JOURNAL OF APPROXIMATION THEORY 16, 384–408 (1976)

Approximation with Sums of Exponentials in L_p $[0, \infty)$*

DAVID W. KAMMLER

Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901

Communicated by E. W. Cheney

Received June 14, 1974

We consider the problem of approximating a given f from $L_p[0, \infty)$ by means of the family $V_n(S)$ of exponential sums; $V_n(S)$ denotes the set of all possible solutions of all possible nth order linear homogeneous differential equations with constant coefficients for which the roots of the corresponding characteristic polynomials all lie in the set S. We establish the existence of best approximations, show that the distance from a given f to $V_n(S)$ decreases to zero as n becomes infinite, and characterize such best approximations with a first-order necessary condition. In so doing we extend previously known results that apply in $L_p[0, 1]$.

1. INTRODUCTION

Given $\mathbf{b} = (b_1, ..., b_n)$ and $\mathbf{c} = (c_1, ..., c_n)$ from C^n (or from R^n if we choose to work with real valued functions), we define $Y_n(\mathbf{b}, \mathbf{c}, t)$ to be the solution of the initial value problem

$$[D^n + c_1 D^{n-1} + \cdots + c_{n-1} D + c_n]\, y(t) = 0, \qquad t \geqslant 0 \tag{1}$$

$$D^{j-1} y(0) = b_j, \qquad j = 1, 2, ..., n, \tag{2}$$

where $D = d/dt$ is the differential operator. A function y that satisfies (1) but does not satisfy any such equation of lower order will be called an exponential sum with order n. We let $P_n[\mathbf{c}, \lambda]$ denote the characteristic polynomial of the differential operator of (1) and let

$$\Lambda_n[\mathbf{c}] = \{\lambda \in C: P_n[\mathbf{c}, \lambda] = 0\} \tag{3}$$

denote the corresponding spectral set. Given a set $S \subseteq C$, we form the collection $V_n(S)$ of all possible exponential sums $Y_n(\mathbf{b}, \mathbf{c}, -)$ with order at

* This research was sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under AFOSR Grant Number 74-2653. The United States government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation hereon.

most n for which $\Lambda_n[\mathbf{c}] \subseteq S$, $n = 1, 2, \ldots$, with $V_0(S)$ defined to be the set whose only element is the zero function and with

$$V_\infty(S) = \bigcup_{n=1}^{\infty} V_n(S)$$

defined to be the collection of all possible exponential sums having spectra contained in S.

We define the space $L_p[0, \infty)$ with the associated norm $\| \ \|_p$, in the usual manner for $1 \leqslant p \leqslant \infty$ and let $C_0[0, \infty)$ denote the space of continuous functions that vanish at ∞ with the uniform norm $\| \ \|_\infty$. From the usual representation theorem for the solutions of (1) (e.g., as given in [3, p. 80]) we see that $V_\infty(S)$ is a proper subset of each of the spaces $C_0[0, \infty)$ and $L_p[0, \infty)$, $1 \leqslant p \leqslant \infty$, if and only if S is a subset of the open left half plane

$$L_0 = \{z \in C\colon \operatorname{Re} z < 0\} \tag{4}$$

so that L_0 forms a natural universal spectral set for these spaces. In addition to $V_\infty(L_0)$, the space $L_\infty[0, \infty)$ also contains those exponential sums y from $V_\infty(\bar{L}_0)$ (where $\bar{L}_0$ is the closure of L_0) that satisfy some equation of the form (1) having a characteristic polynomial with no repeated roots along the imaginary axis $\bar{L}_0 \backslash L_0$.

Our problem may now be stated as follows. Given $n = 1, 2, \ldots$, $S \subseteq C$, $1 \leqslant p \leqslant \infty$, and $f \in L_p[0, \infty)$, we would like to find a best $\| \ \|_p$-approximation to f from $V_n(S)$, i.e., we would like to find some $y_0 \in V_n(S)$ such that

$$\| f - y_0 \|_p = \inf\{\| f - y \|_p : y \in V_n(S)\}. \tag{5}$$

The related problem of approximation on a finite interval instead of on a semiinfinite interval stems from the work of Rice [19], and has been studied in some detail, cf. [4, 5–7, 11, 12, 22]. The above also represents a generalization of the problem of best least squares time domain approximation (corresponding to the special case where $p = 2$ and $S = L_0$), which stems from the work of Aigrain and Williams [1] and which is of interest and importance in the field of circuit analysis, cf. [2, 10, 16], and the references cited therein.

We shall infer the existence of good approximations to a given f by showing that $V_\infty(S)$ is a dense subset of $C_0[0, \infty)$ (and thus, of $L_p[0, \infty)$, $1 \leqslant p < \infty$) when S is any nonvoid subset of L_0, and under suitable smoothness and rate of decay hypotheses on f bound the rate at which the distance from f to $V_n(L_0)$ decays to zero as n becomes infinite. For fixed n we shall establish the existance of a best approximation to f from $V_n(S)$ when S satisfies a mild closure hypothesis. We shall characterize such a best approximation with a first-order necessary condition. Finally, we shall show that, in principle, a best approximation to f on $[0, \infty)$ can be obtained from a sequence $\{y_\nu\}$ that is so constructed that y_ν is a best approximation to f on the finite subinterval $[0, \sigma_\nu]$, $\nu = 1, 2,\ldots$ where $\{\sigma_\nu\}$ is an unbounded sequence of positive real numbers.

2. Existence of Good Approximations

Before proving a Weirstrass type density theorem we prepare two lemmas.

Lemma 1. *Let $1 \leqslant p \leqslant \infty$, and for $m = 0, 1, 2,\ldots$ let*

$$e_m(t) = t^m e^{-t}, \qquad t \geqslant 0. \tag{6}$$

Then $\| e_m \|_p \leqslant m!$.

Proof. Using the Binet formula for the gamma function [21, p. 249] it can be shown that

$$\Gamma(1+s) = [2\pi s]^{1/2}\, s^s e^{-s+\varphi(s)}, \qquad s > 0,$$

where $\varphi(s)$ is a positive nonincreasing continuous function of s for $s > 0$. Thus, for $1 \leqslant p < \infty$ and $m = 1, 2,\ldots$ we have

$$\begin{aligned}
[\| e_m \|_p / m!]^p &= [m!]^{-p} \int_0^\infty t^{mp} e^{-pt}\, dt \\
&= [\Gamma(1+m)]^{-p}\, p^{-1-mp}\, \Gamma(1+mp) \\
&= p^{-1/2} \cdot [2\pi m]^{1/2-p/2} \cdot e^{\varphi(mp)-p\varphi(m)} \\
&\leqslant 1,
\end{aligned}$$

so that the lemma holds for these values of m, p. Separate arguments show that it also holds when $p = \infty$ or $m = 0$. ▮

Lemma 2. *Let $\lambda, \delta \in C$, let $\alpha = -\operatorname{Re} \lambda$, and assume that $\alpha > 0$, $\operatorname{Re} \delta \leqslant 0$, and $|\delta| < \alpha$. Let m be a fixed nonnegative integer, and let*

$$\begin{aligned}
y_n(t) &= t^m e^{\lambda t} \cdot \sum_{k=0}^{n-1} (\delta t)^k / k!, \qquad n = 1, 2,\ldots \\
y(t) &= t^m e^{\lambda t} \cdot e^{\delta t}.
\end{aligned}$$

Then $\{y_n\}$ $\| \ \|_p$-converges to y, $1 \leqslant p \leqslant \infty$.

Proof. Using Taylor's formula we have

$$y(t) - y_n(t) = t^m e^{\lambda t}(\delta t)^n \int_0^1 [(1 - \sigma)^{n-1}/(n - 1)!]\, e^{\delta t \sigma}\, d\sigma,$$

and since $\operatorname{Re} \delta = -\alpha$, this yields the pointwise bound

$$| y(t) - y_n(t)| \leqslant t^m e^{-\alpha t} \,|\, \delta t \,|^n/n! = \alpha^{-m} \,|\, \delta/\alpha \,|^n\, e_{n+m}(\alpha t)/n!,$$

where e_{n+m} is again as in (6). In conjunction with the norm bound of Lemma 1, this implies that $\| y - y_n \|_p \to 0$ as $n \to \infty$, provided that $| \delta/\alpha | < 1$. ▮

THEOREM 1. *Let S be a nonvoid subset of the open left half plane L_0. Then $V_\infty(S)$ is dense in each of the spaces $L_p[0, \infty)$, $1 \leqslant p < \infty$, and $C_0[0, \infty)$.*

Proof. Let λ be chosen from S so that $\alpha = -\operatorname{Re} \lambda$ is positive, and let $f \in L_p[0, \infty)$ be given with $f \in C_0[0, \infty)$ if $p = \infty$. We shall show that we may $\| \;\|_p$-approximate f as closely as we please with the exponential sums from $V_\infty(\{\lambda\})$. We define the transform

$$F(s) = (\alpha s)^{-1/p} f(-\alpha^{-1} \log s), \qquad 0 < s \leqslant 1 \tag{7}$$

of f so that $F \in L_p[0, 1]$ with

$$\||\, F \,\||_p = \| f \|_p, \tag{8}$$

where $\||\;\||_p$ denotes the norm in $L_p[0, 1]$. The function F can be $\||\;\||_p$-approximated as closely as we please by some function $G \in C[0, 1]$ for which $G(0) = 0$ (even when $p = \infty$, in which case F itself can be continuously extended to $[0, 1]$ by setting $F(0) = 0$.) Using the Müntz–Szász theorem [8, p. 197] we see that such a function G, and therefore, F can be $\||\;\||_p$-approximated as closely as we please by using a function of the form

$$H(s) = (\alpha s)^{-1/p}\, Q(s), \tag{9}$$

where Q is a polynomial with $Q(0) = 0$. In view of the norm preserving property (8) of the transformation (7) it follows that we may $\| \;\|_p$-approximate f as closely as we please by using an exponential sum of the form

$$h(t) = Q(e^{-\alpha t}) \tag{10}$$

(which transforms into (9)).

To complete the proof we must show that we may $\| \;\|_p$ approximate any such function (10) as closely as we please with an exponential sum from

$V_\infty(\{\lambda\})$, and since $V_\infty(\{\lambda\})$ is a linear space, it is sufficient to show that this may be done for every simple exponential

$$h(t) = e^{\lambda_0 t}, \qquad t \geqslant 0, \tag{11}$$

for which $\lambda_0 \leqslant -\alpha$. When λ_0 is so close to λ that $|\lambda_0 - \lambda| < \alpha$, this is an immediate consequence of Lemma 2. When this is not the case, we define

$$\delta = (\lambda_0 - \lambda)/m,$$

where the positive integer m is chosen so large that $|\delta| < \alpha$, and set

$$\lambda_k = [k\lambda + (m-k)\lambda_0]/m, \qquad k = 0, 1,..., m.$$

This being the case, $|\delta| < |\operatorname{Re}\lambda_k|$ for $k = 0, 1,..., m$, and by using Lemma 2 we see that each element of $V_\infty(\{\lambda_{k-1}\})$ can be $\|\ \|_p$-approximated as closely as we please with elements of $V_\infty(\{\lambda_k\})$, $k = 1, 2,..., m$. It follows that the function $h \in V_\infty(\{\lambda_0\})$ can be $\|\ \|_p$-approximated as closely as we please by using elements of $V_\infty(\{\lambda_m\}) = V_\infty(\{\lambda\})$ so that the proof is complete. ▌

By suitably modifying the admissible polynomials Q allowed in (9) and (10) we obtain the following corollary (which for the case $p = 2$ may be found in [20]).

COROLLARY. *Let* $0 < \lambda_1 < \lambda_2 < \cdots$ *and assume that* $\sum_{\nu=1}^{\infty} 1/\lambda_\nu$ *diverges. Then the set of exponential sums that may be written as finite linear combinations of the functions* $e^{-\lambda_\nu t}$, $\nu = 1, 2,...,$ *is dense in each of the spaces* $L_p[0, \infty)$, $1 \leqslant p < \infty$, *and* $C_0[0, \infty)$.

Note. When $p = 2$, a variety of special identities (e.g., Parseval's identity) may be exploited in showing that any function of the form (11) lies in the closure of $V_\infty(\{\lambda\})$, cf. [9, p. 95–96] or [18, p. 154–155]. Indeed if h is given by (11) and we use

$$\varphi_k(t) = t^{k-1}e^{\lambda t}, \qquad k = 1,..., n,$$

as a basis for $V_n(\{\lambda\})$, then Gram's lemma [8, p. 194] shows that the $\|\ \|_2$-distance from h to $V_m(\{\lambda\})$ is given by

$$d_2[h, V_n(\{\lambda\})] = [G(\varphi_1,..., \varphi_n, h)/G(\varphi_1,..., \varphi_n)]^{1/2},$$

where G denotes the Grammian of its arguments. Arguments analogous to those customarily used in the proof of the Müntz–Szász theorem (cf. [8, p. 195–196]) then can be used to simplify this expression with the final result being

$$d_2[h, V_n(\{\lambda\})] = |2\operatorname{Re}\lambda|^{-1/2} \cdot |(\lambda - \lambda_0)/\lambda + \lambda_0)|^n. \tag{12}$$

In addition to forming a basis for yet another proof of Theorem 1 in the special case where $p = 2$, (12) provides a convincing illustration of the bad conditioning that is inherent in the exponential sum approximation problem when n is large (e.g., in view of (12) a term e^{-t} in an exponential sum y can be replaced by a suitable element from $V_n(\{-2\})$ without changing the $\| \|_2$-norm of y by more than $3^{-n}/2$). ▮

Application. As an interesting application of Theorem 1 we shall infer the existence of a solution the following circuit synthesis problem. Suppose we are given an arbitrary function $f(t)$, $t \geqslant 0$, (which is taken from $L_p[0, \infty)$ if $1 \leqslant p < \infty$ and from $C_0[0, \infty)$ if $p = \infty$), and a very unusual "kit" consisting of infinitely many identical resistors $R, R,\ldots$; capacitors $C, C,\ldots$; dry cells $V, V,\ldots$; and a single switch. The problem is to use these elementary components to build a circuit having a voltage transient response $V(t)$, $t \geqslant 0$, which $\| \|_p$-approximates f to within some prescribed tolerance $\epsilon > 0$. To see how our problem might be solved, we first examine the circuit of Fig. 1, which has the voltage transient response

$$V(t) = \frac{R_1 V_0}{R_0 + R_1 + R_2} e^{-(t/R_1C_1)} + \frac{R_2 V_0}{R_0 + R_1 + R_2} e^{-(t/R_2C_2)} - \frac{R_1' V_0}{R_0' + R_1' + R_2'} e^{-(t/R_1'C_1')} - \frac{R_2' V_0}{R_0' + R_1' + R_2'} e^{-(t/R_2'C_2')}, \tag{13}$$

cf. [15, p. 30–34]. By connecting our cells in series, we can arrange for V_0 to be any integral multiple of the basic cell potential and we can arrange for the R_i, R_i' and C_i, C_i' to take arbitrary positive rational multiples of R and C, respectively, by using suitable series and parallel connections of the given resistors and capacitors. Analogous considerations hold when the circuit of Fig. 1 is extended by the insertion of additional R_i, C_i and R_i',

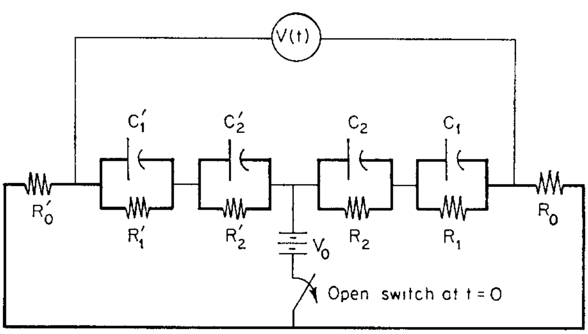

FIG. 1. A circuit for which the voltage transient response $V(t)$ is the exponential sum (13).

C_i' loops, and thus, we see that we can use our kit to build a circuit having any voltage transient response of the form

$$V(t) = A_1 e^{-\alpha_1 t} + \cdots + A_n e^{-\alpha_k t},$$

where each A_i is a rational multiple of V and each α_i is a positive rational multiple of $(RC)^{-1}$. Since the set of all such exponential sums is dense in $L_p[0,\infty)$, $1 \leqslant p < \infty$, and in $C_0[0,\infty)$, it is clear that an appropriate circuit can be constructed from the elements of the given kit. ▮

Theorem 1 provides no measure of the rate at which the $\| \, \|_p$-distance $d_p[f, V_n(S)]$ from f to $V_n(S)$ approaches zero as n becomes infinite. By imposing suitable smoothness and rate-of-decay conditions on f, we obtain the following "Jackson" and "Bernstein" type estimates.

THEOREM 2. *Let m be a positive integer, let $f \in C^m[0, \infty)$, and assume that f is expressible in the form*

$$f(t) = e^{-\alpha t} \varphi(e^{-\alpha t}), \qquad t \geqslant 0, \tag{14}$$

where $\alpha > 0$ and $\varphi \in C^m[0, 1]$. Then there is some constant C (depending only on f and m) such that

$$d_p[f, V_n(L_0)] \leqslant C \cdot n^{-m}, \qquad 1 \leqslant p \leqslant \infty, n = 1, 2, \ldots. \tag{15}$$

Moreover, if the function $\varphi(s)$ in (14) can be chosen to be analytic for $0 \leqslant s \leqslant 1$, then

$$d_p[f, V_n(L_0)] \leqslant A \cdot q^n, \qquad 1 \leqslant p \leqslant \infty, n = 1, 2, \ldots, \tag{16}$$

where $A > 0$ and $0 < q < 1$ are suitable constants (depending only on f and m).

Proof. For $1 \leqslant p \leqslant \infty$ let

$$\begin{aligned} F_p(s) &= 0, \qquad \text{if} \quad s = 0 \\ &= (\alpha s)^{-1/p} \cdot s \cdot \varphi(s), \qquad \text{if} \quad 0 < s \leqslant 1, \end{aligned} \tag{17}$$

so that F_p is the transform (7) of f. For $n = 1, 2, \ldots$, let Q_n be the unique polynomial of degree $n - 1$ or less that best approximates φ in the uniform norm, $|\!|\!| \, |\!|\!|_\infty$, on $[0, 1]$, and let

$$g_n(t) = e^{-\alpha t} Q_n(e^{-\alpha t}), \qquad t \geqslant 0,$$

so that g_n is an exponential sum from $V_n(L_0)$ with the corresponding transform (7) given by

$$G_{n,p}(s) = (\alpha s)^{-1/p} \cdot s \cdot Q_n(s). \tag{18}$$

In view of the norm preserving property (8) of the transformation (7) and the identities (17) and (18) we see that

$$\begin{aligned} \| f - g_n \|_p &= ||| F_p - G_{n,p} |||_p \\ &\leqslant ||| F_p - G_{n,p} |||_\infty \\ &\leqslant \alpha^{-1/p} ||| \varphi - Q_n |||_\infty \\ &\leqslant (1 + \alpha^{-1}) ||| \varphi - Q_n |||_\infty . \end{aligned} \tag{19}$$

By using (19) in conjuction with Jackson's theorem [17, p. 89] and Bernstein's theorem [17, p. 183], we obtain the asymptotic estimates (15) and (16) in the respective cases where $\varphi \in C^m[0, 1]$ and where φ is analytic on $[0, 1]$. ▮

Note. In the formulation and proof of the above theorem the spectral set L_0 could be replaced by the set of negative real numbers. More generally, L_0 could be replaced by the left ray $R_\alpha = \{\theta\alpha : \theta > 0\}$ with $\alpha \in L_0$ being used in the hypothesis (14). ▮

Note. The hypothesis: f is expressible in the form (14), or equivalently, that for a suitable choice of $\alpha > 0$ the function

$$\varphi(s) = f(-\alpha^{-1} \log s)/s, \qquad 0 < s \leqslant 1, \tag{20}$$

can be extended to a function $\varphi \in C^m[0, 1]$, may be replaced by the somewhat simpler hypothesis: f and its first m derivatives decay so rapidly that for some $\beta > 0$

$$f^{(k)}(t) = o(e^{-\beta t}), \qquad k = 0, 1,..., m \tag{21}$$

as $t \to \infty$. From (21) it follows that the function φ of (20) lies in $C^m[0, 1]$ with $\varphi(0) = \varphi'(0) = \cdots = \varphi^{(m)}(0) = 0$. ▮

3. Existence of Best Approximations

We will find it convenient to relate the exponential sum approximation problem in $L_p[0, \infty)$ to that in $L_p[0, \sigma]$ when $\sigma > 0$ is large (but finite). In so doing we make use of the seminorm $\| \ \|_{p,\sigma}$, which we define on $L_p[0, \infty)$ in such a manner that

$$\| f \|_{p,\sigma} = \| f\chi_\sigma \|_p \tag{22}$$

where

$$\begin{aligned} \chi_\sigma(t) &= 1, \qquad \text{if} \quad 0 \leqslant t \leqslant \sigma \\ &= 0, \qquad \text{if} \quad t > \sigma, \end{aligned}$$

is the characteristic function of $[0, \sigma]$. In proving our basic existence theorem we shall need the following result (cf. [4, p. 164]), which is given in [11, Theorem 1 and Lemma 2].

LEMMA 3. *Let* $1 \leqslant p \leqslant \infty$, *let* $0 < \sigma < \infty$, *and let* $\{y_\nu\}$ *be any* $\| \|_{p,\sigma}$-*bounded sequence of exponential sums from* $V_n(C)$. *Then there is a compact set* $K \subset C$ *and a decomposition*

$$y_\nu = v_\nu + x_\nu, \qquad \nu = 1, 2,..., \tag{23}$$

of a suitable subsequence of $\{y_\nu\}$ *(which we continue to denote by* $\{y_\nu\}$*) such that*:

(i) $v_\nu \in V_n(K)$ *for* $\nu = 1, 2,...$,

(ii) $\{v_\nu\}$ $\| \|_{p,\sigma}$-*converges to some exponential sum* $v \in V_n(K)$,

(iii) *only finitely many nonzero terms of the sequence* $\{x_\nu\}$ *lie in any set* $V_{2n}(S)$ *when* $S \subset C$ *is compact, and*

(iv) $\{x_\nu\}$ *is ultimately* $\| \|_{p,\sigma}$-*orthogonal to every* $f \in L_p[0, \infty)$ *in the sense that the inequality*

$$\liminf \|f - x_\nu\|_{p,\sigma} \geqslant \|f\|_{p,\sigma} \tag{24}$$

holds for all such f. ▮

Note. In the case where $p < \infty$, it can be shown that Lemma 3 remains valid when $\sigma = \infty$ provided we replace the universal spectral set C with the universal spectral set L_0 (with the proof being based upon the above version of Lemma 3, the corollary to Theorem 3, and Lemma 5.) When $p = \infty$, Lemma 3 has no such extension, e.g., the $\| \|_\infty$-bounded sequence

$$y_\nu(t) = e^{-t/\nu} \cos(t/\nu), \qquad \nu = 1, 2,...$$

from $C_0[0, \infty)$ has a decomposition (23) satisfying conditions (i), (ii), (iii) (with C replaced by L_0) only when $v_\nu = 0$ and $x_\nu = y_\nu$ for all but finitely many values of ν, in which case (24) (with $\sigma = \infty$) fails to hold for the function $f = 2\chi_1$, where χ_1 is the characteristic function of $[0, 1]$. ▮

THEOREM 3. *Let* $S \subset C$, *let* $1 \leqslant p \leqslant \infty$, *let* n *be a positive integer, and let* L *denote the open left half plane* L_0 *if* $p < \infty$ *and the closed left half plane* $\bar{L}_0$ *if* $p = \infty$. *Then every* $f \in L_p[0, \infty)$ *has a best* $\| \|_p$-*approximation from* $V_n(S)$ *if and only if* $S \cap L$ *is closed in* L.

Proof. Let $f \in L_p[0, \infty)$ be chosen and let the minimizing sequence $\{y_\nu\}$ be selected from $V_n(S)$ in such a manner that

$$\lim \|f - y_\nu\|_p = \inf\{\|f - y\|_p : y \in V_n(S)\}. \tag{25}$$

Such a sequence $\{y_\nu\}$ is $\| \ \|_p$ bounded, and therefore, $\| \ \|_{p,\sigma}$ bounded whenever $0 < \sigma < \infty$. This being the case we may effect the decomposition (23) of Lemma 3 and after passing to a subsequence, if necessary, assume that $\{v_\nu\}$ $\| \ \|_{p,\sigma}$-converges to some $v \in V_n(K)$, where K is a compact subset of $\bar{S}$. Together with (24) and (25) this shows that

$$\begin{aligned} \|f - v\|_{p,\sigma} &\leqslant \liminf \|f - v - x_\nu\|_{p,\sigma} \\ &= \liminf \|f - v_\nu - x_\nu\|_{p,\sigma} \\ &\leqslant \lim \|f - y_\nu\|_p \\ &= \inf\{\|f - y\|_p : y \in V_n(S)\}, \end{aligned}$$

holds for a fixed positive σ, and since the resulting limit v is independent of σ we have

$$\|f - v\|_p \leqslant \inf\{\|f - y\|_p : y \in V_n(S)\}. \tag{26}$$

From (26) we see that v is $\| \ \|_p$-bounded so that $v \in V_n(L)$. This being the case, if $S \cap L$ is closed in L we have

$$v \in V_n(\bar{S}) \cap V_n(L) = V_n(\bar{S} \cap L) = V_n(S \cap L) \subseteq V_n(S),$$

which together with (26) shows that v is a best $\| \ \|_p$-approximation to f from $V_n(S)$.

Conversely, suppose that $S \cap L$ is not closed in L so that there is some sequence $\{\lambda_\nu\}$ from $S \cap L$ that converges to a point $\lambda \in (\bar{S} \backslash S) \cap L$. We shall set

$$y_\nu(t) = e^{\lambda_\nu t}, \qquad t \geqslant 0, \quad \nu = 1, 2,... \tag{27}$$

and show that $\{y_\nu\}$ is a $\| \ \|_p$-minimizing sequence for some function in $L_p[0, \infty)$ that has no best $\| \ \|_p$-approximation in $V_n(S)$. In the case where $\operatorname{Re} \lambda < 0$, we need only set

$$y_\infty(t) = e^{\lambda t}, \qquad t \geqslant 0 \tag{28}$$

and note that $\lim \|y_\infty - y_\nu\|_p = 0$, while $\|y_\infty - y\|_p > 0$ holds for each $y \in V_n(S)$, i.e., there is no best $\| \ \|_p$-approximation for y_∞ in $V_n(S)$. In the remaining case, where $\operatorname{Re} \lambda = 0$ and $p = \infty$, we set

$$f(t) = e^{\lambda t}\{1 - \operatorname{sgn}[\sin(\pi/t)]\}, \qquad t > 0. \tag{29}$$

By construction, $f \in L_\infty[0, \infty)$ and $\|f - y\|_\infty \geqslant 1$ holds for every function $y \in C[0, \infty)$ with equality only if

$$y(1/m) = e^{\lambda/m}, \qquad m = 1, 2,.... \tag{30}$$

In particular, (30) holds for an exponential sum y only if y is the function y_∞ of (28) (since two entire functions that agree on a bounded infinite point set must be identical) so that $\| f - y \|_\infty > 1$ whenever $y \in V_n(S)$. Finally, using (27) and (29) we see that

$$| f(t) - y_\nu(t)| = | y_\nu(t)| \leqslant 1, \qquad \text{when} \quad t > 1$$

and

$$\begin{aligned} |f(t) - y_\nu(t)| &\leqslant |f(t) - e^{\lambda t}| + | e^{\lambda t} - e^{\lambda_\nu t} | \\ &\leqslant 1 + O(| \lambda - \lambda_\nu |), \qquad \text{when} \quad 0 < t \leqslant 1, \end{aligned}$$

so that $\{y_\nu\}$ is a minimizing sequence from $V_n(S)$ with $\lim \| f - y_\nu \|_\infty = 1$, i.e., there is no best $\| \ \|_p$-approximation for f in $V_n(S)$. ∎

COROLLARY. *Let p, n, L be as in the theorem, and let K, S be disjoint subsets of L with K being compact and S being closed. Then there is a constant $\delta > 0$ such that the inequality*

$$\| v - x \|_p \geqslant \delta \| v \|_p \tag{31}$$

holds whenever $v \in V_n(K)$ and $x \in V_n(S)$.

Proof. For each nonzero $v \in V_n(K)$ we determine the largest constant $\delta(v)$ for which (31) holds when x is a best $\| \ \|_p$-approximation to v from $V_n(S)$. By taking the minimum of all such constants, $\delta(v)$, as v ranges over the compact set of $\| \ \|_p$-normalized exponential sums from $V_n(K)$ we obtain the desired constant δ of the corollary. ∎

4. A First-Order Necessary Condition for a Best Approximation

When $y = Y_n(\mathbf{b}, \mathbf{c}, -)$ is a best $\| \ \|_p$-approximation to a given $f \in L_p[0, \infty)$ from $V_n(S)$, the inequality

$$\| f - Y_n(\mathbf{b}, \mathbf{c}, -) \|_p \leqslant \| f - Y_n(\mathbf{b}', \mathbf{c}', -) \|_p \tag{32}$$

must hold whenever $\Lambda_n[\mathbf{c}'] \subset S$. Following the same basic development given in [12] (for the simpler case where the interval of approximation is compact) we combine (32) with an analysis of the first-order effect of the perturbation $\mathbf{b}' - \mathbf{b}$, $\mathbf{c}' - \mathbf{c}$, and thereby, obtain a necessary condition that serves to characterize a best (or local best) approximation. In so doing we shall first prepare four lemmas.

LEMMA 4. *Let $\delta > 0$ and let L_δ be the half plane $\{z \in C: \operatorname{Re} z < -\delta\}$. For each $n = 0, 1, \dots$ there is a constant $M_n(\delta)$ such that the pointwise bound*

$$|y(t)| \leqslant \|y\|_\infty \cdot M_n(\delta) \cdot e^{-\delta t/2}, \qquad t \geqslant 0 \tag{33}$$

holds for every $y \in V_n(L_\delta)$.

Proof. From [13, Theorem 1] we see that there is some constant $\tau_{n\delta} > 0$ such that

$$\|y^*\|_\infty = \max\{|y^*(s)| : 0 \leqslant s \leqslant \tau_{n\delta}\} \qquad \text{when} \quad y^* \in V_n(L_{\delta/2}).$$

In particular, if we choose $y \in V_n(L_\delta)$ and take $y^*(t) = y(t)\, e^{\delta t/2}$, this identity gives

$$|y(t)\, e^{\delta t/2}| \leqslant \max\{|y(s)|\, e^{\delta s/2}: 0 \leqslant s \leqslant \tau_{n\delta}\} \leqslant \|y\|_\infty \cdot e^{\delta\tau_{n\delta}/2}, \qquad t \geqslant 0,$$

so that (33) holds with $M_n(\delta) = e^{\delta\tau_{n\delta}/2}$. ▮

When $V_n(C)$ is equipped with the $\|\ \|_{p,\sigma}$ norm, the mapping $\mathbf{b}, \mathbf{c} \to Y_n(\mathbf{b}, \mathbf{c}, -)$ is clearly Frechet differentiable, $1 \leqslant p \leqslant \infty$, $0 < \sigma < \infty$. Among other things, the next lemma shows that the mapping remains Frechet differentiable even when we set $\sigma = \infty$, provided that the parameters $\mathbf{b}, \mathbf{c}$ remain in some suitably restricted neighborhood of a point $\mathbf{b}_o$, $\mathbf{c}_o$, for which $\Lambda_n[\mathbf{c}_o] \subset L_0$. (The necessity for this restriction is simply illustrated by means of the exponential sum

$$Y_1(\alpha, \alpha, t) = \alpha e^{-\alpha t},$$

which for $\alpha < 0$, $\alpha = 0$, and $\alpha > 0$ has the $\|\ \|_1$-norm ∞, 0, and 1, respectively.)

LEMMA 5. *Let K be a compact subset of L_0. Then $\|\ \|_{p,\sigma}$, $1 \leqslant p \leqslant \infty$, $1 \leqslant \sigma \leqslant \infty$, are uniformly equivalent norms on $V_n(K)$.*

Proof. We establish the lemma by inferring the existence of positive constants m, M (depending only on n, K) such that the inequalities

$$m\|v\|_\infty \leqslant \|v\|_{p,\sigma} \leqslant M\|v\|_\infty \tag{34}$$

hold whenever $1 \leqslant p \leqslant \infty$, $\sigma \geqslant 1$, and $v \in V_n(K)$. We first choose $\delta \in (0, 2]$ so small that the translated set $K + \delta$ lies in L_0 so that the pointwise bound (33) holds for all $v \in V_n(K)$. It follows that

$$\|v\|_{p,\sigma} \leqslant (2M_n(\delta)/\delta) \cdot \|v\|_\infty,$$

whenever $1 \leqslant p \leqslant \infty$, $\sigma \geqslant 1$, and $v \in V_n(K)$, so that the right inequality of (34) holds with $M = 2M_n(\delta)/\delta$. Again using (33) we see that

$$\| v \|_\infty = \| v \|_{\infty,\tau},$$

whenever $\tau = 2\log[M_n(\delta)]/\delta$, so that $\| \ \|_{\infty,\tau}$ and $\| \ \|_\infty$ are equivalent on $V_n(K)$. From [11, Lemma 1] we also see that $\| \ \|_{\infty,\tau}$ and $\| \ \|_{1,1}$ are equivalent on $V_n(K)$. It follows that there is some $m > 0$ such that the left inequality of (34) holds for all $v \in V_n(K)$ in the extreme case where $\sigma = 1$ and $p = 1$, and therefore, in the general case where $1 \leqslant p \leqslant \infty$ and $\sigma \geqslant 1$ as well. ▌

In the process of characterizing a best $\| \ \|_p$-approximation, we shall make use of the fact that the norm functional in $L_p[0, \infty)$ has a Gateaux variation when $1 < p < \infty$ and a one sided Gateaux variation when $p = 1$ or $p = \infty$, with the explicit representation given in the following lemma.

LEMMA 6. *Let $1 \leqslant p \leqslant \infty$ and let $\epsilon, h \in L_p[0, \infty)$. Then*

$$\| \epsilon + \alpha h \|_p = \| \epsilon \|_p + \alpha \cdot \Phi_p[\epsilon, h] + o(\alpha) \tag{35}$$

as α decreases to zero through positive values where

$$\begin{aligned} \Phi_p[\epsilon, h] &= \int_0^\infty T[\epsilon, h; t]\, dt, \qquad \text{if } p = 1 \\ &= \| \epsilon \|_p^{1-p} \int_0^\infty | \epsilon(t)|^{p-1}\, T[\epsilon, h; t]\, dt, \qquad \text{if } 1 < p < \infty \\ &= \lim_{\delta \to 0+} \operatorname{ess\,sup}\{T[\epsilon, h; t] : t \geqslant 0 \quad \text{and} \quad |\epsilon(t)| \geqslant \| \epsilon \|_\infty - \delta\}, \\ &\qquad\qquad \text{if } p = \infty, \end{aligned} \tag{36}$$

when $\| \epsilon \|_p > 0$, and $\Phi_p[\epsilon, h] = \| h \|_p$ when $\| \epsilon \|_p = 0$, and where

$$\begin{aligned} T[\epsilon, h; t] &= |h(t)|, \qquad \text{if } \epsilon(t) = 0 \\ &= \operatorname{Re}[h(t)\, \overline{\epsilon(t)}/|\epsilon(t)|], \qquad \text{if } \epsilon(t) \neq 0 \end{aligned} \tag{37}$$

(*with the bar denoting the complex conjugate*).

Proof. Replace the finite interval [0, 1] by the semiinfinite interval $[0, \infty)$ in the proof of [12, Lemma 3]. ▌

Given **b**, **c**, **b***, **c*** from C^n we define

$$h_n(\mathbf{b}, \mathbf{c}, \mathbf{b}^*, \mathbf{c}^*, t) = \sum_{i=1}^n \left\{ b_i^* \frac{\partial}{\partial b_i} + c_i^* \frac{\partial}{\partial c_i} \right\} Y_n(\mathbf{b}, \mathbf{c}, t), \qquad t \geqslant 0, \tag{38}$$

and

$$H_n(\mathbf{b}, \mathbf{c}) = \{h_n(\mathbf{b}, \mathbf{c}, \mathbf{b}^*, \mathbf{c}^*, -) : \mathbf{b}^*, \mathbf{c}^* \in C^n\}. \tag{39}$$

Clearly, $H_n(\mathbf{b}, \mathbf{c})$ is a linear space that contains the n-dimensional space

$$L_n(\mathbf{c}) = \{h_n(\mathbf{b}, \mathbf{c}, \mathbf{b}^*, \mathbf{0}, -): \mathbf{b}^* \in C^n\} \tag{40}$$

of solutions of (1). We define

$$K_n(\mathbf{b}, \mathbf{c}, \mathbf{c}^*) = \{h_n(\mathbf{b}, \mathbf{c}, \mathbf{b}^*, \alpha\mathbf{c}^*, -): \mathbf{b}^* \in C^n, \alpha \geqslant 0\} \tag{41}$$

and refer to this set as the perturbation cone of y (with respect to the parametrization $\mathbf{b}, \mathbf{c}$) in the direction of $\mathbf{c}^*$. We say that y is accessible through this cone with respect to $V_n(S)$ provided there is a differentiable arc $\mathbf{z}:[0, 1] \to C^n$ such that

$$\mathbf{z}(\alpha) = \mathbf{c} + \alpha\mathbf{c}^* + \mathbf{o}(\alpha), \qquad \text{as} \quad \alpha \to 0+ \tag{42}$$

and such that

$$\Lambda_n[\mathbf{z}(\alpha)] \subseteq S, \qquad \text{for} \quad 0 \leqslant \alpha \leqslant 1. \tag{43}$$

(An extended discussion of these concepts is given in [12, p. 177–180].)

Suppose now that the exponential sum $Y_n(\mathbf{b}, \mathbf{c}, -)$ can be decomposed in the form

$$Y_n(\mathbf{b}, \mathbf{c}, -) = Y_{n_1}(\mathbf{b}_1, \mathbf{c}_1, -) + Y_{n_2}(\mathbf{b}_2, \mathbf{c}_2, -), \tag{44}$$

where n_1, n_2 are positive integers with sum n, and where $\mathbf{c}_1$, $\mathbf{c}_2$ are related to $\mathbf{c}$ through the factorization

$$P_n[\mathbf{c}, -] = P_{n_1}[\mathbf{c}_1, -] \cdot P_{n_2}[\mathbf{c}_2, -]$$

of the corresponding characteristic polynomial. When $Y_n(\mathbf{b}, \mathbf{c}, -)$ is accessible through the cone $K_n(\mathbf{b}, \mathbf{c}, \mathbf{c}^*)$ with respect to $V_n(S)$ and the spectral sets $\Lambda_{n_i}[\mathbf{c}_i]$, $i = 1, 2$, are disjoint, then the components $Y_{n_i}(\mathbf{b}_i, \mathbf{c}_i, -)$, $i = 1, 2$, are accessible through the corresponding cones $K_{n_i}(\mathbf{b}_i, \mathbf{c}_i, \mathbf{c}_i^*)$, $i = 1, 2$, where

$$P_n[\mathbf{c} + \alpha\mathbf{c}^*, -] = P_{n_1}[\mathbf{c}_1 + \alpha\mathbf{c}_1^*, -] \cdot P_{n_2}[\mathbf{c}_2 + \alpha\mathbf{c}_2^*, -] + o(\alpha), \tag{45}$$

as we see by using considerations of continuity together with the following lemma.

LEMMA 7. *Let the arc* $\mathbf{z}$: $[0, 1] \to C^n$ *be differentiable, let* n_1, n_2 *be positive integers with* $n_1 + n_2 = n$, *let the arcs* $\mathbf{z}_i : [0, 1] \to C^{n_i}$, $i = 1, 2$, *be continuous, and assume that*

$$P_n[\mathbf{z}(\alpha), -] = P_{n_1}[\mathbf{z}_1(\alpha), -] \cdot P_{n_2}[\mathbf{z}_2(\alpha), -], \qquad 0 \leqslant \alpha \leqslant 1, \tag{46}$$

and

$$\Lambda_{n_1}[\mathbf{z}_1(\alpha)] \cap \Lambda_{n_2}[\mathbf{z}_2(\alpha)] = \varnothing, \qquad 0 \leqslant \alpha \leqslant 1. \tag{47}$$

Then the induced arcs $\mathbf{z}_1(\alpha)$, $\mathbf{z}_2(\alpha)$ *are differentiable for* $0 \leqslant \alpha \leqslant 1$.

Proof. We first show that the arcs $\mathbf{z}_1(\alpha)$, $\mathbf{z}_2(\alpha)$ are differentiable at $\alpha = 0$. By hypothesis, $\mathbf{z}(\alpha)$ is differentiable at $\alpha = 0$, so that (42) holds with $\mathbf{c} = \mathbf{z}(0)$, $\mathbf{c}^* = \mathbf{z}'(0)$. Since $\mathbf{z}_i(\alpha)$ is continuous we can also write

$$\mathbf{z}_i(\alpha) = \mathbf{c}_i + \boldsymbol{\beta}_i(\alpha), \tag{48}$$

where $\mathbf{c}_i = \mathbf{z}_i(0)$, and where $\boldsymbol{\beta}_i : [0, 1] \to C^{n_i}$ is a continuous arc with $\boldsymbol{\beta}_i(0) = 0$, $i = 1, 2$. We must show that each of the ratios $\boldsymbol{\beta}_i(\alpha)/\alpha$ has a finite limit as $\alpha \to 0+$.

For convenience in notation we define

$$Q_n[\mathbf{c}, \lambda] = P_n[\mathbf{c}, \lambda] - \lambda^n = c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n,$$

so that Q_n is a polynomial of degree $n - 1$ or less in λ that depends linearly on the coefficients $\mathbf{c}$. When λ is a root of $P_{n_1}[\mathbf{c}_1, -]$ (and thus, by (46) a root of $P_n[\mathbf{c}, -]$ but by (47) not a root of $P_{n_2}[\mathbf{c}_2, -]$) we see that

$$\begin{aligned}
\alpha Q_n[\mathbf{c}^*, \lambda] &= P_n[\mathbf{c}, \lambda] + \alpha Q_n[\mathbf{c}^*, \lambda] \\
&= P_n[\mathbf{z}(\alpha), \lambda] + o(\alpha) \\
&= P_{n_1}[\mathbf{z}_1(\alpha), \lambda]\, P_{n_2}[\mathbf{z}_2(\alpha), \lambda] + o(\alpha) \\
&= \{P_{n_1}[\mathbf{c}_1, \lambda] + Q_{n_1}[\boldsymbol{\beta}_1(\alpha), \lambda]\}\{P_{n_2}[\mathbf{c}_2, \lambda] + Q_{n_2}[\boldsymbol{\beta}_2(\alpha), \lambda]\} + o(\alpha) \\
&= Q_{n_1}[\boldsymbol{\beta}_1(\alpha), \lambda] \cdot \{P_{n_2}[\mathbf{c}_2, \lambda] + o(1)\} + o(\alpha),
\end{aligned}$$

and thus, that

$$Q_n[\mathbf{c}^*, \lambda] = Q_{n_1}[\boldsymbol{\beta}_1(\alpha)/\alpha, \lambda] \cdot \{P_{n_2}[\mathbf{c}_2, \lambda] + o(1)\} + o(1),$$

as $\alpha \to 0+$. By using essentially the same argument together with the Leibnitz rule for differentiating a product, we see that if $P_{n_1}[\mathbf{c}_1, -]$ has distinct roots $\lambda_1, \ldots, \lambda_l$ with multiplicities $m_1, \ldots, m_l$, respectively, then, as $\alpha \to 0+$, we have

$$Q_n^{(m)}[\mathbf{c}^*, \lambda_i] = \sum_{j=0}^{m} \binom{m}{j} Q_{n_1}^{(j)}[\boldsymbol{\beta}_1(\alpha)/\alpha, \lambda_i]\{P_{n_2}^{(m-j)}[\mathbf{c}_2, \lambda_i] + o(1)\} + o(1),$$

for $0 \leqslant m < m_i$ and $1 \leqslant i \leqslant l$. Since $P_{n_2}[\mathbf{c}_2, \lambda_i] \neq 0$, it follows that $Q_{n_1}^{(j)}[\boldsymbol{\beta}_1(\alpha)/\alpha, \lambda_i]$ has a finite limit as $\alpha \to 0+$ for $0 \leqslant j \leqslant m_i$ and $1 \leqslant i \leqslant l$, and since the expression

$$\max\{|\, Q^{(j)}(\lambda_i)| : 0 \leqslant j \leqslant m_i,\ 1 \leqslant i \leqslant l\}$$

defines a norm on the space of all polynomials Q of degree n_1 or less, it follows that $\boldsymbol{\beta}_1(\alpha)/\alpha$ has a finite limit as $\alpha \to 0+$. Thus, $\mathbf{z}_1(\alpha)$ and analogously, $\mathbf{z}_2(\alpha)$ is differentiable at $\alpha = 0$.

The same argument can be used at an arbitrary point $\alpha \in [0, 1]$ so that the proof is complete. ∎

THEOREM 4. *Let $f \in L_p[0, \infty)$, and let the exponential sum $y_0 = Y_n(\mathbf{b}, \mathbf{c}, -)$ be accessible through the perturbation cone $K_n(\mathbf{b}, \mathbf{c}, \mathbf{c}^*)$ with respect to $V_n(S)$. Then a necessary condition for y_0 to be a best (or local best) $\| \ \|_p$-approximation to f from $V_n(S)$ is that y_0 be a best $\| \ \|_p$-approximation to f from $K_n(\mathbf{b}, \mathbf{c}, \mathbf{c}^*)$ so that*

$$\Phi_p[f - y_0, -h] \geqslant 0 \tag{49}$$

whenever $h \in K_n(\mathbf{b}, \mathbf{c}, \mathbf{c}^*)$.

Proof. We shall assume that $y_0 = Y_n(\mathbf{b}, \mathbf{c}, -)$ is a best (or local best) $\| \ \|_p$-approximation to f from $V_n(S)$ and that the differentiable arc $\mathbf{z}\colon [0, 1] \to C^n$ satisfies (42) and (43). Let $h \in K_n(\mathbf{b}, \mathbf{c}, \mathbf{c}^*)$ be selected. Since $y_0 = h_n(\mathbf{b}, \mathbf{c}, \mathbf{b}, \mathbf{0}, -)$, it follows that $h - y_0 \in K_n(\mathbf{b}, \mathbf{c}, \mathbf{c}^*)$ so that for some choice of $\mathbf{b}^* \in C^n$ and $\alpha_0 > 0$ we have

$$h = h_n(\mathbf{b}, \mathbf{c}, \mathbf{b}^*, \alpha_0\mathbf{c}^*, -) + y_0 . \tag{50}$$

By rescaling $\mathbf{c}^*$, if necessary, we arrange things so that $\alpha_0 = 1$. We must show that $\|f - h\|_p \geqslant \|f - y_0\|_p$, and in so doing we will assume that $h \in L_p[0, \infty)$.

When $\Lambda_n[\mathbf{c}] \subset L_0$ (so that $H_n(\mathbf{b}, \mathbf{c})$ is contained in $L_p[0, \infty)$) we obtain the estimate

$$\| Y_n(\mathbf{b} + \alpha\mathbf{b}^*, \mathbf{z}(\alpha), -) - Y_n(\mathbf{b}, \mathbf{c}, -) - \alpha h_n(\mathbf{b}, \mathbf{c}, \mathbf{b}^*, \mathbf{c}^*, -)\|_p = \mathbf{o}(\alpha) \tag{51}$$

by using Lemma 5 together with the "compact" version of (51), which results when the norm $\| \ \|_p$ is replaced by the seminorm $\| \ \|_{p,1}$. On the other hand, when $\Lambda_n[\mathbf{c}] \not\subset L_0$ we decompose y_0 in the form

$$y_0 = Y_{n_1}(\mathbf{b}_1, \mathbf{c}_1, -) + Y_{n_2}(\mathbf{b}_2, \mathbf{c}_2, -)$$

of (44), where $\Lambda_{n_1}[\mathbf{c}_1] \subset L_0$ but $\Lambda_{n_2}[\mathbf{c}_2] \subset C\backslash L_0$. We simultaneously decompose h in the form $h = h_1 + h_2$, where

$$h_i = h_{n_i}(\mathbf{b}_i, \mathbf{c}_i, \mathbf{b}_i^*, \mathbf{c}_i^*, -) \in V_{2n_i}(\Lambda_{n_i}[\mathbf{c}_i]), \qquad i = 1, 2,$$

and where $\mathbf{c}_1, \mathbf{c}_1^*, \mathbf{c}_2, \mathbf{c}_2^*, \mathbf{c}, \mathbf{c}^*$ are related by (45). A factorization of the form (46)–(47) then can be effected with $\Lambda_{n_i}[\mathbf{z}_i(\alpha)] \subseteq S$ holding for all sufficiently small $\alpha > 0$, and with

$$\mathbf{z}_i(\alpha) = \mathbf{c}_i + \alpha\mathbf{c}_i^* + \mathbf{o}(\alpha)$$

for $i = 1, 2$. By again using Lemma 5 we find that as $\alpha \to 0+$

$$\| Y_{n_1}(\mathbf{b}_1 + \alpha\mathbf{b}_1^*, \mathbf{z}_1(\alpha), -) - Y_{n_1}(\mathbf{b}_1, \mathbf{c}_1, -) - \alpha h_{n_1}(\mathbf{b}_1, \mathbf{c}_1, \mathbf{b}_1^*, \mathbf{c}_1^*, -)\|_p = o(\alpha),$$

since $\Lambda_{n_1}[\mathbf{c}_1] \subset L_0$. By assumption $h \in L_p[0, \infty)$ and since $h_1 \in V_{2n_1}(\Lambda_{n_1}[\mathbf{c}_1]) \subset L_p[0, \infty)$ we see that $h_2 = h - h_1$ is also in $L_p[0, \infty)$. Since $h_2 \in V_{2n_2}(C \backslash L_0)$ the requirement that $h_2 \in L_p[0, \infty)$ forces h_2 to vanish identically except in the case where $p = \infty$ when h_2 may be a linear combination of simple exponentials having exponential parameters on the imaginary axis. In any event it follows that $\mathbf{c}_2^* = \mathbf{0}$ so that $h_2 = h_{n_2}(\mathbf{b}_2, \mathbf{c}_2, \mathbf{b}_2^*, \mathbf{0}, -) = Y_{n_2}(\mathbf{b}_2^*, \mathbf{c}_2, -)$ and with no loss of generality we may assume at this point that $\mathbf{z}_2(\alpha) \equiv \mathbf{c}_2$. This being the case

$$\begin{aligned} &\| Y_{n_1}(\mathbf{b}_1 + \alpha\mathbf{b}_1^*, \mathbf{z}_1(\alpha), -) + Y_{n_2}(\mathbf{b}_2 + \alpha\mathbf{b}_2^*, \mathbf{c}_2, -) \\ &\qquad - Y_n(\mathbf{b}, \mathbf{c}, -) - \alpha h_n(\mathbf{b}, \mathbf{c}, \mathbf{b}^*, \mathbf{c}^*, -)\|_p \\ &\quad = \| Y_{n_1}(\mathbf{b} + \alpha\mathbf{b}_1^*, \mathbf{z}_1(\alpha), -) - Y_{n_1}(\mathbf{b}_1, \mathbf{c}_1, -) \\ &\qquad - \alpha h_{n_1}(\mathbf{b}_1, \mathbf{c}_1, \mathbf{b}_1^*, \mathbf{c}_1^*, -)\|_p = o(\alpha). \end{aligned} \tag{52}$$

Using the fact that $y_0 = Y_n(\mathbf{b}, \mathbf{c}, -)$ is a best (or local best) $\| \ \|_p$-approximation to f from $V_n(S)$ together with either (52) or the corresponding (51) (which we rewrite in the form (52) by setting $n_1 = n$ and $n_2 = 0$) and with (50) we have

$$\begin{aligned} \|f - y_0\|_p &\leqslant \|f - [Y_{n_1}(\mathbf{b}_1 + \alpha\mathbf{b}_1^*, \mathbf{z}_1(\alpha), -) + Y_{n_2}(\mathbf{b}_2 + \alpha\mathbf{b}_2^*, \mathbf{c}_2^*, -)]\|_p \\ &= \|f - y_0 - \alpha h_n(\mathbf{b}, \mathbf{c}, \mathbf{b}^*, \mathbf{c}^*, -)\|_p + o(\alpha) \\ &= \|f - y_0 - \alpha(h - y_0)\|_p + o(\alpha) \\ &= \|(1 - \alpha)(f - y_0) + \alpha(f - h)\|_p + o(\alpha) \\ &\leqslant \|f - y_0\|_p + \alpha[\|f - h\|_p - \|f - y_0\|_p] + o(\alpha), \end{aligned}$$

for all sufficiently small $\alpha > 0$. Hence, $\|f - y_0\|_p \leqslant \|f - h\|_p$ so that y_0 is a best $\| \ \|_p$-approximation to f from the cone $K_n(\mathbf{b}, \mathbf{c}, \mathbf{c}^*)$.

Finally, using Lemma 6 together with the fact that y_0 is a best $\| \ \|_p$-approximation to f from $K_n(\mathbf{b}, \mathbf{c}, \mathbf{c}^*)$ we find

$$\|f - y_0\|_p \leqslant \|f - y_0 - \alpha h\|_p = \|f - y_0\|_p + \alpha\Phi_p[f - y_0, -h] + o(\alpha),$$

as α decreases to zero through positive values so that (49) holds. ∎

When $\Lambda_n[\mathbf{c}]$ lies in the interior of S, y_0 is accessible through every cone $K_n(\mathbf{b}, \mathbf{c}, \mathbf{c}^*)$ so that (49) holds for all h in $H_n(\mathbf{b}, \mathbf{c})$. When some point of $\Lambda_n[\mathbf{c}]$ lies on the boundary of S this need no longer be the case, but in any event

(49) must hold for every h in the degenerate cone $K_n(\mathbf{b}, \mathbf{c}, \mathbf{0}) = L_n(\mathbf{c})$ of solutions of (1). In both of these two limiting cases (49) holds for all h in some linear space so that in (49) h may be replaced by θh when θ is any complex scalar with unit magnitude. Using this together with (36)–(37) and (49) we obtain the following corollary.

COROLLARY 1. *Let the exponential sum* $y_0 = Y_n(\mathbf{b}, \mathbf{c}, -)$ *be a best (or local best)* $\| \ \|_p$*-approximation to* f *from* $V_n(S)$, *let* $\epsilon = f - y_0$, *and assume that* $\| \epsilon \|_p > 0$. *Then for each* h *in the n-dimensional linear space* $L_n(\mathbf{c})$ *we have*

$$\int_{\epsilon\neq 0} h(t) \operatorname{sgn} \overline{\epsilon(t)}\, dt \leqslant \int_{\epsilon=0} | h(t)|\, dt, \qquad \text{if} \quad p = 1 \tag{53a}$$

$$\int_{\epsilon\neq 0} h(t) \mid \epsilon(t)|^{p-1} \operatorname{sgn} \overline{\epsilon(t)}\, dt = 0, \qquad \text{if} \quad 1 < p < \infty \tag{53b}$$

$$\lim_{\delta\to 0+} \operatorname{ess\,sup}\{\operatorname{Re}[h(t) \operatorname{sgn} \overline{\epsilon(t)}]: t \geqslant 0 \quad \text{and} \quad | \epsilon(t)| \geqslant \| \epsilon \|_\infty - \delta\} \geqslant 0$$
$$\text{if} \quad p = \infty. \tag{53c}$$

(Here sgn(z) is defined to be 0 or $z/| z |$ according as $z = 0$ or $z \neq 0$, respectively.) If in addition each element of $\Lambda_n[\mathbf{c}]$ is an interior point of S, then (53) holds for each h in the linear space $H_n(\mathbf{b}, \mathbf{c})$. ∎

Following arguments analogous to those given in [4, p. 179] and [12, p. 183] we may show that under mild hypotheses a best approximation y_0 has full order.

COROLLARY 2. *Let* $1 \leqslant p < \infty$, *let* y_0 *be a best (or local best)* $\| \ \|_p$*-approximation to* f *from* $V_n(S)$ *with* $\| f - y_0 \|_p > 0$, *and assume that* S *possesses some finite accumulation point in* L_0, *i.e., that* S *contains a sequence of distinct points* $\lambda_1, \lambda_2, \ldots$ *with limit* λ *in* L_0. *In the case where* $p = 1$, *assume in addition that* $f(t) \neq y_0(t)$ *holds for almost all* t. *Then* y_0 *has full order* n, *i.e.*, $y_0 \in V_n(S)\backslash V_{n-1}(S)$.

Proof. Suppose that the zero function is a local best $\| \ \|_p$-approximation to f from $V_1(S)$. Then from Corollary 1 we see that

$$\int_0^\infty | f(t)|^{p-2} \overline{f(t)}\, y(t)\, dt = 0 \tag{54}$$

holds whenever $y \in V_1(S)$, and thus, whenever y is any finite linear combination of the exponentials $\exp(\lambda_i t)$. Using this together with Lemma 5 and the fact that $\lambda \in L_0$, we see that (54) also holds whenever $y \in V_\infty(\{\lambda\})$.

Since $V_\infty(\{\lambda\})$ is dense in $L_p[0, \infty)$, it then follows that $\|f\|_p = 0$ so that the corollary holds when $n = 1$. Finally, since the zero function is a local best $\| \ \|_p$-approximation to $f - y_0$ from $V_1(S)$ only if $\|f - y_0\|_p = 0$ it follows that y_0 cannot be a best (or local best) $\| \ \|_p$-approximation to f from $V_n(S)$ when $y_0 \in V_{n-1}(S)$ unless $\|f - y_0\|_p = 0$. ∎

Note. In carrying out this argument it is essential that the finite accumulation point lie within the interior of the left half plane. For example, $y_0 \equiv 0$ is the best $\| \ \|_\infty$-approximation to $f \equiv 1$ from $V_1(S)$ when $S = \{z \in C\colon \operatorname{Re} z = 0 \text{ and } \operatorname{Im} z \neq 0\}$. ∎

By specializing the above corollaries to the case of unconstrained least squares approximation (where $p = 2$ and $S = L_0$) we obtain the following generalization of the Aigrain–Williams condition of [1, p. 598].

COROLLARY 3. *Let $f \in L_2[0, \infty)$, let the exponential $y_0 = Y_n(\mathbf{b}, \mathbf{c})$ be a best or local best least squares approximation to f on $[0, \infty)$ from $V_n(L_0)$, and assume that $\|f - y_0\|_2 > 0$. Then y_0 has full order, i.e., $y_0 \in V_n(L_0)\backslash V_{n-1}(L_0)$, and*

$$Y_0^{(i-1)}(\overline{\lambda_j}) = F^{(i-1)}(\overline{\lambda_j}), \qquad i = 1, 2,\ldots, 2k_j, \quad j = 1, 2,\ldots, l, \tag{55}$$

where

$$Y_0(s) = \int_0^\infty e^{st} y_0(t)\, dt, \qquad F(s) = \int_0^\infty e^{st} f(t)\, dt, \quad \operatorname{Re} s < 0,$$

are the Laplace transforms of y_0, f, respectively, where the parameters l, k_j, λ_j are taken from the canonical factorization

$$P(\mathbf{c}, \lambda) = (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_l)^{k_l}, \tag{56}$$

of the characteristic polynomial for the nth order differential operator that annihilates y_0 (and where the bar denotes the complex conjugate.)

Proof. By using Corollary 2 and the factorization (56) we see that y_0 has full order and that $H_n(\mathbf{b}, \mathbf{c})$ is spanned by the $2n$ functions

$$h_{ij}(t) = t^{i-1} e^{\lambda_j t}, \qquad i = 1, 2,\ldots, 2k_j, \quad j = 1, 2,\ldots, l, \tag{57}$$

(cf. [12, Lemma 1].) Since $S = L_0$, (53b) holds for each h_{ij} of (57) and

$$\begin{aligned} F^{(i-1)}(\bar\lambda_j) - Y_0^{(i-1)}(\bar\lambda_j) &= (d/ds)^{i-1} \int_0^\infty [f(t) - y_0(t)]\, e^{st}\, dt \Big|_{s=\bar\lambda_j} \\ &= \int_0^\infty [f(t) - y_0(t)]\, t^{i-1} e^{\bar\lambda_j t}\, dt \\ &= 0, \qquad i = 1, 2,\ldots, 2k_j, \quad j = 1, 2,\ldots, l. \ ∎ \end{aligned}$$

EXAMPLE. We shall find a (actually the) best $\| \|_2$-approximation to the unit step

$$f(t) = 1, \quad \text{if } 0 \leqslant t \leqslant 1$$
$$= 0, \quad \text{if } t > 1,$$

from $V_1(L_0)$. We let $y(t) = Ae^{\lambda t}$ denote such a best approximation and compute the Laplace transforms

$$F(s) = \int_0^\infty f(t)\, e^{st}\, dt = [e^s - 1]/s, \qquad Y(s) = \int_0^\infty y(t)\, e^{st}\, dt = -A/(s + \lambda).$$

The Aigrain–Williams equations (55) require that

$$[e^{\bar\lambda} - 1]/\bar\lambda = -A/(\lambda + \bar\lambda), \quad [(\bar\lambda - 1)\, e^{\bar\lambda} + 1]/(\bar\lambda)^2 = A/(\lambda + \bar\lambda)^2,$$

or equivalently, that

$$A = -e^{\bar\lambda}(\lambda + \bar\lambda)^2/\lambda, \qquad e^{-\bar\lambda} = 1 - \bar\lambda - (\bar\lambda)^2/\lambda. \tag{58}$$

From Theorem 3 we know that a best $\| \|_2$-approximation exists and from Corollary 3 above we know that any such best $\| \|_2$-approximation must satisfy (55). Thus, we know that there exists some $\lambda \in L_0$ and $A \in C$ that satisfy (58). In this case there is only one such real valued solution

$$\lambda = -1.25643120..., \qquad A = 1.43066372... \tag{59}$$

that can be shown to give the unique best $\| \|_2$-approximation to f from $V_1(C)$. ∎

For some applications in the physical and biological sciences, one is interested in obtaining a best uniform approximation to a given continuous real valued function f by means of a real valued exponential sum y having real exponents. Such an exponential sum may be parametrized in the form

$$y(t) = \sum_{i=1}^{l} \sum_{j=1}^{k_i} a_{ij} t^{j-1} \exp(\lambda_i t), \tag{60}$$

where

$$\begin{gathered} k_1 + k_2 + \cdots + k_l = k \leqslant n, \\ \lambda_1 < \lambda_2 < \cdots < \lambda_l, \\ a_{ij} \in R, \qquad \text{with} \qquad a_{ik_i} \neq 0. \end{gathered} \tag{61}$$

(When we work on $[0, \infty)$ we also require $\lambda_l \leqslant 0$ with $k_l = 1$ whenever $\lambda_l = 0$ so that $\| y \|_\infty < \infty$.) Within this context we may formulate an alternation type characterization of such a best approximation. In so doing, we say that the bounded real valued function $\epsilon \in C[0, \infty)$ essentially alternates at least m times on $[0, \infty)$ provided that for each $\delta > 0$ there exist points $0 < t_0 < t_1 < \cdots < t_m$ and some $s = \pm 1$ such that

$$s \cdot (-1)^i \epsilon(t_i) \geqslant \| \epsilon \|_\infty - \delta, \qquad i = 0, 1,..., m,$$

(cf. [14] for an extended discussion).

COROLLARY 4. *Let the real valued exponential sum y_0 with the parametrization of* (60)–(61) *be a best $\| \ \|_\infty$-approximation from $V_n(S)$ to the given bounded real valued function $f \in C[0, \infty)$, and let $\epsilon = f - y_0$.*

(a) *If the exponents $\lambda_{i_1} < \lambda_{i_2} < \cdots < \lambda_{i_p}$ of* (60) *and* (61) *are in the interior of S with respect the topology of L_0, then either* $\limsup |f(t)| = \| \epsilon \|_\infty$ *as $t \to +\infty$, or else ϵ essentially alternates at least $n + k_{i_1} + k_{i_2} + \cdots + k_{i_p}$ times on* $[0, +\infty)$.

(b) *If the exponents $\lambda_{i_1} < \lambda_{i_2} < \cdots < \lambda_{i_p}$ of* (60)–(61) *are in the interior of S with respect to the topology of* $(-\infty, 0)$, *then either* $\limsup |f(t)| = \| \epsilon \|_\infty$ *as $t \to +\infty$, or else ϵ essentially alternates at least $n + p$ times on* $[0, \infty)$.

Proof. By using [14, Theorem 2] we may extend the proof of [12, Corollary 3 to Theorem 2] to apply in the above context where the interval of approximation is $[0, \infty)$ rather than $[0, 1]$. ∎

In the special case where the exponential parameters λ_i are real valued but otherwise unconstrained, the above corollary requires the error curve corresponding to a best uniform approximation to a continuous function f to essentially alternate at least $n + l$ times if $\lambda_l < 0$, and at least $n + l - 1$ times if $\lambda_l = 0$. Thus, the Braess necessary condition of [5, Satz 1] must be weakened when the interval of approximation is extended from $[0, 1]$ to $[0, \infty)$ due to the weakened form of (b) that results when $\lambda_l = 0$.

EXAMPLE. Let $n \geqslant 1$ be selected and let the continuous real valued function f be defined on $[0, \infty)$ in such a manner that f varies linearly between the $n + 1$ points $(t, f(t)) = (j, 1 + (-1)^{n-j})$, $j = 0, 1,..., n$, with $f(t) = 2$ for $t \geqslant n$. We shall show that $y_0(t) \equiv 1$ with $\| f - y_0 \|_\infty = 1$ is the unique best $\| \ \|_\infty$-approximation to f from the set of real valued functions in $V_n(R)$. Indeed, if $y \in V_n(R)$ and $\| f - y \|_\infty \leqslant 1$, then y may be parametrized in the form (60)–(61) with $\lambda_l = 0$ and $k_l = 1$. But for any such fixed choice of the parameters λ_i, k_i the function y_0 is the unique best uniform approximation to f on $[0, n]$ (and thus, on $[0, +\infty)$) from the k-dimensional linear space spanned by the Haar system $\varphi_{ij}(t) = t^{j-1} \exp(\lambda_i t)$, $1 \leqslant j \leqslant k_i$, $1 \leqslant i \leqslant l$, since $f - y_0$ alternates $n \geqslant k$ times on $[0, n]$. In this case we have the minimum number $n + l - 1 = n$ alternations of the optimum error curve with $n - 1$ alternations being lost because y_0 is in the set $V_1(R)$ and with one additional alternation being lost because the exponential parameter $\lambda_1 = 0$ does not lie in the interior of $(-\infty, 0)$. ∎

Note. Using Lemmas 4–6, one can extend the sufficiency condition of [12, Theorem 3] to the present context, where the interval of approximation is $[0, \infty)$.

5. Approximation on Compact Subintervals of $[0, \infty)$

For computational purposes it is desirable to work on a compact interval of approximation rather than on the whole semiinfinite interval $[0, \infty)$. In principle, it is always possible to obtain a best $\| \, \|_p$-approximation on $[0, \infty)$ from a sequence of best $\| \, \|_p$-approximations on compact subintervals of $[0, \infty)$ as we see from the following result.

THEOREM 5. *Let $1 \leqslant p \leqslant \infty$, let $f \in L_p[0, \infty)$, and let the positive integer n be given. Let $S \subseteq C$ be closed, let $\{\sigma_\nu\}$ be an unbounded strictly increasing sequence of positive real numbers, and for each $\nu = 1, 2, \ldots$ let y_ν be a best $\| \, \|_{p,\sigma_\nu}$-approximation to f from $V_n(S)$ (where $\| \, \|_{p,\sigma}$ again denotes the seminorm* (22)). *Then there is some subsequence of $\{y_\nu\}$ that may be decomposed in the form* (23) *satisfying conditions* (i)–(iv) *of Lemma* 3 *for positive σ, with the limit function v being a best $\| \, \|_p$-approximation to f from $V_n(S)$. Moreover, if S is a compact subset of L_0, or if $v \in V_n(L_0) \setminus V_{n-1}(L_0)$, then some subsequence of $\{y_\nu\}$ $\| \, \|_p$-converges to this best $\| \, \|_p$-approximation, v.*

Proof. Since $\{\sigma_\nu\}$ is unbounded and since y_ν is a best $\| \, \|_{p,\sigma_\nu}$-approximation to f from $V_n(S)$, we see that for positive σ, we have

$$\begin{aligned} \limsup \| y_\nu \|_{p,\sigma} &\leqslant \limsup \| y_\nu \|_{p,\sigma_\nu} \\ &\leqslant \limsup \{ \| f \|_{p,\sigma_\nu} + \| f - y_\nu \|_{p,\sigma_\nu} \} \leqslant 2 \| f \|_p , \end{aligned}$$

so that $\{y_\nu\}$ is $\| \, \|_{p,\sigma}$-bounded. After passing to a subsequence, if necessary, we may effect the decomposition (23) satisfying conditions (i)–(iv) of Lemma 3 with $\{v_\nu\}$ $\| \, \|_{p,\sigma}$-converging to some fixed $v \in V_n(S)$ for each choice of $\sigma > 0$. If we let y_∞ be some best $\| \, \|_p$-approximation to f from $V_n(S)$ and use (24) we find

$$\begin{aligned} \| f - v \|_{p,\sigma} &\leqslant \liminf \| f - v - x_\nu \|_{p,\sigma} = \liminf \| f - y_\nu \|_{p,\sigma} \\ &\leqslant \liminf \| f - y_\nu \|_{p,\sigma_\nu} \leqslant \liminf \| f - y_\infty \|_{p,\sigma_\nu} \\ &= \| f - y_\infty \|_p , \end{aligned}$$

holds for positive σ, and it follows that v is a best $\| \ \|_p$-approximation to f from $V_n(S)$.

In the special case where S is a compact subset of L_0, the condition (iii) of Lemma 3 requires that $x_\nu = 0$, and thus, that $y_\nu = v_\nu$ for all but finitely many values of ν. This being the case, some subsequence of $\{y_\nu\}$ $\| \ \|_{p,\sigma}$-converges to v, and in view of Lemma 5, the convergence must also take place with respect to the norm $\| \ \|_p$. Finally, if $v \in V_n(L_0)\backslash V_{n-1}(L_0)$, then the $\| \ \|_{p,\sigma}$-convergence of $\{v_\nu\}$ to v requires that all but a finite number of the v_ν must lie in $V_n(K)$ when K is any compact subset of L_0 for which $v \in V_n(K^0)$ (where K^0 denotes the interior of K with respect to C), cf. [12, Theorem 1]. This being the case, we may replace the closed set S by the compact set $S \cap K$ and again conclude that some subsequence of $\{y_\nu\}$ $\| \ \|_p$-converges to v. ∎

COROLLARY. *Let $1 < p < \infty$, let $S = L_0$, and let $\{y_\nu\}$ be selected as in the theorem. Then some subsequence of $\{y_\nu\}$ $\| \ \|_p$-converges to a best $\| \ \|_p$-approximation to f from $V_n(C)$.*

Proof. If $f \in V_n(L_0)$, then $y_\nu = f$ for each ν. If $f \notin V_n(L_0)$, then the best $\| \ \|_p$-approximation v, of the theorem must have full order (as we see from Corollary 2 to Theorem 4) and thus, lie in $V_n(L_0)\backslash V_{n-1}(L_0)$. ∎

Note. When $p = 1$ or $p = \infty$, the sequence $\{y_\nu\}$ of Theorem 5 need not possess any $\| \ \|_p$-convergent subsequence. For example, the unique best $\| \ \|_{\infty,\nu}$ approximation to the function

$$f(t) = te^{-t}, \qquad t \geqslant 0$$

from $V_1(R)$ has the form

$$y_\nu(t) = a_\nu e^{\lambda_\nu t}, \qquad a_\nu > 0, \quad \lambda_\nu > 0 \tag{62}$$

(as we see by using the alternation characterization of [12, Corollary 3 to Theorem 2]). In this case, we must clearly have $\lim a_\nu = (2e)^{-1}$ and $\lim \lambda_\nu = 0$ so that for each $\sigma > 0$, $\{y_\nu\}$ $\| \ \|_{\infty,\sigma}$-converges to the unique best $\| \ \|_\infty$-approximation

$$y(t) = (2e)^{-1}$$

for f from $V_1(R)$, but since $\| y_\nu \|_\infty = \infty$ for each ν, there is no subsequence of $\{y_\nu\}$ that $\| \ \|_\infty$-converges to y.

As a second example, by using arguments analogous to those presented in the analysis of [12, Example 1] we see that the unique best $\| \ \|_1$-approximation to the function

$$\begin{aligned} f(t) &= 1, \qquad \text{if } m - 2^{-m} \leqslant t \leqslant m, \quad m = 1, 2, \ldots \\ &= 0, \qquad \text{otherwise} \end{aligned}$$

from $V_1(L_0)$ is the function $y \equiv 0$. Again we find that the best $\| \|_{1,\nu}$-approximation to f takes the form (62), where now, $\lim a_\nu = 0$ and $\lim \lambda_\nu = +\infty$ so that $\{y_\nu\}$ $\| \|_{1,\sigma}$-converges to y for each $\sigma > 0$, but no subsequence of $\{y_\nu\}$ has the $\| \|_1$-limit, y. ∎

6. Acknowledgment

I am endebted to the referee for a number of helpful suggestions and for the above proof of Lemma 2.

References

1. P. R. Aigrain and E. M. Williams, Synthesis of n-reactance networks for desired transient response, *J. Appl. Phys.* **20** (1949), 597–600.
2. R. Bellman, "Methods of Nonlinear Analysis," Vol. I. Academic Press, New York, 1970.
3. G. Birkhoff and G.-C. Rota, "Ordinary Differential Equations," Blaisdell, Waltham, Massachusetts, 1962.
4. C. de Boor, On the approximation by γ-polynomials, *in* "Approximations with Special Emphasis on Spline Functions," (I. J. Schoenberg, Ed.), p. 157–183. Academic Press, New York, 1969.
5. D. Braess, Über die Approximation mit Exponentialsummen, *Computing* **2** (1967), 309–321.
6. D. Braess, Über die Vorzeichenstruktur der Exponentialsummen, *J. Approximation Theory* **3** (1970), 101–113.
7. D. Braess, Die Konstruktion der Tschebyscheff-Approximation bei der Anpassung mit Exponentialsummen, *J. Approximation Theory* **3** (1970), 261–273.
8. E. W. Cheney, "Introduction to Approximation Theory," McGraw-Hill, New York, 1966.
9. R. Courant and D. Hilbert, "Methods of Mathematical Physics," Vol. I. Interscience, New York, 1965.
10. R. K. Harman and F. W. Fairman, Exponential approximation via a closed form Gauss-Newton method, *IEEE Trans. Circuit Theory*, **CT-20** (1973), 361–369.
11. D. W. Kammler, Existence of best approximations by sums of exponentials, *J. Approximation Theory* **9** (1973), 78–90.
12. D. W. Kammler, Characterization of best approximations by sums of exponentials, *J. Approximation Theory* **9** (1973), 173–191.
13. D. W. Kammler, A minimal decay rate for solutions of stable nth order homogeneous differential equations with constant coefficients, *Proc. Amer. Math. Soc.* **48** (1975), 145–151.
14. D. W. Kammler, An alternation characterization of best uniform approximations on noncompact intervals, *J. Approximation Theory* **16** (1976), 97–104.
15. E. Kreyszig, "Advanced Engineering Mathematics," Wiley, New York, 1972.
16. R. N. McDonough and W. H. Huggins, Best least squares representation of signals by exponentials, *IEEE Trans. Autom. Control*, **AC-13** (1968), 408–412.
17. I. P. Natanson, "Constructive Function Theory," Vol. I. Ungar, New York, 1964.
18. I. P. Natanson, "Constructive Function Theory," Vol. II. Ungar, New York, 1965.

19. J. R. Rice, Chebyshev approximation by exponentials, *SIAM J. Appl. Math.* **10** (1962). 149–161.
20. F. J. Servais, On the expansion of experimental functions in series of descending exponentials of real argument, *Nucl. Sci. Eng.* **34** (1968), 196–197.
21. E. T. Whittaker and G. N. Watson, "A Course of Modern Analysis," Cambridge Univ. Press, London/New York, 1962.
22. H. Werner, Tchebyscheff approximation with sums of exponentials, *in* "Approximation Theory," (A. Talbot, Ed.), Academic Press, New York, 1970.