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# EMBEDDING IN R-CLOSED SPACES

Alan DOW* and Jack PORTER**

*Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA*

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Some problems in the theory of R-closed spaces are solved by showing that every regular space can be embedded in a minimal regular space and there is an R-closed space with no coarser minimal regular topology. A class of spaces is found so that when fed into the Jones' machinery for producing non-Tychonoff, regular spaces, the output is non-Tychonoff R-closed and minimal regular spaces. Also, an example of a strongly minimal regular space that is not locally R-closed is given.

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## 1. Introduction

A regular (includes Hausdorff) space $X$ is *R-closed* if $X$ is closed in every regular space containing $X$ as a subspace, is *minimal regular* if $X$ has no strictly coarser regular topology, and is *strongly minimal regular* if $X$ is R-closed and has a closed basis consisting of R-closed subspaces. It is well-known, cf., [2, 16], that a compact Hausdorff space is strongly minimal regular, a strongly minimal regular space is minimal regular, and a minimal regular space is R-closed. Many facts are known about R-closed spaces [2, 4, 5, 7, 11, 12, 15, 16] but a number of problems remain unsolved. Three of these problems are solved in this paper.

Jones [9] has developed machinery that produces non-Tychonoff, regular spaces when fed with non-normal, Tychonoff spaces. In Section 2, a class of non-normal, Tychonoff spaces is developed so that when one of these spaces is fed into the Jones machinery, noncompact R-closed and minimal regular spaces result. Also, in the second section, two examples are given. First, an example of an R-closed space with no coarser minimal regular topology; this solves a problem in [2]. This example is modified to obtain an example of a strongly minimal regular space that

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is not locally R-closed (i.e., some point does not have a neighborhood base of R-closed spaces).

Herrlich [8] showed there are regular spaces that can not be densely embedded in an R-closed space. In Section 3, an arbitrary regular space is shown to be embeddable in a minimal regular space and, hence, in an R-closed space. This result solves two problems in [2] in the affirmative. In the last section, the paper is concluded with a couple of unsolved problems.

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In the remainder of this section, the definitions and results necessary for the sequel are introduced. Also, at the end of this section, a new sufficient condition for a subspace of an R-closed space to be R-closed is given.

The set of all integers is denoted by $\mathbb{Z}$. Ordinals are assumed to have the usual order topology and the usual interval notation for ordinals is used, e.g., if $\alpha < \beta$, then

$$[\alpha, \beta) = \{\gamma : \gamma \text{ is ordinal and } \alpha \leq \gamma < \beta\}.$$

Cardinals are defined as initial ordinals. If $\kappa$ is a cardinal, then $\kappa + 1$ denotes the successor ordinal to $\kappa$ whereas $\kappa^+$ denotes the successor cardinal to $\kappa$. In particular, $\kappa + 1$ is a compact Hausdorff space. As usual, $\omega$ denotes the first infinite ordinal and $\omega_1$ denotes the first uncountable ordinal. A *regular-open* subset $A$ of a space $X$ satisfies $A = \operatorname{int}(\operatorname{cl} A)$; $A$ is *regular-closed* if $X \backslash A$ is regular-open.

A *regular* filter $\mathcal{F}$ on a space $X$ is a filter with the property that for $A \in \mathcal{F}$, there is an open set $U \in \mathcal{F}$ such that $\operatorname{cl} U \subseteq A$. Clearly, a regular filter is an open filter, i.e., has an open filter base. A *maximal regular* filter is a maximal element in the set of all regular filters partially ordered by inclusion. It follows easily that the neighborhood filter $\mathcal{N}_p$ of a point $p$ in a regular space is a maximal filter. The *adherence* of a filter $\mathcal{F}$, denoted as $\operatorname{ad} \mathcal{F}$, is defined as $\bigcap\{\operatorname{cl} A : A \in \mathcal{F}\}$. A filter $\mathcal{F}$ is *fixed* if $\operatorname{ad} \mathcal{F} \neq \emptyset$; otherwise, $\mathcal{F}$ is said to be *free*. Thus, for a regular filter $\mathcal{F}$, $\operatorname{ad} \mathcal{F} = \bigcap \mathcal{F}$. A filter $\mathcal{F}$ on a space $X$ *traces* on $A \subseteq X$ if $F \cap A \neq \emptyset$ for all $F \in \mathcal{F}$; otherwise, $\mathcal{F}$ is said to *miss* $A$. Note that if $\mathcal{F}$ is a regular filter on a space $X$ and traces on $A$, then $\mathcal{F}|A = \{F \cap A : F \in \mathcal{F}\}$ is a regular filter on $A$.

Many of the basic properties of R-closed and minimal regular spaces are contained in [2]. Some of the properties needed in the paper are now listed.

**1.1.** (a) *A regular space $X$ is R-closed iff every regular filter on $X$ is fixed.*

(b) *A regular space $X$ is minimal regular iff every regular filter on $X$ with a unique adherent point is convergent.*

(c) *A Tychonoff, R-closed space is compact.*

(d) *The continuous image of an R-closed space onto a regular space is R-closed.*

(e) *The product of an R-closed* (*resp. minimal regular*) *space with a compact Hausdorff space is R-closed* (*resp. minimal regular*).

(f) *A clopen subspace of an R-closed* (*resp. minimal regular*) *space is R-closed* (*resp. minimal regular*).

(g) *A regular space that is the finite union of R-closed (resp. minimal regular) spaces is R-closed (resp. minimal regular).*

The problem of determining which subsets of R-closed spaces are also R-closed is still unsolved. From the well-known examples of noncompact, R-closed spaces (e.g., Example 4.18 in [2]), it is clear that regular-closed subspaces of an R-closed space are not necessarily R-closed. By 1.1(f), clopen subsets of R-closed spaces are R-closed. Another sufficient condition is presented in the next result.

**1.2.** *If $X$ is R-closed and $U$ (resp. $A$) is an open (resp. closed) subset of $X$ such that* bd $U$ *(resp.* bd $A$*) is R-closed, then* cl $U$ *(resp. $A$) is R-closed.*

**Proof.** Since the proofs are similar, only the open case is proven. Let $B = \text{cl}\, U$. Assume $\mathcal{F}$ is a free regular filter on $B$. Since bd $U$ is R-closed, then $\mathcal{F}$ misses bd $U$. So, there is $V \in \mathcal{F}$ such that $V \cap \text{bd}\, U = \emptyset$ and $V$ is open in $B$. Since $V \subseteq U$, then $V$ is open in $X$. Let $\mathcal{G}$ be the filter on $X$ generated by $\mathcal{F} | U$. Then $\mathcal{G}$ is a regular filter and $\bigcap \mathcal{G} = \bigcap \mathcal{F} = \emptyset$. This is a contradiction as $X$ is R-closed.

A regular space is *locally R-closed* if every point has a neighborhood base consisting of R-closed subspaces. There are strongly minimal regular spaces which are not locally R-closed (see 2.7 in this paper). Also, there are R-closed, locally R-closed spaces which are not minimal regular (see the space $J(X)$ described before 2.5 in this paper). However, the next result is a consequence of 1.2.

**1.3.** *If $X$ is R-closed and has an open basis with R-closed boundaries, then $X$ is both locally R-closed and strongly minimal regular.*

**Proof.** It is immediate from 1.2 that $X$ is locally R-closed. If $A \subseteq X$ is closed and $p \notin A$, there is an open set $U$ of $p$ such that bd $U$ is R-closed and $A \cap \text{cl}\, U = \emptyset$. Let $B = X \backslash U$. Then $p \notin B$, $A \subseteq B$, and $\text{bd}\, B = \text{bd}\, U$ is R-closed. By 1.2, $B$ is R-closed. So, $X$ is strongly minimal regular.

## 2. Jones' machinery

In 1973, Jones [9] developed for each non-normal regular space $X$, a non-Tychonoff, regular space $J(X)$ which contains $X$ as a closed subspace. Let $A$ and $B$ be disjoint closed subsets of $X$ that can not be separated by disjoint open sets. Then $H = A \backslash \text{int}\, A$ and $K = B \backslash \text{int}\, B$ are disjoint nowhere dense closed subsets of $X$ that can not be separated by disjoint open sets. Let $Y = X \times \mathbb{Z}$ with this identification: if $x \in H$ and $n$ is even, identify $(x, n)$ and $(x, n+1)$ and if $x \in K$ and $n$ is odd, identify $(x, n)$ and $(x, n+1)$. Let $X_n = X \times \{n\}$, $H_n = H \times \{n\}$, and $K_n = K \times \{n\}$. Suppose $\{p_+, p_-\} \cap Y = \emptyset$. Let $DJ(X) = Y \cup \{p_+, p_-\}$ where $U \subseteq DJ(X)$ is open if $U \cap Y$ is open in $Y$ and $p_+ \in U$ (resp. $p_- \in U$) implies there is $n > 0$ such that $\bigcup\{X_m : m > n\} \subseteq U$ (resp. $\bigcup\{X_m : m < -n\} \subseteq U$). Let $J(X) = \{p_+\} \cup \{X_m : m \geq 0\}$.

**2.1** [9]. *Suppose $X$ is regular and $H$ and $K$ are disjoint closed sets that can not be separated by disjoint open sets. Then $J(X)$ and $DJ(X)$ are regular, non-Tychonoff spaces. Also, if $f$ is a real-valued continuous function on $DJ(X)$, then $f(p_+) = f(p_-)$.*

We now give a sufficient condition for $J(X)$ to be R-closed and $DJ(X)$ to be minimal regular.

**2.2.** *Suppose $X$ is a regular space with a pair of disjoint closed sets $H$ and $K$ that satisfies:*

(∗) *Every free regular filter on $X$ traces on $H$ and $K$.*

*Then $J(X)$ is R-closed.*

**Proof.** Assume $J(X)$ is not R-closed. So, there is a free regular filter $\mathcal{F}$ on $J(X)$.

*Case 1.* $\mathcal{F}$ misses each $X_n$. For each $n \in \omega$, there is $U_n \in \mathcal{F}$ such that $(X_0 \cup \cdots \cup X_n) \cap U_n = \emptyset$. Thus, if $U$ is a neighborhood of $p_+$, then for some $n > 0$, $J(X) \backslash (X_0 \cup \cdots \cup X_n) \subseteq U$; so, $U_n \subseteq U$. Hence, $\mathcal{F}$ converges to $p_+$, contradicting that $\mathcal{F}$ is free.

*Case 2.* $\mathcal{F}$ traces on some $X_n$. Then $\mathcal{F}|X_n$ is a free regular filter on $X_n$. Hence, $\mathcal{F}$ traces on $H_n$ and $K_n$. If $n$ is even (resp. odd), then $\mathcal{F}$ traces on $H_{n+1}$ (resp. $K_{n+1}$). So, $\mathcal{F}$ traces on $X_{n+1}$. By induction, it follows that $\mathcal{F}$ traces on $X_m$ for $m \geq n$. Hence, $p_+ \in \operatorname{ad} \mathcal{F}$ contradicting that $\mathcal{F}$ is free.

**2.3.** *Suppose $X$ is a regular space with a pair of disjoint closed sets $H$ and $K$ that satisfies:*

(∗∗) *Every regular filter on $X$ with a unique adherent point and which misses one of $H$ or $K$ is a maximal regular filter.*

*Then $DJ(X)$ is minimal regular.*

**Proof.** First, we will show that $X$ satisfies (∗). Assume $X$ does not satisfy (∗), i.e., there is a free regular filter $\mathcal{F}$ on $X$ that misses $H$. Let $y \in X \backslash H$ and $\mathcal{G} = \{F \cup U : F \in \mathcal{F}, U \in \mathcal{N}_y\}$. Then $\operatorname{ad} \mathcal{G} = \{y\}$ and $\mathcal{G}$ is a regular filter on $X$ missing $H$. By (∗∗), $\mathcal{G}$ is a maximal regular filter on $X$; this is impossible as $\mathcal{G}$ is properly contained in $\mathcal{N}_y$. So, $X$ satisfies (∗). To show $DJ(X)$ is minimal regular, let $\mathcal{F}$ be a regular filter on $DJ(X)$ with a unique adherent point $y$.

*Case 1.* $y = p_+$ (the $y = p_-$ case is similar). There is $m \in \omega$ such that $\mathcal{F}$ misses $\{X_k : k \leq -m\} \cup \{p_-\}$. Assume $\mathcal{F}$ traces on some $X_n$. Then $\mathcal{F}_n = \mathcal{F}|X_n$ is a free regular filter on $X_n$. Thus, $\mathcal{F}_n$ traces on $H_n$ and $K_n$. If $n$ is even (resp. odd), then $\mathcal{F}_n$ traces on $K_{n-1}$ (resp. $H_{n-1}$). So, $\mathcal{F}$ traces on $X_{n-1}$, and by induction, it follows that $\mathcal{F}$ traces on $X_k$ for all $k \leq n$. This is impossible as $\mathcal{F}$ does not trace on $X_{-m}$. Hence, $\mathcal{F}$ does not trace on $X_n$ for all $n \in \mathbb{Z}$. It follows that $\mathcal{F}$ converges to $p_+$.

*Case 2.* $y \in X_n$ for some $n$. Assume $\mathscr{F}$ traces on both $H_n$ and $K_n$. Now $y \notin H_n$ or $y \notin K_n$ as $H_n \cap K_n = \emptyset$. Suppose $y \notin H_n$. If $n$ is even (resp. odd), then $\mathscr{F}$ traces on $H_{n+1}$(resp. $H_{n-1}$) and $y \notin X_{n+1}$ (resp. $X_{n-1}$) as $y \notin H_n$. So, $\mathscr{F}_{n+1}$(resp. $\mathscr{F}_{n-1}$) is a free regular filter on $X_{n+1}$(resp. $X_{n-1}$). By induction, it follows that $\mathscr{F}$ traces on $X_k$ for all $k \geq n$ (resp. $k \leq n$). This is a contradiction as it shows that $p_+ \in \operatorname{ad} \mathscr{F}$ (resp. $p_- \in \operatorname{ad} \mathscr{F}$). So, $\mathscr{F}_n$ misses $H_n$ or $K_n$. By (**), $\mathscr{F}_n$ is a maximal regular filter on $X_n$. Since $y \in \bigcap \mathscr{F}$, then $\mathscr{F} \subseteq \mathscr{N}_y$. Let $\mathscr{N}_y^n = \mathscr{N}_y | X_n$. So, $\mathscr{F}_n \subseteq \mathscr{N}_y^n$; by maximality of $\mathscr{F}_n$, $\mathscr{F}_n = \mathscr{N}_y^n$. If $y \notin H_n \cup K_n$, then $X_n \backslash (H_n \cup K_n) \in \mathscr{N}_y$; it follows that $\mathscr{N}_y = \mathscr{F}$ and $\mathscr{F}$ converges to $y$. If $y \in H_n \cup K_n$, then $y \in X_{n+1}$ (or $X_{n-1}$) and $\mathscr{F}_{n+1}$ (or $\mathscr{F}_{n-1}$) is a maximal regular filter on $X_{n+1}$ (or $X_{n-1}$); so, $\mathscr{F}_{n+1} = \mathscr{N}_y^{n+1}$ (or $\mathscr{F}_{n-1} = \mathscr{N}_y^{n-1}$). Thus, $X_n \cup X_{n+1} \in \mathscr{N}_y$ (or $X_n \cup X_{n-1} \in \mathscr{N}_y$); it follows that $\mathscr{N}_y = \mathscr{F}$ and $\mathscr{F}$ converges to $y$.

**2.4.** *Let $X$ be a regular space with disjoint closed sets $H$ and $K$ that can not be separated by disjoint open sets. Suppose $X$ also satisfies:*

(1) *For every pair of disjoint regular-closed sets, one is R-closed, and*

(2) *$X$ has an open base with R-closed boundaries.*

(a) *If $X$ is Tychonoff, then $X$ satisfies* (*).

(b) *If $X$ satisfies* (*), *then $X$ satisfies* (**).

**Proof.** To show (a), assume $\mathscr{F}$ is a free regular filter on $X$ that does not trace on $H$. Then there is an open set $U \in \mathscr{F}$ such that $H \cap \operatorname{cl} U = \emptyset$. Also, there is an open set $V \in \mathscr{F}$ such that $\operatorname{cl} V \subseteq U$. Now $\operatorname{cl} V$ is not R-closed since $\mathscr{F}$ is free. But $H \subseteq \operatorname{cl}(X \backslash \operatorname{cl} U)$ and $\operatorname{cl}(X \backslash \operatorname{cl} U) \cap \operatorname{cl} V = \emptyset$. So, $\operatorname{cl}(X \backslash \operatorname{cl} U)$ is R-closed and Tychonoff and, hence, is compact. But the compact set $H$ in a regular space can be separated from a disjoint closed set by disjoint open sets; this is a contradiction.

To prove (b), suppose $\mathscr{F}$ is a regular filter on $X$ with a unique adherent point $y$ that misses $H$. Assume $\mathscr{F}$ does not converge to $y$, then there is an open neighborhood $U$ of $y$ such that $\operatorname{bd} U$ is R-closed, and $\mathscr{F}$ traces on $X \backslash U$. Let $\mathscr{G}$ be the filter generated by $\{F \backslash U : F \in \mathscr{F}\}$. Since $\mathscr{F} | (X \backslash U)$ is a free regular filter on $X \backslash U$ and $\operatorname{bd} U$ is R-closed, then $X \backslash \operatorname{cl} U \in \mathscr{F} | (X \backslash U)$. Thus, $\mathscr{G}$ is a free regular filter on $X$. But $\mathscr{G}$ misses $H$, a contradiction, as $X$ satisfies (*).

Let $X = ((\omega_1 + 1) \times (\omega + 1)) \backslash \{(\omega_1, \omega)\}$ where $H = \{(\alpha, \omega) : \alpha < \omega_1\}$ and $K = \{(\omega_1, \alpha) : \alpha < \omega\}$. The non-normal Tychonoff plank $X$ is locally compact and for every pair of disjoint regular-closed sets, one is compact; also, $H$ and $K$ can not be separated by disjoint open sets. By 2.4, $J(X) = \{p_+\} \cup \{X_n : n \geq 0\}$ (resp. $DJ(X) = \{p_+, p_-\} \cup \{X_n : |n| \in \omega\}$) is R-closed (resp. minimal regular); $J(X)$ (resp. $DJ(X)$) is a well-known example that is not minimal regular (resp. compact), cf. [3].

**2.5.** (a) *The R-closed subspaces of $J(X) \backslash \{p_+\}$ are compact subspaces.*

(b) *$J(X)$ has only one coarser minimal regular topology and this topology is compact Hausdorff.*

**Proof.** (a) This follows by 1.1(c) and the fact that every subspace of $J(X)\backslash\{p_+\}$ is Tychonoff.

(b) Let $\tau$ denote the usual topology of $J(X)$ and $\tau^*$ denote the topology on $J(X)$ which is the one point compactification of the locally compact space $J(X)\backslash\{p_+\}$. Now $\tau^*$ is a coarser regular topology which is compact Hausdorff. Suppose that $\tau'\subseteq\tau$ and $\tau'$ is a minimal regular topology on $J(X)$. Let $y\in J(X)\backslash\{p_+\}$, and let $V$ be a compact $\tau$-neighborhood of $y$. By 1.2 $J(X)\backslash(\text{int}_\tau V)$ is R-closed since $\text{bd}_\tau[J(X)\backslash\text{int}_\tau V]$ is compact. Therefore $J(X)\backslash(\text{int}_\tau V)$ is $\tau'$-closed and so $\text{int}_\tau V$ is $\tau'$ open. So,

$$\tau^*|(J(X)\backslash\{p_+\})=\tau'|(J(X)\backslash\{p_+\})=\tau|(J(X)\backslash\{p_+\}).$$

If $U$ is a $\tau^*$-open neighborhood of $p_+$, then $J(X)\backslash U$ is compact and, hence, closed in $\tau'$. This shows that $U$ is $\tau'$-open and proves $\tau^*\subseteq\tau'$. By minimal regularity of $\tau'$, it follows that $\tau^*=\tau'$.

**Remark.** If a space $P$ satisfies 2.4 and is Tychonoff (and hence locally compact), then the same proof for 2.5(b) yields that $J(P)$ has only one coarser minimal regular topology and this topology is compact Hausdorff. This would seem to indicate that one should not expect the Jones machinery to yield a positive solution to Problem 17(b) in [2] (finding an R-closed space with at least two coarser minimal regular topologies).

Let $Z=\{(x,\gamma)\in DJ(X)\times(\omega+1)$: if $\gamma\in\omega$, then $x=p_+$ or $x=(\alpha,\beta,n)$ where $n\geqslant-\gamma\}$. For $\gamma\in\omega+1$, let $Y_\gamma=\{(x,\zeta)\in Z:\zeta=\gamma\}$ and $T_\gamma=\{x:(x,\gamma)\in Y_\gamma\}$. Note that $Z$ is a subspace of the minimal regular space $DJ(X)\times(\omega+1)$ (see 1.1(e)), for $n\in\omega$, $T_n$ is homeomorphic to $J(X)$, and $T_\omega$ is homeomorphic to $DJ(X)$. Now, $U\subseteq Z$ is open iff $U\cap Y_\gamma$ is open in $Y_\gamma$ for each $\gamma\in\omega+1$ and if $(x,\omega)\in Y_\omega\cap U$, there are an open neighborhood $W$ of $x$ in $DJ(X)$ and $k\in\omega$ such that $Z\cap(W\times[k,\omega+1))\subseteq U$.

**2.6.** *$Z$ is R-closed and has no coarser minimal regular topology.*

**Proof.** First, we show $Z$ is R-closed. Let $\mathcal{F}$ be a regular filter on $Z$ such that $(p_-,\omega)\notin\bigcap\mathcal{F}$. There are an open neighborhood $U$ of $(p_-,\omega)$ and $F\in\mathcal{F}$ such that $F\cap U=\emptyset$. Thus, for some integer $m$,

$$F\subseteq\bigcup\{Y_k:0\leqslant k\leqslant m\}\cup(T_m\times[m,\omega])$$

which is an R-closed space by 1.1(e, g). Thus, $\bigcap\mathcal{F}\neq\emptyset$. This shows $Z$ has no free regular filters and, hence, is R-closed. Assume $Z'$ is $Z$ with a coarser minimal regular topology. Since for $n\in\omega$, $Y_n$ is clopen in $Z$, then $Y_n$ and $Z\backslash Y_n$ are R-closed by 1.1(f). Thus, $Y_n$ and $Z\backslash Y_n$ are closed subsets of $Z'$. Hence, $Y_n$ is a clopen subset of $Z'$. Let $Y_n'$ be $Y_n$ with the $Z'$ topology. By 1.1(f), $Y_n'$ is minimal regular. Since $Y_n$ is homeomorphic to $J(X)$, then by 2.5(b) $Y_n'$ is a compact Hausdorff space;

hence, every neighborhood of $(p_+, n)$ in $Z'$ traces on $X_{-n}\times\{n\}$. Let $U$ be an open neighborhood of $(p_+, \omega)$ in $Z'$. For some $n\in\omega$, $(p_+, m)\in U$ for $m\geq n$. So, $U\cap(X_{-m}\times\{m\})\neq\emptyset$ for all $m\geq n$. This implies that $(p_-, \omega)\in \mathrm{cl}_{Z'}U$. So, $(p_+, \omega)$ and $(p_-, \omega)$ can not be separated by disjoint open sets; hence, $Z'$ is not Hausdorff, a contradiction.

Now, 2.6 gives a negative answer to Problem 17(a) in [2] by showing that $P$ = regular does not have this property: every $P$-closed space has a coarser minimal $P$ topology. Also, when $P$ = Urysohn [13] the property is not satisfied; however, the property is satisfied when $P$ = Hausdorff [10] or Tychonoff [1].

Now, we modify $Z$ to obtain a strongly minimal regular space $W$ that is not locally R-closed. In $Z$, for each $n\in\omega$, identify $\{(\omega_1, n, 0, \gamma)\in Z: \gamma\in\omega+1\}$ to a point. Let $W$ be the corresponding quotient space of $Z$ and $\phi: Z\to W$ the corresponding quotient map.

**2.7.** *W is strongly minimal regular but not locally R-closed.*

**Proof.** It is straightforward to show that $W$ is regular; hence, by 1.1(d), $W$ is R-closed. To prove $W$ is strongly minimal regular, we need to find for each $y\in W$, a neighborhood base $\mathcal{B}$ such that $W\backslash B$ is R-closed for each $B\in\mathcal{B}$.

*Case 1.* $y\notin\{(p_-, \omega)\}\cup\{(p_+, \gamma): \gamma\in\omega+1\}$. In this case $W$ is locally compact at $y$. So, there is an open neighborhood base $\mathcal{B}$ of $y$ such that $\mathrm{bd}\, B$ is compact for $B\in\mathcal{B}$. Since $\mathrm{bd}\, B=\mathrm{bd}(W\backslash B)$ and $W$ is R-closed, then by 1.2, $W\backslash B$ is R-closed.

*Case 2.* $y=(p_-, \omega)$. Now for each $m\in\omega$, $P_m=\{Y_k: 0\leq k\leq m\}\cup T_m\times[m, \omega]$ is R-closed (see proof of 2.6). But $\phi(P_m)$ is R-closed and $\{W\backslash\phi(P_m): m\in\omega\}$ is a neighborhood base of $(p_-, \omega)$.

*Case 3.* $y=(p_+, \gamma)$ for $\gamma\in\omega+1$. For each even integer $m\geq 0$, let

$$S_m=\{(\alpha, \beta, n, \delta)\in Z: n\leq m, \delta\in\omega+1\}\cup\{(p_-, \omega)\}.$$

Now, we will show that $\phi(S_m)$ is R-closed. Assume there is a free regular filter $\mathcal{F}$ on $\phi(S_m)$. Since $\phi(Y_\omega\cap S_m)$ is homeomorphic to $J(X)$, an R-closed space, there is an $F\in\mathcal{F}$ such that $F\cap\phi(Y_\omega\cap S_m)=\emptyset$. There are sets $F_0, F_1, \ldots, F_{m+1}=F\in\mathcal{F}$ such that $F_i\subseteq \mathrm{cl}\, F_i\subseteq F_{i+1}$ for $0\leq i\leq m$. If $F_0$ meets $\phi(H_m\times\{\gamma\})$ (recall that $H_m=\{(\alpha, \omega, m)\in J(X): \alpha\in\omega_1\}$) in an unbounded subspace, then by the same proof as in Case 2 of 2.2, $F$ meets $\phi(K_0\times\{\gamma\})$ (recall that $K_0=\{(\omega_1, \beta, 0)\in J(X): \beta\in\omega\}$). Hence, $F$ meets $\phi(K_0\times\{\omega\})$, a subspace of $\phi(Y_\omega\cap S_m)$, a contradiction. Thus, $F_0\cap\phi(H_m\times\{\gamma\})$ is a bounded set. Thus, there is a bounded subset $C\subseteq H_m$ (the closure of a countable union of bounded subsets of $H_m$ is bounded) such that

$$F_0\cap\phi(H_m\times(\omega+1))\subseteq\phi(C\times(\omega+1)).$$

So, there is a $F'\in\mathcal{F}$ such that $F'\subseteq F_0$ and $F'\cap\phi(H_m\times(\omega+1))=\emptyset$. Thus, $\mathcal{F}|F'$ is a free regular filter base on $W$ as $\phi(H_m\times(\omega+1))$ is the boundary of $\phi(S_m)$; this is a contradiction as $W$ is R-closed. This shows that $\phi(S_m)$ is R-closed for each even integer. Since $Z\setminus Y_\gamma$ is R-closed for $\gamma \in \omega$, then $\phi(Z\setminus Y_\gamma) \cup \phi(S_m)$ is R-closed. But $\{W\setminus(\phi(Z\setminus Y_\gamma) \cup \phi(S_m))$: even $m \in \omega\}$ is a neighborhood base for $(p_+, \gamma)$. Also, for $m \in \omega$, $\bigcup\{Y_\gamma: 0 \leq \gamma \leq m\}$ is R-closed and

$$\{W\setminus(\phi(S_m) \cup \phi(\bigcup\{Y_\gamma: 0 \leq \gamma \leq m\})): \text{even } m \in \omega\}$$

is a neighborhood base for $(p_+, \omega)$.

Finally, we will show that $W$ is not locally R-closed at $(p_-, \omega)$. Let $U$ be a R-closed neighborhood of $(p_-, \omega)$ such that $U \subseteq \phi(S_{-2})$ (see above Case 3 for definition of $S_m$). There is $m \in \omega$ such that $\phi(X_{-m} \times \{m\}) \subseteq U \cap \phi(Y_m)$. Also, $U \cap \phi(Y_m)$ is clopen in $U$ and, hence, R-closed. Thus, $U \cap \phi(Y_m)$ is homeomorphic to an R-closed subspace $D$ of $J(X)\setminus\{p_+\}$. By 2.5, $D$ is compact; however, $D$ contains a closed, nonnormal subspace, namely, $\phi(X_{-m} \times \{m\})$, a contradiction. So, $(p_-, \omega)$ does not have a neighborhood base of R-closed subspaces.

## 3. Embeddings

In this section, every regular space is shown to be embeddable in a minimal regular space; this result solves Problems 8 and 9 (the non-dense part) in [2].

Let $X$ be an infinite regular space. Overall, the plan is simple – $X$ is embedded in a regular space $Y$ such that every element of $\mathcal{N} = \{\mathcal{G}: \mathcal{G}$ is a non-convergent regular filter on $X$ with a unique adherent point$\}$ extends to a regular filter with *two* adherent points in $Y$. At the same time this is accomplished, no new non-convergent regular filters on $Y\setminus X$ with unique adherent points are introduced. Thus, $X$ is embedded in a minimal regular space $Y$. As the construction is rather long and complicated, a quick sketch is presented here. First, $X$ is embedded in a space $Y_1$ which is $X$ plus many spokes coming out of $X$, one for each element of $\mathcal{N}$; each spoke is attached to $X$ via a gluing space. In constructing $Y_1$, two new sets of problem regular filters are introduced; one set arises from the spokes and the other from the gluing spaces. The problem filters arising from the spokes are handled by compactifying the spokes and obtaining a space $Y_2$ containing $Y_1$. Unfortunately, $Y_2$ may not be regular; so, a regular subspace $Y_3$ of $Y_2$ is selected so that $X$ is a subspace of $Y_3$. Finally, the problem filters arising from the gluing spaces are handled by compactifying; the resulting space $Y$ is minimal regular and contains $X$ as a subspace.

*Preliminaries.* For each $\mathcal{G} \in \mathcal{N}$, there is regular-closed neighborhood $V_\mathcal{G}$ of the unique adherent point of $\mathcal{G}$ such that $G\setminus V_\mathcal{G} \neq \emptyset$ for all $G \in \mathcal{G}$. Let $\mathcal{M} = \{$filter generated by $\mathcal{G}|(X\setminus V_\mathcal{G})$: $\mathcal{G} \in \mathcal{N}\}$. So, $\mathcal{M}$ is a family of free filters (not necessarily regular) on $X$. Let $m = |\mathcal{M}|$, $\mathcal{M} = \{\mathcal{F}_\alpha: \alpha < m\}$, and $\kappa = [|X| \cdot m]^+$. The space $T = (\kappa+1) \times (\kappa+1)\setminus\{(\kappa, \kappa)\}$ is a non-normal, locally compact space in which for every pair of disjoint regular-closed sets, one is compact. Using Jones' machinery, $S = DJ(T)$ is a minimal regular space. For each $\alpha \in m$, a copy of $S$ (the spoke) will be attached to $X$ via a gluing space. Let $D^* = \{d_\alpha : \alpha \leq \kappa\}$ be homeomorphic to $\kappa + 1$, $D = \{d_\alpha : \alpha < \kappa\}$, and $E = \{(\kappa, \beta, 0) \in S : \beta < \kappa\}$.

*Construction of* $Y_1$. Let

$$Y_1 = X \cup (X \times D \times m) \cup [(\kappa+1) \times D^* \backslash \{(\kappa, d_\kappa)\}] \times m \cup S \times m$$

where for each $\alpha \in m$ the corresponding points of $\kappa \times \{d_\kappa\} \times \{\alpha\}$ and $E \times \{\alpha\}$ are identified, i.e., for $\beta \in \kappa$, $\{(\beta, d_\kappa, \alpha), (\kappa, \beta, 0, \alpha)\}$ is identified to a point. So, for each $\alpha \in m$,

$$Z_\alpha = X \times D \times \{\alpha\} \cup [(\kappa+1) \times D^* \backslash \{(\kappa, d_\kappa)\}] \times \{\alpha\} \cup S \times \{\alpha\}$$

is attached to $X$. The $S \times \{\alpha\}$ is the spoke and

$$X \times D \times \{\alpha\} \cup [(\kappa+1) \times D^* \backslash \{(\kappa, d_\kappa)\}] \times \{\alpha\}$$

is the gluing space. A subset $U \subseteq X \cup Z_\alpha$ is open iff

(i) $U \cap X$ is open in $X$,

(ii) there is an open interval $C = (d_\delta, d_\kappa)$ in $D$ for some $\delta < \kappa$ such that $(U \cap X) \times C \times \{\alpha\} \subseteq U$ and if $U \cap X \in \mathscr{F}_\alpha$, then $(\delta, \kappa] \times (C \cup \{d_\kappa\}) \times \{\alpha\} \subseteq U$ and

$$[\{p_+, p_-\} \cup ((\delta, \kappa] \times (\delta, \kappa] \backslash \{(\kappa, \kappa)\}) \times \mathbb{Z}] \times \{\alpha\} \subseteq U,$$

(iii) for $x \in X$, $U \cap (\{x\} \times D \times \{\alpha\})$ is open in $\{x\} \times D \times \{\alpha\}$,

(iv) if $(\kappa, d_\beta, \alpha) \in U$, there is $V \in \mathscr{F}_\alpha$ such that $V \times \{d_\beta\} \times \{\alpha\} \subseteq U$,

(v) $U \cap (S \times \{\alpha\})$ is open in $S \times \{\alpha\}$, and

(vi) $U \cap (\kappa+1) \times D^* \times \{\alpha\} \backslash \{(\kappa, d_\kappa, \alpha)\}$ is open in $(\kappa+1) \times D^* \times \{\alpha\} \backslash \{(\kappa, d_\kappa, \alpha)\}$.

Now, $U \subseteq Y_1$ is defined to be open if $U \cap (X \cup Z_\alpha)$ is open in $X \cup Z_\alpha$ for each $\alpha \in m$. The new space $Y_1$ isolates and separates the problem filters in $\mathscr{M}$. It is straightforward to show that $Y_1$ is regular at all points of $Y_1 \backslash (X \cup \{p_+, p_-\} \times m)$.

*Construction of* $Y_2$. Let $Y_2 = Y_1 \cup (S \times \beta m \backslash m)$ where $\beta m$ is the Stone–Čech compactification of $m$ with the discrete topology. Note that $S \times m \subset Y_2$. Before the topology on $Y_2$ is defined, two other sets, $A_U$ and $\hat{U}_{n,\delta}$, must be defined first. Let $U$ be an open subset of $X$. Define $A_U = \{\alpha \in m : U \in \mathscr{F}_\alpha\}$, and for $\delta \in \kappa$ and $n$ a positive integer define

$$\begin{aligned}\hat{U}_{n,\delta} = {} & U \cup (U \times (d_\delta, d_\kappa) \times m) \cup ((\delta, \kappa] \times (d_\delta, d_\kappa] \times A_U) \\ & \cup [(\delta, \kappa] \times (\delta, \kappa] \backslash \{(\kappa, \kappa)\} \times [-2n, 2n+1] \times \mathrm{cl}_{\beta m} A_U] \\ & \cup [(S \backslash \bigcup \{T_k : -2n \leq k \leq 2n+1) \times \mathrm{cl}_{\beta m} A_U] \\ & \cup [(\delta, \kappa] \times \kappa \times \{-2n, 2n+1\} \times \mathrm{cl}_{\beta m} A_U]\end{aligned}$$

where $T_k$ is the $k$th plank of $S$. Now, define $V \subseteq Y_2$ to be open iff $V \cap Y_1$ is open in $Y_1$, $V \cap (S \times \beta m)$ is open in $S \times \beta m$, and there are $\delta \in \kappa$ and $n \in \omega$ such that $(V \cap X)\hat{}_{n,\delta} \subseteq V$. An examination of $\hat{U}_{n,\delta}$ for $n > 0$, $\delta \in \kappa$ and $U$ open in $X$ reveals that $\hat{U}_{n,\delta} \backslash X$ is clopen in $Y_2 \backslash X$. Unfortunately, $Y_2$ is not necessarily regular; so, a subspace $Y_3$ of $Y_2$ is defined.

*Construction of* $Y_3$. Let $U \subseteq X$ be open, $A \subseteq m$, $\delta < \kappa$, and $n \in \omega$. Note that $\hat{U}_{n,\delta} \cap S \times \mathrm{cl}_{\beta m} A = \emptyset$ iff $\mathrm{cl}_{\beta m} A \cap \mathrm{cl}_{\beta m} A_U = \emptyset$ iff $A \cap A_U = \emptyset$. Define $R = \{t \in \beta m \setminus m$: for each $A \in t$, there is an $x \in X$ such that for all open $U \subseteq X$ with $x \in U$, $A_U \cap A \neq \emptyset\}$. It is easy to show that (1) $R$ is closed in $\beta m$ and (2) if $t \in R$, $A \in t$, and $x$ is a point of $X$ such that $A_U \cap A \neq \emptyset$ whenever $x \in U \subseteq X$ and $U$ is open, then $A_U \cap A$ is infinite. Let $F = \{p_+, p_-\} \times R$, and define $Y_3 = Y_2 \setminus F$ and $H = Y_3 \setminus (X \cup S \times \beta m)$. Note that $Y_3 \setminus X$ is a regular space.

**3.1.** $Y_3$ *is regular.*

**Proof.** It is straightforward to show that $Y_3$ is regular at points of $Y_3 \setminus (X \cup (\{p_+, p_-\} \times \beta m))$. So, let $y \in (X \cup \{p_+, p_-\} \times \beta m) \cap Y_3$.

*Case 1.* $y = x \in X$. Now a neighborhood base for $x$ is $\mathcal{U}_x = \{\hat{U}_{n,\delta}$: $n \in \omega$, $\delta \in \kappa$, $U$ is an open neighborhood of $x$ in $X\}$. Let $U$, $V$ be open subsets of $x$ in $X$ such that $\mathrm{cl}_X U \subseteq V$ and $n \in \omega$, $\delta \in \kappa$. It suffices to show that $\mathrm{cl}_{Y_3} \hat{U}_{n+1,\delta} \subseteq \hat{V}_{n,\delta}$. This will follow from the fact that $\mathrm{cl}_{Y_3} \hat{U}_{n+1,\delta} \subseteq \hat{U}_{n,\delta} \cup \mathrm{cl}_X U$. Clearly, $\mathrm{cl}_{Y_3} \hat{U}_{n+1,\delta} \setminus X \subseteq \hat{U}_{n,\delta}$. Let $z \in W = X \setminus \mathrm{cl}_X U$. Then $A_U \cap A_W = \emptyset$ implying $\hat{U}_{n+1,\delta} \cap \hat{W}_{0,0} = \emptyset$. So, $z \notin \mathrm{cl}_{Y_3} U_{n+1,\delta}$.

*Case 2.* $y = (p_+, t) \in Y_3$ (the proof for the case $y = (p_-, t)$ is similar). Now, $t \in \beta m \setminus R$. So, there is $A \in t$ such that for each $x \in X$, there exists an open set $U_x$ of $x$ in $X$ so that $A_{U_x} \cap A = \emptyset$. Since $[(S \times \mathrm{cl}_{\beta m} A) \setminus F] \cap (\hat{U}_x)_{0,0} = \emptyset$ and $(p_+, t)$ has a regular neighborhood base in $(S \times \mathrm{cl}_{\beta m} A) \setminus F$, then it follows that $Y_3$ is regular at $(p_+, t)$.

**3.2.** *If* $U_1$ *and* $U_2$ *are open subsets of* $Y_3$ *such that* $\mathrm{cl}_{Y_3} U_1 \subseteq U_2 \subseteq \mathrm{cl}_{Y_3} U_2 \subseteq H$, *then there is a clopen subset* $C$ *of* $Y_3$ *such that* $U_1 \subseteq C \subseteq H$.

**Proof.** Since $\mathrm{cl}_{Y_3} U_2 \cap X = \emptyset$, then for $\alpha \in m$ and $x \in X$, there are open $W_x$ of $x$ in $X$ and $\delta_x \in \kappa$ such that

$$\mathrm{cl}_{Y_3} U_2 \cap (W_x \times (d_{\delta_x}, d_\kappa) \times \{\alpha\}) = \emptyset.$$

Let $\alpha' = \sup\{\delta_x : x \in X\}$. Then

$$(X \times (d_{\alpha'}, d_\kappa) \times \{\alpha\}) \cap \mathrm{cl}_{Y_3} U_2 = \emptyset.$$

Thus, $(\{\kappa\} \times (d_{\alpha'}, d_\kappa) \times \{\alpha\}) \cap \mathrm{cl}_{Y_3} U_1 = \emptyset$. Since $\mathrm{cl}_{Y_3} U_2 \cap E \times \{\alpha\} = \emptyset$, there is a $\alpha'' < \kappa$ such that

$$((\kappa + 1) \times (d_{\alpha''}, d_\kappa) \times \{\alpha\}) \cap \mathrm{cl}_{Y_3} U_1 = \emptyset.$$

Let $\beta = \sup\{\alpha', \alpha'' : \alpha \in m\}$. Then

$$C = (X \times [d_0, d_\beta] \times m) \cup ((\kappa + 1) \times [d_0, d_\beta] \times m)$$

is clopen in $Y_3$ and $\mathrm{cl}_{Y_3} U_1 \subseteq C \subseteq H$.

*Construction of* $Y$. Let $\mathcal{C} = \{C \subseteq H$: $C$ is clopen in $Y_3\}$. Let $Y_4 = Y_3 \cup \beta H$ (assuming that $\beta H \setminus H \cap Y_3 = \emptyset$), and define $V \subseteq Y_4$ to be open if $V \cap Y_3$ is open in $Y_3$ and

$V\cap\beta H$ is open in $\beta H$. Consider the subspace $Y=Y_3\cup\bigcup\{\mathrm{cl}_{\beta H}C: C\in\mathscr{C}\}$. If $U$ is open in $X$, $0<n\in\omega$, and $\delta<\kappa$, let $U'_{n,\delta}=\hat{U}_{n,\delta}\cup\bigcup\{\mathrm{cl}_{\beta H}(\hat{U}_{n,\delta}\cap C): C\in\mathscr{C}\}$. It is straightforward to show that $Y$ is regular. Also since $\hat{U}_{n,\delta}\backslash X$ is clopen in $Y_2\backslash X$ it is easy to check that $U'_{n,\delta}\backslash X$ is clopen in $Y\backslash X$. Let $H'=Y\backslash(X\cup S\times\beta m)$.

**3.3.** *Let $\mathscr{D}$ be a regular filter on $Y$.*

(a) *If $\mathscr{D}$ traces on $X$ and $\mathscr{D}|_X$ is not convergent on $X$, then* $\mathrm{ad}_Y\mathscr{D}$ *contains at least two points.*

(b) *If $H'\in\mathscr{D}$ and $\mathscr{D}$ has at most one adherent point in $Y$, then $\mathscr{D}$ converges in $Y$ to some point in $H'$.*

**Proof.** (a) Suppose that $|\mathrm{ad}_X(\mathscr{D}|_X)|<2$ and that $x\in X$ is not an adherent point of $\mathscr{D}|_X$. Let $\mathscr{E}=\mathscr{D}|_X$ and $\mathscr{N}_x$ be the neighborhood filter of $x$ in $X$. If $\mathscr{E}$ has no adherent point, let $\mathscr{G}=\{K\cup N: K\in\mathscr{E}, N\in\mathscr{N}_x\}$. Otherwise let $\mathscr{G}=\mathscr{E}$. Then $\mathscr{G}$ is a non-convergent filter on $X$ with a unique adherent point and $\mathscr{G}\in\mathscr{N}$. Let $\mathscr{F}$ be the filter on $X$ generated by $\mathscr{G}|_{(X\backslash V_{\mathscr{G}})}$. Then $\mathscr{F}\in\mathscr{M}$ and $\mathscr{F}=\mathscr{F}_\alpha$ for some $\alpha\in m$. Now, $\mathscr{F}_\alpha\supset\mathscr{D}|_X$. If $U\in\mathscr{D}$ is open, then for some $n\in\omega$ and $\delta<\kappa$, $(U\cap X)'_{n,\delta}\subseteq U$. But $U\cap X\in\mathscr{F}_\alpha$ implies $\{(p_+,\alpha),(p_-,\alpha)\}\subseteq(U\cap X)'_{n,\delta}\subseteq U$. Thus $\mathrm{ad}_Y\mathscr{D}\supseteq\{(p_+,\alpha),(p_-,\alpha)\}$.

(b) There are open sets $U_1,U_2\in\mathscr{D}$ such that $\mathrm{cl}_YU_1\subseteq U_2\subseteq\mathrm{cl}_YU_2\subseteq H'$. Now, $\mathrm{cl}_YU_i=\mathrm{cl}_Y(U_i\cap H)$ for $i=1,2$ and

$$\mathrm{cl}_{Y_3}(U_1\cap H)\subseteq U_2\cap H\subseteq\mathrm{cl}_{Y_3}(U_2\cap H)\subseteq H.$$

By 3.2, there is a $C\in\mathscr{C}$ such that $\mathrm{cl}_{Y_3}(U_1\cap H)\subseteq C$ implying

$$\mathrm{cl}_Y(U_1)=\mathrm{cl}_Y(U_1\cap H)\subseteq\mathrm{cl}_YC=\mathrm{cl}_{\beta H}C$$

is compact. Since $\mathscr{D}$ contains a compact set, $\mathrm{cl}_YC$, $\mathscr{D}$ converges to a point in $\mathrm{cl}_YC\subseteq H'$.

**Theorem 3.4.** *$Y$ is a minimal regular space containing $X$ as a closed subspace.*

**Proof.** Clearly $X$ is a closed subspace of $Y$. To show $Y$ is minimal regular, let $\mathscr{D}$ be a regular filter on $Y$ with at most one adherent point. Let $y$ be the adherent point of $\mathscr{D}$ if it exists.

*Case 1.* $\mathscr{D}$ misses $X\cup((S\times\beta m)\backslash F)$. This is clearly equivalent to assuming $H'\in\mathscr{D}$. Therefore, by 3.3(b) $\mathscr{D}$ converges to $y$ in $H'$.

*Case 2.* $\mathscr{D}$ misses $X$ but traces on $(S\times\beta m)\backslash F$. Choose $W,V\in\mathscr{D}$ such that $\mathrm{cl}_YW\subseteq V$ and $\mathrm{cl}_YV\cap X=\emptyset$. Let $A=\{\alpha\in m: (p_+,\alpha)\in V$ or $(p_-,\alpha)\in V\}$. For each $x\in X$, choose a neighborhood $U(x)$ of $x$ such that $(U(x))'_{n,\delta}\cap W=\emptyset$ for some $\delta\in\kappa$ and $n\in\omega\backslash\{0\}$. Therefore, for $(p_+,t)\in F$ or $(p_-,t)\in F$, $A\notin t$; in particular, $(S\times\mathrm{cl}_{\beta m}A)\cap F=\emptyset$. Also, note that if $(p_+,t)\in\mathrm{cl}_YW$, then $(p_+,t)\in V$ implying $((\bigcup\{T_k: k\geq l\})\cup\{p_+\})\times B\subseteq V$ for some $l\in\omega$ and $B\in t$. But since $B\in t$ and $B\subseteq A$, then $A\in t$. Likewise if $(p_-,t)\in\mathrm{cl}_YW$, then $A\in t$. Now, we will show that, for some

$N \in \omega \backslash \{0\}$,

$$(*) \qquad \text{cl}_{S\times\beta m}(W \cap (S \times \beta m)) \subseteq (\bigcup\{T_k : -N < k < N\}) \times \beta m \cup S \times \text{cl}_{\beta m} A.$$

Assume that (*) is not true. Then, by symmetry, we can assume for each $i \in \omega$ that

$$W \cap [(\bigcup\{T_k : k > i\}) \times (\beta m \backslash \text{cl}_{\beta m} A)] \neq \emptyset.$$

Since $W$ is open in $S \times \beta m$, $W \cap [(\bigcup\{T_k : k > i\}) \times (m \backslash A)] \neq \emptyset$. For each $i \in \omega$, choose some $\alpha_i \in m \backslash A$ such that $W \cap [(\bigcup\{T_k : k > i\}) \times \{\alpha_i\}] \neq \emptyset$. Let $B = \{\alpha_i : i \in \omega\}$. If $B$ is finite, then for some $j \in \omega$, $(p_+, \alpha_j) \in \text{cl}_Y W \subseteq V$ where $\alpha_j \in B \backslash A$; however, this is contrary to the definition of $A$. So $B$ is infinite and there is some $t \in \text{cl}_{\beta m} B \backslash B$. Thus $(p_+, t) \in \text{cl}_{S\times\beta m}(W \cap (S \times \beta m))$. If $(p_+, t) \in \text{cl}_Y W$, then by the comment preceding (*), $A \in t$ which is impossible as $B \in t$ and $B \subseteq m \backslash A$. So, $(p_+, t) \in F$. Hence, there is some $x \in X$ such that for all neighborhoods $V$ of $x$ in $X$, $A_V \cap B \neq \emptyset$ which implies $(U(x))'_{n,\delta} \cap W \neq \emptyset$, a contradiction. Therefore, (*) is true for some $N \in \omega \backslash \{0\}$. This shows that $\text{cl}_{S\times\beta m}(W \cap (S \times \beta m)) \subseteq (S \times \beta m) \backslash F$ and that $\mathscr{E} = \mathscr{D}|_{S\times\beta m}$ is a regular filter base on $S \times \beta m$.

As $S \times \beta m$ is minimal regular and $\mathscr{D}$ has at most one adherent point, $\mathscr{E}$ converges to a point $y$ in $(S \times \beta m) \backslash F$. Suppose, in order to get a contradiction, that $\mathscr{D}$ does not converge to $y$ in $Y$. There is, therefore, a neighborhood $W$ of $y$ such that $V \backslash W \neq \emptyset$ for all $V \in \mathscr{D}$. If $y \in E \times \beta m$ (= boundary in $Y$ of $(S \times \beta m) \backslash F$), then we may assume that $W$ is clopen because each point of $E \times \beta m$ has a clopen neighborhood base. On the other hand if $y \in \text{int}_Y((S \times \beta m) \backslash F)$, then we may suppose that $\text{cl}_Y W \subset (S \times \beta m) \backslash F$. In this case, since $\mathscr{D}|_{S\times\beta m}$ converges to $y$ in $S \times \beta m$, there is a $V \in \mathscr{D}$ such that $V \cap S \times \beta m \subset W$. This implies that $V \backslash W = V \backslash (S \times \beta m)$ and is open in $Y$. In either of the above cases, $\{V \backslash W : V \in \mathscr{D}$ and $V \cap (S \times \beta m) \subset W\}$ is a base for a regular filter $\mathscr{D}_1$ on $Y$ and $H' \in \mathscr{D}_1$. By Case 1, $\mathscr{D}_1$ has an adherent point distinct from $y$ which contradicts that $\mathscr{D}$ has a unique adherent point.

*Case 3.* $\mathscr{D}$ traces on $X$. By 3.3(a) $\mathscr{D}|_X$ converges in $X$ to $y \in X$. Let $U'_{n,\delta}$ be an arbitrary neighborhood of $y$ in $Y$ and suppose that for each $V \in \mathscr{D}$, $V \backslash U'_{n,\delta} \neq \emptyset$. However, there is a $V \in \mathscr{D}$ such that $V \backslash U'_{n,\delta} \cap X = \emptyset$ since $\mathscr{D}|_X$ converges. Since $U'_{n,\delta} \backslash X$ is clopen in $Y \backslash X$, it follows that $\mathscr{D}_1 = \{W \backslash U'_{n,\delta} : W \in \mathscr{D}$ and $W \backslash U'_{n,\delta} \cap X = \emptyset\}$ is a regular filter base in $Y$.

Since $\mathscr{D}_1$ misses $X$, Cases 1 and 2 imply that $\mathscr{D}_1$ has an adherent point in $Y \backslash X$. This, of course, contradicts that $\mathscr{D}$ has a unique adherent point; hence, $\mathscr{D}$ converges in $Y$.

## 4. Unsolved problems

Theorem 3.4 completely solves the non-dense embedding problem for regular spaces in R-closed spaces. The dense embedding problem has been solved but not in terms of a topological property. That is, Harris [6] has found a necessary and sufficient condition, in terms of generalized proximities, for a regular space to be densely embeddable in an R-closed space. On the other hand, Porter and Votaw [14] have shown that every regular space can be densely embedded in a regular space which is nearly R-closed. However, neither solution is topological.

**Problem 4.1.** Find a topological characterization of those regular spaces that can be densely embedded in R-closed spaces.

One obvious class of regular spaces that can be densely embedded in R-closed spaces is the class of spaces that are the topological sum of a Tychonoff space and an R-closed space; the authors are unaware of any other general class of regular spaces that can be densely embedded in R-closed spaces.

A regular space with an open basis of sets with R-closed boundaries is called *rim R-closed*. The results in the second half of the first section show that rim R-closed, R-closed spaces are especially nice. It seems natural to form the next problem.

**Problem 4.2.** Prove or disprove that a rim R-closed space can be densely embedded in an R-closed space.

Of course, a solution of Problem 4.1 should yield a solution to Problem 4.2. Another related problem about rim R-closed is the following:

**Problem 4.3.** Prove or disprove there exists a noncompact, rim R-closed, R-closed space.

An affirmative answer to a problem by Banaschewski [1] (i.e., the existence of a noncompact space in which every closed set is R-closed) would certainly solve Problem 4.3.

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