{g-1} \pi^{2g-2+d}} \sum_{n=1}^{\infty} \frac{\prod_j \sin(\pi n t_j)}{n^{2g-2+d}}. \tag{5.3}$$

Let us confine ourselves now to the case where there is only one boundary component, with weight t; where $t \neq 0$, the moduli space $\bar{\mathcal{S}}_g(t) = \mathcal{S}_g(t)$ is smooth. We may then rewrite the formula for $\text{vol}(\bar{\mathcal{S}}_g(t))$ as follows. We have

$$\text{vol}(\bar{\mathcal{S}}_g(t)) = 2 \frac{1}{2^{g-1} \pi^{2g-1}} \sum_{n=1}^{\infty} \frac{\sin(\pi n t)}{n^{2g-1}} \qquad \text{from (5.3)} \tag{5.4}$$

$$= (-1)^g 2^g P_{2g-1}(t/2) \qquad [3, (22)], \tag{5.5}$$

where P_m is the mth *Bernoulli polynomial*, a polynomial of degree m. (In other words, (5.4) is the Fourier series for a polynomial in t.)

Remark. Our result (Theorem 4.10) yields the formula (5.4) only when t is *rational*; however, it follows from Proposition 3.12 that $\text{vol}(\bar{\mathcal{S}}_g(t))$ is a piecewise continuous function of t, so the values for $t \in \mathbb{Q}$ suffice to establish the formula for general t.

The following argument given in [3, Sect. 6] (see also [14]) shows how to use formula (5.5) to extract information about the ring structure of $H^*(\bar{\mathscr{S}}_g(1))$. If t is close to 1, there is a fibration

$$S^2 \to \bar{\mathscr{S}}_g(t) \xrightarrow{q} \bar{\mathscr{S}}_g(1). \tag{5.6}$$

Moreover, if we denote by $[\omega_t]$ the cohomology class on $\bar{\mathscr{S}}_g(t)$ corresponding to the symplectic form, and by $[\omega_1]$ the corresponding class on $\bar{\mathscr{S}}_g(1)$, then we have

$$[\omega_t] = q^*[\omega_1] + (1-t)\, e. \tag{5.7}$$

Here, $e \in H^2(\bar{\mathscr{S}}_g(t), \mathbb{Z})$ restricts on each fibre of q to the generator of $H^2(S^2, \mathbb{Z})$, and additionally $e^2 = q^*(\beta/4)$.

We thus see that the coefficients of the different powers of t in the formula (5.5) for the symplectic volume of $\bar{\mathscr{S}}_g(t)$ yield the intersection pairings of powers of α and β on $\bar{\mathscr{S}}_g(1)$:

$$\begin{aligned}
\mathrm{vol}(\bar{\mathscr{S}}_g(t)) &= \frac{[\omega_t]^{3g-2}}{(3g-2)!}\,[\bar{\mathscr{S}}_g(t)] \\
&= \frac{(q^*[\omega_1] + (1-t)\, e)^{3g-2}}{(3g-2)!}\,[\bar{\mathscr{S}}_g(t)] \\
&= \frac{1}{(3g-2)!} \sum_{m=0}^{3g-2} \binom{3g-2}{m} (1-t)^m \\
&\quad \times e^m (q^*[\omega_1])^{3g-2-m}\,[\bar{\mathscr{S}}_g(t)] \\
&= \frac{1}{(3g-2)!} \sum_{2n=0}^{3g-4} \binom{3g-2}{2n+1} (1-t)^{2n+1} \\
&\quad \times (\beta/4)^n (\alpha/2)^{3g-3-2n}\,[\bar{\mathscr{S}}_g(1)].
\end{aligned} \tag{5.8}$$

In this way, the intersection pairings of α and β may be read from the Bernoulli polynomial (5.5).

5.2. *Holomorphic Sections of Line Bundles*

In addition, work of Thaddeus [11] relates these intersection pairings to the dimension of the space of holomorphic sections $H^0(\bar{\mathscr{S}}_g(1), \mathscr{L}_1^{k/2})$ for even positive integers k. The Riemann–Roch theorems tells us that

$$\dim H^0(\bar{\mathscr{S}}_g(1), \mathscr{L}_1^{k/2}) = (\mathrm{ch}\, \mathscr{L}_1)^{k/2}\, \mathrm{Td}(\bar{\mathscr{S}}_g(1))[\bar{\mathscr{S}}_g(1)], \tag{5.9}$$

and ch $\mathscr{L}_1^{k/2} = \exp(k/2)\,\alpha = \exp k[\omega_1]$, while [11, above (28)]

$$\operatorname{Td} \bar{\mathscr{S}}_g(1) = \exp \alpha \left(\frac{\sqrt{\beta}/2}{\sinh(\sqrt{\beta}/2)} \right)^{2g-2}. \tag{5.10}$$

(This is an *even* function of $\sqrt{\beta}$, and thus a power series in β.) Thus the dimension $\dim H^0(\bar{\mathscr{S}}_g(1), \mathscr{L}_1^{k/2})$ may be obtained from the pairings $\alpha^{3g-3-2n}\beta^n[\bar{\mathscr{S}}_g(1)]$.

Finally we give a brief treatment of how these results yield the dimension of the space of holomorphic sections of powers of a certain line bundle $\mathscr{L}_0$ over the (non-smooth) moduli space $\bar{\mathscr{S}}_g$. This line bundle $\mathscr{L}_0$ is distinguished by $c_1(\mathscr{L}_0|_{\mathscr{S}_g}) = [\omega]$, where $[\omega]$ is the cohomology class represented by the symplectic form ω on $\mathscr{S}_g$. A theorem of Bertram and Szenes [2] (using the *Hecke correspondence*) relates $\dim H^0(\bar{\mathscr{S}}_g, \mathscr{L}_0^k)$ to the holomorphic Euler characteristic of certain vector bundles over $\bar{\mathscr{S}}_g(1)$:

THEOREM 5.1 [2, Theorem 2.4]. *The dimension of the space of holomorphic sections of $\mathscr{L}_0^k$ is given by*

$$\dim H^0(\bar{\mathscr{S}}_g, \mathscr{L}_0^k) = \sum_j (-1)^j \dim H^j(\bar{\mathscr{S}}_g(1), S^k\mathscr{U}_p).$$

(Here, $S^k\mathscr{U}_p$ is the kth symmetric power of $\mathscr{U}_p$.)

By the Riemann–Roch theorem, we thus have

PROPOSITION 5.2. *The dimension of the space of holomorphic sections of $\mathscr{L}_0^k$ is given by*

$$\dim H^0(\bar{\mathscr{S}}_g, \mathscr{L}_0^k) = \operatorname{ch}(S^k\mathscr{U}_p)\operatorname{Td}(\bar{\mathscr{S}}_g(1))\,[\bar{\mathscr{S}}_g(1)].$$

But we recall that

$$c_1(\det \mathscr{U}_p) = c_1(\mathscr{L}_1) = \alpha, \qquad c_2(\operatorname{ad}\mathscr{U}_p) = \beta. \tag{5.11}$$

A short calculation then yields

$$\operatorname{ch}(S^k\mathscr{U}_p) = \exp(k\alpha/2)\frac{\sinh((k+1)\sqrt{\beta}/2)}{\sinh(\sqrt{\beta}/2)}. \tag{5.12}$$

One thus finds finally

PROPOSITION 5.3. *The dimension of the space of holomorphic sections of the line bundle $\mathscr{L}_0^k$ can be expressed in terms of the intersection pairings of α and β on $\mathscr{S}_g(1)$ via the formula*

$$\dim H^0(\bar{\mathscr{S}}_g, \mathscr{L}_0^k) = \exp\frac{(k+2)\alpha}{2}\left(\frac{\sqrt{\beta}/2}{\sinh(\sqrt{\beta}/2)}\right)^{2g-2} \frac{\sinh((k+1)\sqrt{\beta}/2)}{\sinh(\sqrt{\beta}/2)}[\bar{\mathscr{S}}_g(1)]. \tag{5.13}$$

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