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The fundamental progroupoid of a general topos

Eduardo J. Dubuc

Departmento de Mathematica, Universidad de Buenos Aires, Pabellion 1 - Ciudad Universitaria, 1428 Buenos Aires, Argentina

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ABSTRACT

It is well known that the category of covering projections (that is, locally constant objects) of a locally connected topos is equivalent to the classifying topos of a strict progroupoid (or, equivalently, a localic prodiscrete groupoid), the *fundamental progroupoid*, and that this progroupoid represents first degree cohomology. In this paper we generalize these results to an arbitrary topos. The fundamental progroupoid is now a localic progroupoid, and cannot be replaced by a localic groupoid. The classifying topos is no longer a Galois topos. Not all locally constant objects can be considered as covering projections. The key contribution of this paper is a novel definition of covering projection for a general topos, which coincides with the usual definition when the topos is locally connected. The results in this paper were presented in a talk at the Category Theory Conference, Vancouver, July 2004.

0. Introduction

It is well known that if $\mathcal{E}$ is a locally connected topos then the category of covering projections (that is, locally constant objects) is equivalent to the classifying topos of a progroupoid (or, equivalently, a localic prodiscrete groupoid), the *fundamental progroupoid* $\pi(\mathcal{E})$, and that this progroupoid represents first degree cohomology. In this paper we generalize these results to an arbitrary topos.

The subject that concern us here was developed (in the context of a Grothendieck topos over $\mathcal{S}ets$) for a pointed locally connected topos by Grothendieck–Verdier in a series of commented exercises in Expose IV of the SGA4 [1], and by Artin–Mazur [2]. Later Moerdijk [15] treats the subject over a general base topos $\mathcal{S}$, and replace *progroups* by *prodiscrete localic groups*. Bunge [3] does the unpointed case and works with *prodiscrete localic groupoids*. See also Bunge–Moerdijk [4], and the Appendix in Dubuc [6] for a resume of this theory.

The salient feature of the theory is that covering projections are considered as a *full* subcategory of the topos, and this fact is essential in the proofs of the validity of the statements. Covering projections cannot be considered as a full subcategory when the topos is not locally connected. And even more, not all locally constant objects should be considered as covering projections.

The principal source of inspiration for our work was the paper of Hernandez Paricio [8], where he treats successfully the case of non-locally connected topological spaces. We interpret his work in terms of descent theory in the Appendix.

Given any topos, there is no problem to construct the topos of locally constant objects trivialized by a (fixed) cover. The problem is that when the topos is not locally connected, the resulting topos is not atomic because it fails to be both locally connected and boolean.

In Section 1 we prepare the ground for our work by explicitly establishing an equivalence between the usual definition of locally constant object and a certain descent datum. Section 2 contains the key contribution of this paper, which is a novel definition of covering projection for a general topos (Definition 2.12). When the topos is locally connected, every locally

E-mail address: edubuc@dm.uba.ar.

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constant object is a covering projection in our sense. We construct the topos of covering projections (corresponding to a fixed cover) and show that it is atomic (Theorem 2.19). The resulting localic groupoid (Theorem 2.20) appears to be the first genuine application of Joyal–Tierney results [10, VIII, 3. Theorem 1] to the Galois theory of locally constant objects. Here for the first time a non-prodiscrete localic groupoid appears in this theory, as well as an atomic topos which is not a Galois topos. In Sections 3, 4 and 5 we show that our notion of covering projection is well behaved and adequate to a treatment with inverse limit of topoi techniques. We construct the category of all covering projections, the topos it generates, and the fundamental (in this case localic) progroupoid. We show an equivalence between the classifying topos of this progroupoid and the topos of covering projections (Theorem 5.5). Finally, in Section 6 we prove that torsors (for a discrete group) are covering projections in our sense, and that the fundamental localic progroupoid represents first degree cohomology (Theorem 6.6).

Comparison between the locally connected, spatial and general cases.

In the case of locally connected topoi, given a (fixed) cover, the points of the topos of covering projections are essential, and the corresponding groupoid is an ordinary (discrete) groupoid. This determines a fundamental ordinary progroupoid. The transition morphisms are surjective on triangles, a fact that allows us to replace this progroupoid by a prodiscrete localic groupoid. Equivalent to this, the transition morphisms between the topoi are connected. This implies that the classifying topos is a Galois Topos.

In the case of a non-locally connected topological space, given a (fixed) cover, the corresponding groupoid is still discrete, and we still have an ordinary fundamental progroupoid. However, it cannot be replaced by a localic groupoid because the transition morphisms are not surjective on triangles, or, equivalently, the transition morphisms between the topoi are not connected. This implies that the classifying topos is not locally connected, thus no longer a Galois topos.

In the case of a general topos, given a (fixed) cover, the points of the topos of covering projections are not essential, and the corresponding groupoid is a localic (non-discrete) groupoid. This determines a fundamental localic progroupoid. The topoi in the system are (of course) atomic, but no longer Galois.

Context. Throughout this paper $\mathcal{S} = Sets$ denotes the topos of sets. However, we argue in a way that should be valid if $\mathcal{S}$ is an arbitrary Grothendieck topos, but let the interested reader verify this. All topoi $\mathcal{E}$ are assumed to be Grothendieck topoi (over $\mathcal{S}$), the structure map will be denoted by $\gamma : \mathcal{E} \to \mathcal{S}$ in all cases.

Recall that a geometric morphism $\mathcal{E} \xrightarrow{f} \mathcal{F}$ is said to be *essential* if the inverse image functor f^* itself has a left adjoint $f_!$, *locally connected* when $f_!$ is $\mathcal{F}$-indexed, *connected* if the inverse image functor f^* is full and faithful, and *atomic* if f^* is logical. A topos is said to be locally connected, connected, or atomic, when the structure morphism is so. A topos is atomic if and only if it is locally connected and boolean. We refer to [1] Expose IV, 4.3.5, 4.7.4, 7.6 and 8.7., [13], and [10].

1. Locally constant objects and descent data

By a *cover* $\mathcal{U} = \{U_i\}_{i\in I}$ in a topos $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$ we mean an epimorphic family $U_i \to 1$ in $\mathcal{E}$, $I \in \mathcal{S}$. As usual, this is an alternative notation for a map $\zeta \colon U \to \gamma^* I$, with $U \to 1$ epimorphic. Notice that covers are 3-tuples $\mathcal{U} = (U, I, \zeta)$.

Assumption 1.1. We assume that $U_i \neq \emptyset\ \forall i \in I$.

The concept of *locally constant object* is a direct translation into the topos context of the classical notion of *covering projection* [17, Ch. 2, Sec. 1]. It is defined in SGA4 Expose IX, 2.0.

Definition 1.2. Given a topos $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$ and a cover $\mathcal{U} = \{U_i\}_{i\in I}$, a *trivialization* of an object $X \in \mathcal{E}$ is a family of sets $\{S_i\}_{i\in I}$ together with isomorphisms in $\mathcal{E}$, $\{\gamma^* S_i \times U_i \xrightarrow{\theta_i} X \times U_i\}_{i\in I}$ over U_i. Alternatively, it is an arrow $S \to I$ in $\mathcal{S}$, and an isomorphism $\theta \colon \gamma^* S \times_{\gamma^* I} U \to X \times U$ over U. We say that X is $\mathcal{U}$-split by the trivialization.

Trivializations will be denoted by the letter θ in all cases.

Definition 1.3. An object X in a topos $\mathcal{E}$ is *locally constant* if it is $\mathcal{U}$-split for some cover $\mathcal{U}$ in $\mathcal{E}$.

In [3] Bunge introduces a push-out of topoi whose underline category is the category of locally constant objects split by a (fixed) cover, and whose arrows are maps which preserve the trivializations.

Definition 1.4 (*M. Bunge*). Given a cover $\zeta \colon U \to \gamma^* I$ in a topos $\mathcal{E}$, the category $\mathcal{P}_{\mathcal{U}}$ of locally constant objects split by $\mathcal{U}$ is given by the following push-out topos:

$$\begin{tikzcd} \mathcal{E}_{/U} \arrow[r, "\varphi_{\mathcal{U}}"] \arrow[d, "\rho_{\mathcal{U}}"] & \mathcal{E} \arrow[d, "\upsilon_{\mathcal{U}}"] \\ \mathcal{S}_{/I} \arrow[r, "\varrho_{\mathcal{U}}"] & \mathcal{P}_{\mathcal{U}} \end{tikzcd}$$

where $\rho_{\mathcal{U}}$ and $\varphi_{\mathcal{U}}$ are given by $\rho^*_{\mathcal{U}}(S \to I) = \gamma^* S \times_{\gamma^* I} U$ and $\varphi^*_{\mathcal{U}}(X) = X \times U$. By the constructions of push-outs of topoi, the category $\mathcal{P}_{\mathcal{U}}$ is the following:

Objects: $(X, S \to I, \theta)$, is a 3-tuple, with $X \in \mathcal{E}$, $S \to I \in \mathcal{S}_{/U}$ and $X \times U \xrightarrow{\theta} \gamma^* S \times_{\gamma^* I} U$ an isomorphism over U.

Arrows: $(X, S \to I, \theta) \to (Y, T \to I, \theta)$, is a pair of morphisms $X \xrightarrow{f} Y \in \mathcal{E}$, $S \xrightarrow{\vartheta} T \in \mathcal{S}$ over I, $\{S_i \xrightarrow{\vartheta_i} T_i\}$, compatible with the trivialization data:

$$\begin{tikzcd} \gamma^* S_i \times U_i \arrow[r, "\theta_i"] \arrow[d, "\gamma^*\vartheta_i \times U_i"] & X \times U_i \arrow[d, "f \times U_i"] \\ \gamma^* T_i \times U_i \arrow[r, "\theta_i"] & Y \times U_i \end{tikzcd}$$

The functors $\upsilon^*_{\mathcal{U}} : \mathcal{P}_{\mathcal{U}} \to \mathcal{E}$ and $p^*_{\mathcal{U}} : \mathcal{P}_{\mathcal{U}} \to \mathcal{S}_{/I}$ are the projections.

It is important to remark that the map $X \xrightarrow{f} Y$ completely determines the function $S \xrightarrow{\vartheta} T$. That is, the latter, if it exists, is unique. Arrows in $\mathcal{P}_{\mathcal{U}}$ can be considered as maps in $\mathcal{E}$ satisfying a condition. However, in spite of this uniqueness, we shall also say that ϑ lifts f to an arrow in $\mathcal{P}_{\mathcal{U}}$.

When the topos $\mathcal{E}$ is locally connected, we can assume that $\rho_{\mathcal{U}}$ is connected and locally connected (that is, all the objects U_i connected). It follows then that the point $\mathcal{S}_{/I} \xrightarrow{p_{\mathcal{U}}} \mathcal{P}_{\mathcal{U}}$ is locally connected and surjective, in particular, of effective descent [3]. Thus, $\mathcal{P}_{\mathcal{U}}$ is equivalent to the classifying topos of the (discrete because the point, being locally connected is representable) groupoid of automorphisms of $p_{\mathcal{U}}$ [10], in particular, $\mathcal{P}_{\mathcal{U}}$ is an atomic topos. It also follows that $\upsilon_{\mathcal{U}}$ is connected, thus $\mathcal{P}_{\mathcal{U}} \subset \mathcal{E}$ is a full subcategory via the inverse image functor $\upsilon^*_{\mathcal{U}}$ [4]. The reader can also check the appendix in [6], where all this is proved "by hand", without recourse to the results in [10].

The known theory for the locally connected case stops here for non-locally connected topoi. It is no longer possible to assume that $\rho_{\mathcal{U}}$ is connected and locally connected. Now $\mathcal{P}_{\mathcal{U}}$ is no longer a full subcategory of $\mathcal{E}$, and furthermore, the point $\mathcal{S}_{/I} \xrightarrow{p_{\mathcal{U}}} \mathcal{P}_{\mathcal{U}}$ fails to be of effective descent. We analyzed the situation and found that although $p_{\mathcal{U}}$ is surjective, the problem is that $\mathcal{P}_{\mathcal{U}}$ is not atomic because it fails to be both locally connected and boolean.

Trivializations versus descent data

A locally constant object together with a trivialization structure is the object constructed by descent on a certain descent datum associated to the Cech nerve of the covering. The descent data corresponding to locally constant objects form a topos equivalent to $\mathcal{P}_{\mathcal{U}}$. We found that an explicit use of this equivalence is not only technically useful, but also contributes to a better understanding of the concept of locally constant object (particularly in the case of covering projections of a topological space, see Appendix).

Consider a topos $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$ and a cover $\mathcal{U} = (U, I, \zeta)$, $\zeta : U \to \gamma^* I$, $I \in \mathcal{S}$, $U \in \mathcal{E}$.

Let $U_\bullet$ (resp. $I_\bullet$) be the simplicial object (resp. set) whose n-simplexes are given by $U_n = U \times U \times \cdots U$ (resp. $I_n = I \times I \times \cdots I$).

Cech nerve 1.5. Let $\mathcal{U}_\bullet \subset I_\bullet$ be the Cech nerve of $\mathcal{U}$, that is, the simplicial set $\mathcal{U}_n = \{(i_0, i_1, \ldots i_n) \mid U_{i_0} \times U_{i_1} \cdots \times U_{i_n} \neq \emptyset\}$ (notice that $\mathcal{U}_0 = I$).

Since the cover will remain fixed in this section, to simplify the notation we shall omit a subindex $\mathcal{U}$ on the arrows. The map $\zeta : U \to \gamma^* I$ induces a morphism of simplicial objects $\zeta_\bullet : U_\bullet \to \gamma^* I_\bullet$ which factors through $U_\bullet \to \gamma^* \mathcal{U}_\bullet \subset \gamma^* I_\bullet$. Actually, $\gamma^* \mathcal{U}_\bullet$ is the image of $\zeta_\bullet$. We abuse the notation and write $\zeta_\bullet : U_\bullet \to \gamma^* \mathcal{U}_\bullet$. This morphism determines a morphism of simplicial topoi $\rho_\bullet : \mathcal{E}_{/U_\bullet} \longrightarrow \mathcal{S}_{/\mathcal{U}_\bullet}$.

Consider the truncated simplicial topoi which determine the descent situations (see for example 3.2 *Descent.* in [14]):

$$\begin{tikzcd} \mathcal{E}_{/U\times U\times U} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\rho_2"] & \mathcal{E}_{/U\times U} \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[d, "\rho_1"] & \mathcal{E}_{/U} \arrow[r, "\varphi"] \arrow[d, "\rho"] & \mathcal{E} \arrow[d, "\upsilon"] \arrow[dr, bend left, "\mu"] & \\ \mathcal{S}_{/\mathcal{U}_2} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{S}_{/\mathcal{U}_1} \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & \mathcal{S}_{/I} \arrow[r, "\varrho"] \arrow[rr, bend right, "\delta"] & \mathcal{P}_{\mathcal{U}} \arrow[r] & \mathcal{D}_{\mathcal{U}} \end{tikzcd} \tag{1.6}$$

where $\mathcal{S}_{/I} \xrightarrow{\delta} \mathcal{D}_{\mathcal{U}}$ is defined to be the descent topos. It is well known that the morphism $\mathcal{E}_{/U} \xrightarrow{\varphi} \mathcal{E}$ is of effective descent. It follows the existence of the arrow $\mathcal{E} \xrightarrow{\mu} \mathcal{D}_{\mathcal{U}}$. $\mathcal{P}_{\mathcal{U}}$ is the push-out topos 1.4, and the morphism $\mathcal{P}_{\mathcal{U}} \to \mathcal{D}_{\mathcal{U}}$ follows by the universal property of this push-out. We shall now examine in more detail diagram (1.6).

Facts 1.7. (1) The morphism φ is given by

$$\varphi^*(X) = X \times U : \qquad \varphi^*(X) = \{X \times U_i\}_{i \in I}.$$

(2) The morphism ρ is given by

$$\rho^*(S \to I) = \gamma^* S \times_{\gamma^* I} U : \rho^*(\{S_i\}_{i\in I}) = \{\gamma^* S_i \times U_i\}_{i\in I}.$$
$$\rho_1^*(S \to \mathcal{U}_1) = \gamma^* S \times_{\gamma^* \mathcal{U}_1} (U \times U) :$$
$$\rho^*(\{S_{(i,j)}\}_{(i,j)\in \mathcal{U}_1}) = \{\gamma^* S_{(i,j)} \times (U_i \times U_j)\}_{(i,j)\in\mathcal{U}_1}.$$

(3) The morphism υ "forgets" the family and the trivialization, and is given by

$$\upsilon^*(X,\ S \to I,\ \theta) = X, \quad \upsilon^* \text{is a faithful functor.}$$

(4) The morphism ϱ "forgets" the object and the trivialization, and is given by

$$\varrho^*(X,\ S \to I,\ \theta) = S \to I, \quad \varrho^* \text{ is a faithful functor.}$$

(5) An object of $\mathcal{D}_\mathcal{U}$ is a pair $(S \to I,\ \lambda)$, where λ is a $\mathcal{U}_\bullet$-descent datum on the object $S \to I$ in the topos $\mathcal{S}_{/I}$. Such a descent datum is an isomorphism in the topos $\mathcal{S}_{/\mathcal{U}_1}$, and it consists of the following data:

bijections $\lambda_{ji} : S_i \to S_j,\ (i, j) \in \mathcal{U}_1,$ such that

$$\lambda_{ii} = id,\ i \in I,\ \lambda_{k,i} = \lambda_{kj} \circ \lambda_{ji},\ (i, j, k) \in \mathcal{U}_2.$$

(6) The morphism δ "forgets" the descent datum, and is given by

$$\delta^*(S \to I,\ \lambda) = S \to I, \quad \delta^* \text{ is a faithful functor.}$$

(7) An object in the image of the functor ρ^* is an object in the topos $\mathcal{E}_{/U}$ of the form $\gamma^* S \times_{\gamma^* I} U \longrightarrow U$, $\{\gamma^* S_i \times U_i \longrightarrow U_i\}_{i\in I}$. A $U_\bullet$-descent datum σ on such an object is an isomorphism in the topos $\mathcal{E}_{/U\times U}$, and it consists of a family of isomorphisms in $\mathcal{E}$, $\{\sigma_{j,i}\}_{(i,j)\in\mathcal{U}_1}$:

$$\begin{tikzcd}
\gamma^* S_i \times U_i \times U_j \arrow[r, "\sigma_{j,i}"] \arrow[d] & \gamma^* S_j \times U_j \times U_i \arrow[d] \\
U_i \times U_j \arrow[r, "\tau"] & U_j \times U_i
\end{tikzcd}$$

satisfying the appropriate identity and cocycle conditions (the arrow τ is the symmetry isomorphism). The commutativity of the square implies that $\sigma_{j,i}$ is completely determined by its first projection. We abuse the notation and write this projection with the same letter: $\gamma^* S_i \times U_i \times U_j \xrightarrow{\sigma_{j,i}} \gamma^* S_j$.

(8) Given a descent datum as in (5), the morphism of simplicial topoi $\rho_\bullet$ induces a descent datum as in (7):

$$\sigma = \rho_1^* \lambda, \qquad \sigma_{j,i} = \gamma^* \lambda_{j,i} \times \tau.$$

(9) The fact that $\mathcal{E}_{/U} \xrightarrow{\varphi} \mathcal{E}$ is of effective descent means (in particular) that given a descent datum as in (7), there exist a (unique up to isomorphism) object X in $\mathcal{E}$ together with an isomorphism $\gamma^* S \times_{\gamma^* I} U \xrightarrow{\theta} X \times U$ over U, $\{\gamma^* S_i \times U_i \xrightarrow{\theta_i} X \times U_i\}_{i\in I}$ (Thus, X is a locally constant object $\mathcal{U}$-split by a trivialization θ). Moreover, this isomorphism θ is compatible with the descent datum σ and the trivial descent datum on $X \times U \to U$, $\{X \times U_i \to U_i\}_{i\in I}$, given by $X \times U_i \times U_j \xrightarrow{X\times\tau} X \times U_j \times U_i$. That is:

$$\begin{tikzcd}
\gamma^* S_i \times U_i \times U_j \arrow[r, "\sigma_{j,i}"] \arrow[d, "\theta_i \times U_j"] & \gamma^* S_j \times U_j \times U_i \arrow[d, "\theta_j \times U_i"] \\
X \times U_i \times U_j \arrow[r, "X\times\tau"] & X \times U_j \times U_i
\end{tikzcd}$$

The object X corresponding to a descent datum as in (8) furnishes the inverse image for the morphisms $\mathcal{E} \to \mathcal{D}_\mathcal{U}$ and $\mathcal{P}_\mathcal{U} \to \mathcal{D}_\mathcal{U}$.

Proposition 1.8. *The push-out topos $\mathcal{P}_\mathcal{U}$ of locally constant objects split by $\mathcal{U}$ (Definition 1.4) is equivalent to the following category:*

Objects: $(S \to I,\ \sigma)$, is a pair, where $S \to I \in \mathcal{S}_{/I}$, and σ is a $U_\bullet$-descent datum on $\gamma^* S \times_{\gamma^* I} U \longrightarrow U$ in $\mathcal{E}$ (1.7 (7)).

Arrows: $(S \to I,\ \sigma) \to (T \to I,\ \eta)$, is a family of functions $\{S_i \xrightarrow{\vartheta_i} T_i\}_{i\in I}$ compatible with the descent data:

$$\begin{tikzcd}
\gamma^* S_i \times U_i \times U_j \arrow[r, "\sigma_{j,i}"] \arrow[d, "\gamma^*\vartheta_i \times U_i \times U_j"] & \gamma^* S_j \times U_j \times U_i \arrow[d, "\gamma^*\vartheta_j \times U_j \times U_i"] \\
\gamma^* T_i \times U_i \times U_j \arrow[r, "\eta_{j,i}"] & \gamma^* S_j \times U_j \times U_i
\end{tikzcd}$$

Proof. We have a functor defined by the assignment $(X, S \to I, \theta) \mapsto (S \to I, \sigma)$, where the descent data $\sigma_{j,\,i}$ is given by the composite:

$$\gamma^* S_i \times U_i \times U_j \xrightarrow{\theta_i \times U_j} X \times U_i \times U_j \xrightarrow{X \times \tau} X \times U_j \times U_i \xrightarrow{\theta_j^{-1} \times U_i} \gamma^* S_j \times U_j \times U_i.$$

The statement in 1.7(9) says that this functor is an equivalence of categories. □

All the details in the proof of this proposition can be checked in an straightforward way, and it is interesting to do so to understand exactly how the two types of data match.

1.9

When dealing with locally constant objects we shall use the trivialization or the descent data indistinctly. We shall write $X = (X, S \to I, \theta)$, or $X = (S \to I, \sigma)$, indicating as usual with X either the object X or the whole structure.

Facts 1.10. (1) The morphism $\mathcal{S}_{/I} \xrightarrow{\varrho} \mathcal{P}_{\mathcal{U}}$ "forgets" the descent datum, and is given by

$$\varrho^*(S \to I,\ \sigma) = S \to I.$$

Clearly it is surjective (actually a surjective family of points).

(2) Given a set $T \in \mathcal{S}$, the inverse image of the structure morphism $\mathcal{P} \xrightarrow{\gamma} \mathcal{S}$ in T is the trivial descent datum on the constant family $\gamma^* T \times U \to U$, given by $\gamma^* T \times U_i \times U_j \xrightarrow{\gamma^* T \times \tau} \gamma^* T \times U_j \times U_i$. Thus, $\gamma^* T = (\gamma^* T,\ \gamma^* T \times \tau)$. Clearly, if $\mathcal{E}$ is connected, so is $\mathcal{P}$. □

2. Covering projections associated to a (fixed) cover

Let $X = (S \to I,\ \sigma)$ be a locally constant object trivialized by a cover $U \to \gamma^* I$.

Definition 2.1. An *action triple* for X is a 3-tuple $(u,\ v,\ s)$ where $C \xrightarrow{u} U_i$, $C \xrightarrow{v} U_j$, $C \in \mathcal{E}$, $C \neq \emptyset$, and $S_i \xrightarrow{s} S_j$ a bijective function, such that

$$\begin{tikzcd}
\gamma^* S_i \times U_i \times U_j \arrow[r, "\sigma_{j,\,i}"] & \gamma^* S_j \times U_j \times U_i \\
\gamma^* S_i \times C \arrow[u, "\gamma^* S_i \times (u,\,v)"] \arrow[r, "\gamma^* s \times C"] & \gamma^* S_j \times C \arrow[u, "\gamma^* S_j \times (v,\,u)"']
\end{tikzcd}$$

Remark 2.2. Notice that $C \neq \emptyset$ implies that for given $(u,\ v)$, if there exists a bijection s to complete an action triple, this bijection is unique. □

Remark 2.3. Given an action triple $(u,\ v,\ s)$ and an arrow $C' \xrightarrow{f} C$, $C' \neq \emptyset$, the pair $(u',\ v')$, where $u' = uf,\ v' = vf$, can be completed into an action triple $(u',\ v',\ s')$, with $s' = s$. □

Remark 2.4. Any pair $(u,\ v)$ with C, a connected object, can always be completed by a bijection s into an action triple. □

The proof of the following proposition is rather straightforward, and it is left to the reader.

Proposition 2.5. *Let $X \to Y$, $(S \to I, \sigma) \xrightarrow{\vartheta} (T \to I, \eta)$, $S \xrightarrow{\vartheta} T$ (see 1.8) be a morphism between locally constant objects. Then:*

(a) *If ϑ is surjective, given any action triple $(u,\ v,\ s)$, $S_i \xrightarrow{s} S_j$ for X, the pair $(u,\ v)$ can be completed into an action triple $(u,\ v,\ t)$, $T_i \xrightarrow{t} T_j$ for Y, $t\vartheta_i = \vartheta_j s$.*

(b) *If ϑ is injective, given any action triple $(u,\ v,\ t)$, $T_i \xrightarrow{t} T_j$ for Y, the pair $(u,\ v)$ can be completed into an action triple $(u,\ v,\ s)$, $S_i \xrightarrow{t} S_j$ for X, $t\vartheta_i = \vartheta_j s$.* □

Definition 2.6. The descent data σ determines an equivalence relation on the set S as follows: Given $x,\ y \in S,\ x \in S_i,\ y \in S_j$, then $x \sim_\sigma y$ if there exists an action triple $(u,\ v,\ s)$ such that $y = sx$. That is,

$$\begin{tikzcd}
\gamma^* S_i \times C \arrow[rr, "\gamma^* s \times C"] & & \gamma^* S_j \times C \\
& C \arrow[ul, "\gamma^* x \times C"] \arrow[ur, "\gamma^* y \times C"'] &
\end{tikzcd}$$

This relation is reflexive (given $x \in U_i$, take $C = U_i$, $u = id$, $v = id$, so that $(u,\ v) = \Delta$, and $s = id$, this establishes $x \sim_\sigma x$) and clearly symmetric. Its transitive closure is an equivalence relation, that we denote also by $\sim_\sigma$.

Remark 2.7. Notice that if $x \sim_\sigma y$, we can always choose an action triple such that the arrow $C \xrightarrow{(u,v)} U_i \times U_j$ is a monomorphism. □

The *generating pairs* are those pairs $x \sim_\sigma y$ which are related by a single action triple. We shall now characterize in an explicit way the transitive closure.

Consider the (small) set of "*arrows*" $i \xrightarrow{\phi} j$ between the elements of I: $\mathbb{G}_{\mathcal{U}} \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} I$, $\delta_0\phi = i$, $\delta_1\phi = j$, where $i \xrightarrow{\phi} j$ denotes a sequences of spans:

$$\phi = ((U_{i_0} \xleftarrow{u_0} C_0 \xrightarrow{v_0} U_{i_1}),\ (U_{i_1} \xleftarrow{u_1} C_1 \xrightarrow{v_1} U_{i_2}), \ldots (U_{i_{n-1}} \xleftarrow{u_{n-1}} C_{n-1} \xrightarrow{v_{n-1}} U_{i_n}))$$

$(i_k, i_{k+1}) \in \mathcal{U}_1$, and $C_k \overset{(u_k,v_k)}{\hookrightarrow} U_{i_k} \times U_{i_{k+1}}$, $C_k \neq \emptyset$, a subobject of $U_{i_k} \times U_{i_{k+1}}$, $i_0 = i$, $i_n = j$, $n > 0$.

It follows from Remark 2.7:

Proposition 2.8. *Given any $x \in S_i$, $y \in S_j$,*

$$x \sim_\sigma y \iff \exists\, i \xrightarrow{\phi} j \in \mathbb{G}_{\mathcal{U}} \quad \text{such that } s_\phi x = y,$$

where $s_\phi x$ means that ϕ can be completed into a sequence of action triples $((u_0, v_0, s_0), (u_1, v_1, s_1), \ldots (u_{n-1}, v_{n-1}, s_{n-1}))$, and $s_\phi x = s_{n-1} \ldots s_1 s_0 x$. □

Proposition 2.9. *Let $X = (S \to I, \sigma)$ be any locally constant object, and $R \subset S$ be an equivalence class of $\sim_\sigma$. Given any subobject $Y = (T \to I, \sigma)$, $T \subset S$, if $R \cap T \neq \emptyset$, then $R \subset T$.*

Proof. The proof is immediate (consider Proposition 2.5(b)) □

Corollary 2.10. *A locally constant object $X = (S \to I, \sigma)$ is a connected object in $\mathcal{P}_{\mathcal{U}}$ if and only if $\sim_\sigma$ has a single equivalence class (that is, $x \sim_\sigma y$ for all pairs x, y).* □

Warning: In general the descent datum does not restrict to R, so that equivalence classes are not subobjects.

We see then that all possible families $S \to I$ which admit a connected descent datum are quotients of subsets of $\mathbb{G}_{\mathcal{U}}$. It follows

Proposition 2.11. *The collection of all connected locally constant objects trivialized by a cover $U \to \gamma^* I$ is a (small) set.* □

Covering projections

Not all locally constant objects should be considered as covering projections. However, to require that the $U_\bullet$-descent datum comes from a $\mathcal{U}_\bullet$-descent datum (1.7(8)), as in the case of topological spaces (see Appendix), is too restrictive when the topos is not spatial.

Definition 2.12. We say that a locally constant object $X = (S \to I, \sigma)$ trivialized by a cover $U \to \gamma^* I$ is a *covering projection* if, for each $(i, j) \in \mathcal{U}_1$ (the set of 1-simplexes of the Cech nerve, 1.5), the family $C \xrightarrow{(u,v)} U_i \times U_j$ is an epimorphic family, where (u, v) ranges over all action triples (u, v, s).

The following two propositions follow immediately from Proposition 2.5.

Proposition 2.13. *Any subobject in $\mathcal{P}_{\mathcal{U}}$ of a covering projection is a covering projection.* □

Proposition 2.14. *Any quotient in $\mathcal{P}_{\mathcal{U}}$ of a covering projection is a covering projection.* □

Proposition 2.15. *Any finite limit in $\mathcal{P}_{\mathcal{U}}$ of covering projections is a covering projection.* □

Proof. The terminal object clearly is a covering projection. Let (u, v, s), (u', v', s') be action triples for $X = (S \to I, \sigma)$, $X' = (S' \to I', \sigma')$. The fiber product of C and C' over $U_i \times U_j$ (if non-empty) is an action triple for X and X' simultaneously. By the construction of binary products in $\mathcal{P}_{\mathcal{U}}$ it readily follows that it determines, together with $s \times s'$, an action triple for the product $X \times X'$. The proof follows from this and the fact that in a topos the fiber products of epimorphic families yield an epimorphic family. The case of a general finite limit can be treated exactly in the same way, but it follows anyway from Proposition 2.13. □

Proposition 2.16. *Let $X = (S \to I, \sigma)$ be a covering projection, and $R \subset S$ be an equivalence class of $\sim_\sigma$. Then, the descent datum σ restricts to $R \to I$ and it determines a subobject $A \hookrightarrow X$, $A = (R \to I, \sigma)$. Furthermore, given any subobject $Y = (T \to I, \sigma)$, $Y \hookrightarrow X$, $T \subset S$, if $A \cap Y \neq \emptyset$, (equivalently $R \cap T \neq \emptyset$), then $A \hookrightarrow T$, (equivalently $R \subset T$).*

Proof. Let $(i, j) \in \mathcal{U}_1$. Take an epimorphic family $C \xrightarrow{(u,\, v)} U_i \times U_j$, with (u, v, s) action triples, and consider the following diagram;

$$\begin{tikzcd}
\gamma^*S_i \times U_i \times U_j \arrow[r, "\sigma_{j,\, i}"] & \gamma^*S_j \times U_j \times U_i \\
\gamma^*R_i \times U_i \times U_j \arrow[u, hook] \arrow[r, dashed, "\sigma_{j,\, i}"] & \gamma^*R_j \times U_j \times U_i \arrow[u, hook] \\
\gamma^*R_i \times C \arrow[u, "\gamma^*R_i\times(u,\, v)"'] \arrow[r, "\gamma^*s\times C"] & \gamma^*R_j \times C \arrow[u, "\gamma^*R_j\times(v,\, u)"']
\end{tikzcd}$$

The family $\gamma^*R_i \times C \xrightarrow{\gamma^*R_i\times(u,\, v)} \gamma^*R_i \times U_i \times U_j$ is epimorphic, and the outer diagram commutes by definition. It follows that $\sigma_{j,\, i}$ factors as shown. The second assertion is immediate (see Proposition 2.9). □

Proposition 2.17. *Let $(S \to I, \sigma)$ be a covering projection. The quotient set $S \to S_{/\sim_\sigma}$ has the following property:*

Given any set T, $T \in \mathcal{S}$, and a function $S \xrightarrow{\vartheta} T \times I$ over I (that is, a function $S \xrightarrow{\vartheta} T$, $\{S_i \xrightarrow{\vartheta_i} T\}_{i\in I}$, then ϑ determines a morphism $(S, \sigma) \xrightarrow{\vartheta} (\gamma^*T, \gamma^*T \times \tau)$ in $\mathcal{P}_{\mathcal{U}}$ if and only if it factors $S \longrightarrow S_{/\sim_\sigma} \longrightarrow T$.

Proof. Let $x \in S_i$, $y \in S_j$ be such that $x \sim_\sigma y$ by an action triple (u, v, s). Consider the following diagram:

$$\begin{tikzcd}
\gamma^*T \times U_i \times U_j \arrow[rr, "\gamma^*T\times\tau"] & & \gamma^*T \times U_j \times U_i \\
\gamma^*S_i \times U_i \times U_j \arrow[u, "\gamma^*\vartheta_i\times U_i\times U_j"'] \arrow[rr, "\sigma_{j,\, i}"] & & \gamma^*S_j \times U_j \times U_i \arrow[u, "\gamma^*\vartheta_j\times U_j\times U_i"'] \\
\gamma^*S_i \times C \arrow[u, "\gamma^*S_i\times(u,\, v)"'] \arrow[rr, "\gamma^*s\times C"] & & \gamma^*S_j \times C \arrow[u, "\gamma^*S_j\times(v,\, u)"'] \\
& C \arrow[ul, "\gamma^*x\times C"] \arrow[ur, "\gamma^*y\times C"'] &
\end{tikzcd}$$

(a) The middle square and the lower triangle commute by definition.

(b) The commutativity of the upper square means that ϑ is a morphism in $\mathcal{P}_{\mathcal{U}}$.

(c) If $C \xrightarrow{(u,\, v)} U_i \times U_j$ is a monomorphism, the commutativity of the outer diagram implies that ϑ factors through $S_{/\sim_\sigma}$. Notice that this together with $C \neq \emptyset$ implies that we have:

$$(\gamma^*(\vartheta_i x),\ (v,\ u)) = (\gamma^*(\vartheta_j y),\ (v,\ u)) \iff \vartheta_i x = \vartheta_j y.$$

f morphism $\Rightarrow$ *ϑ factors*: Let $x \sim_\sigma y$. Clearly it is enough to take a generating pair. Furthermore we can assume that the arrow (u, v) is a monomorphism (Remark 2.7). Since the outer diagram commutes, it follows that $\vartheta_i x = \vartheta_j y$.

ϑ factors $\Rightarrow$ *ϑ morphism*: Take an epimorphic family $C \xrightarrow{(u,\, v)} U_i \times U_j$, with (u, v, s) action triples. For all $x \in S_i$ let $y = sx$. The family $C \xrightarrow{\gamma^*x\times(u,\, v)} \gamma^*S_i \times U_i \times U_j$ is epimorphic. Since for all x, (u, v) the outer diagram commutes, it follows that the upper square commutes. □

From Proposition 2.11 we have, in particular:

Proposition 2.18. *The collection of all connected covering projections trivialized by a cover $U \to \gamma^*I$ is a (small) set.* □

Let $\mathcal{G}_{\mathcal{U}} \subset \mathcal{P}_{\mathcal{U}}$ be the full subcategory whose objects are sums of covering projections trivialized by the cover $U \to \gamma^*I$. From Propositions 2.14–2.18 and 1.10(2) it follows

Theorem 2.19. 1. *The category $\mathcal{G}_{\mathcal{U}}$ is an atomic (locally connected and boolean) topos and the full inclusion is the inverse image of a geometric morphism $\mathcal{P}_{\mathcal{U}} \longrightarrow \mathcal{G}_{\mathcal{U}}$. If $\mathcal{E}$ is connected, so is $\mathcal{G}_{\mathcal{U}}$.*

2. *The functor $\mathcal{G}_{\mathcal{U}} \xrightarrow{\varrho^*} \mathcal{S}_I$, $\varrho^*(S \to I, \sigma) = S \to I$ is the inverse image of a surjective point $\mathcal{S}_{/I} \xrightarrow{\varrho} \mathcal{G}_{\mathcal{U}}$.* □

We abuse the notation and write $\mathcal{E} \xrightarrow{\upsilon} \mathcal{G}_{\mathcal{U}}$ and $\mathcal{S}_{/I} \xrightarrow{\varrho} \mathcal{G}_{\mathcal{U}}$ for the composites $\mathcal{E} \xrightarrow{\upsilon} \mathcal{P}_{\mathcal{U}} \longrightarrow \mathcal{G}_{\mathcal{U}}$ and $\mathcal{S}_{/I} \xrightarrow{\varrho} \mathcal{P}_{\mathcal{U}} \longrightarrow \mathcal{G}_{\mathcal{U}}$ respectively. Notice that the point $\mathcal{S}_{/I} \xrightarrow{\varrho} \mathcal{G}_{\mathcal{U}}$ can be thought as a family $\mathcal{S} \xrightarrow{\varrho_i} \mathcal{G}_{\mathcal{U}}$, $\varrho_i^*(S \to I,\ \sigma) = S_i$, of (enough) points indexed by the set I.

Let $I \to \mathbf{G}_{\mathcal{U}}$ be the localic groupoid of the points $\mathcal{S}_{/I} \xrightarrow{\varrho} \mathcal{G}_{\mathcal{U}}$, with (discrete) set of objects I (this groupoid is explicitly constructed in [6, Section 2]), and let $\mathcal{S}_{/I} \longrightarrow \beta\mathbf{G}_{\mathcal{U}}$ be its classifying topos ([10,14,6]). There is a comparison morphism $\beta\mathbf{G}_{\mathcal{U}} \longrightarrow \mathcal{G}_{\mathcal{U}}$ compatible with the respective points. The next theorem follows from Theorem 2.19 by [10, VIII, 3. Theorem 1] (also [5, Theorem 8.4], or, explicitly [6, Theorem 3.6.4]).

Theorem 2.20. *The comparison morphism $\beta\mathbf{G}_{\mathcal{U}} \xrightarrow{\cong} \mathcal{G}_{\mathcal{U}}$ is an equivalence which identifies the point ϱ with the canonical point of $\beta\mathbf{G}_{\mathcal{U}}$.* $\square$

Observation 2.21. Recall that $\mathbf{G}_{\mathcal{U}}$ may be chosen to be *etale complete* [14]. Actually, the construction in [6] yields an etale complete localic groupoid.

The theory presented here generalizes the known theory for the locally connected case, bringing a direct proof for the atomicity of the topos of locally constant objects split by a covering. From Remark 2.4 it immediately follows that for a locally connected topos every locally constant object is a covering projection. Also, in this case, the points $\mathcal{S}_{/I} \longrightarrow \mathcal{P}_{\mathcal{U}} = \mathcal{G}_{\mathcal{U}}$ are essential (therefore representable). This amounts to the fact that the fibers of an inverse limit in $\mathcal{P}_{\mathcal{U}}$ are the inverse limit of the fibers in $\mathcal{S}$ (fact that is not true if the topos is not locally connected). It follows that the localic groupoid $\mathbf{G}_{\mathcal{U}}$ is an ordinary discrete groupoid, and the representation Theorem 2.20 can be easily proved without recourse to Joyal–Tierney results (see [5,6]). We have

Proposition 2.22. *If the topos $\mathcal{E}$ is locally connected, every locally constant object is a covering projection. That is, $\mathcal{G}_{\mathcal{U}} = \mathcal{P}_{\mathcal{U}}$. Furthermore, the points are representable, and the groupoid $\mathbf{G}_{\mathcal{U}}$ is an ordinary groupoid in $\mathcal{S}$.* $\square$

Our Theorem 2.20 for the non-locally connected case appears to be the first genuine application of [10, VIII, 3. Theorem 1] in the Galois theory of locally constant objects. It is also worth noticing that it is the first time a non-prodiscrete localic groupoid appears in this theory, as well as an atomic topos which is not a Galois topos.

3. Geometric morphisms induced by cover refinements

The covers of a topos form a category $Cov(\mathcal{E})$, taking as arrows the family morphism. Given two covers $\mathcal{U} = (U, I, \zeta)$, $\mathcal{V} = (V, J, \xi)$, an arrow is a pair $U \xrightarrow{h} V$, $I \xrightarrow{\alpha} J$, making the following square commutative:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "\zeta"] & V \arrow[d, "\xi"] \\ \gamma^* I \arrow[r, "\gamma^*\alpha"] & \gamma^* J \end{tikzcd}$$

We say that $\mathcal{U}$ *refines* $\mathcal{V}$, and call the morphisms *refinements*. In alternative notation, the arrow $\mathcal{U} \xrightarrow{(h,\alpha)} \mathcal{V}$ corresponds to a family $h = \{U_i \xrightarrow{h_i} V_{\alpha(i)}\}_{i \in I}$.

Any two covers have a common refinement, namely, the family $\{U_i \times V_j\}_{(i,j) \in I \times J, |U_i \times V_j \neq \emptyset}$. We remark that $Cov(\mathcal{E})$ is not a filtered category because given two refinements they cannot be further refined to become a single one. However, covers do form a filtered poset $cov(\mathcal{E})$ directed by the existence of a refinement, and there is a functor $Cov(\mathcal{E}) \to cov(\mathcal{E})$ which identifies the different refinements.

We shall let $\mathcal{T}op_{\mathcal{S}}$ denote the 2-category of Grothendieck topoi and geometric morphisms. We have:

Proposition 3.1. *The construction of the atomic topos $\mathcal{G}_{\mathcal{U}}$ of covering projections (2.19) is functorial : $Cov(\mathcal{E}) \xrightarrow{\mathcal{G}} \mathcal{T}op_{\mathcal{S}}$. Given a refinement $\mathcal{U} \xrightarrow{(h,\alpha)} \mathcal{V}$ in $Cov(\mathcal{E})$, we also denote $\mathcal{G}_{\mathcal{U}} \xrightarrow{(h,\alpha)} \mathcal{G}_{\mathcal{V}}$ the corresponding geometric morphism. The following diagram commutes:*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\upsilon_{\mathcal{U}}"] \arrow[dr, "\upsilon_{\mathcal{V}}"] & \mathcal{G}_{\mathcal{U}} \arrow[d, "{(h,\alpha)}"] & \mathcal{S}_{/I} \arrow[l, "\varrho_{\mathcal{U}}"'] \arrow[d, "\alpha"] \\ & \mathcal{G}_{\mathcal{V}} & \mathcal{S}_{/J} \arrow[l, "\varrho_{\mathcal{V}}"'] \end{tikzcd}$$

Proof. It follows from the universal property of the push-out that there is a geometric morphism $\mathcal{P}_{\mathcal{U}} \xrightarrow{(h,\,\alpha)} \mathcal{P}_{\mathcal{V}}$, and that the statement of the theorem holds for the push-out construction $\mathcal{P}_{\mathcal{U}}$ (1.4). Consider the following diagram of topoi and inverse image functors:

$$\begin{tikzcd}
\mathcal{E} & \mathcal{G}_{\mathcal{U}} \arrow[l, "\upsilon^*_{\mathcal{U}}"] \arrow[r, hook] & \mathcal{P}_{\mathcal{U}} \arrow[r, "\varrho^*_{\mathcal{U}}"] & \mathcal{S}_{/I} \\
& \mathcal{G}_{\mathcal{V}} \arrow[ul, "\upsilon^*_{\mathcal{V}}"] \arrow[r, hook] & \mathcal{P}_{\mathcal{V}} \arrow[u, "(h,\,\alpha)^*"] \arrow[r, "\varrho^*_{\mathcal{V}}"] & \mathcal{S}_{/J} \arrow[u, "\alpha^*"']
\end{tikzcd}$$

It is clear that the theorem follows if we prove that the inverse image functor $\mathcal{P}_{\mathcal{V}} \xrightarrow{(h,\,\alpha)^*} \mathcal{P}_{\mathcal{U}}$ sends covering projections to covering projections. We need an explicit description of this functor.

Remark 3.2. Let $(S \to I,\, \sigma) \in \mathcal{P}_{\mathcal{U}}$ be the value of the functor $(h,\,\alpha)^*$ on an object $(T \to J,\, \eta) \in \mathcal{P}_{\mathcal{V}}$, $(S \to I,\, \sigma) = (h,\,\alpha)^*(T \to J,\, \eta)$, where $S = \alpha^* T$ (notice that $S_i = T_{\alpha(i)}$). By construction, there is a commutative diagram:

$$\begin{tikzcd}
\gamma^* T_{\alpha(i)} \times V_{\alpha(i)} \times V_{\alpha(j)} \arrow[r, "\eta_{\alpha(i),\,\alpha(j)}"] & \gamma^* T_{\alpha(j)} \times V_{\alpha(j)} \times V_{\alpha(i)} \\
\gamma^* S_i \times U_i \times U_j \arrow[u, "id \times h_i \times h_j"] \arrow[r, "\sigma_{i,\,j}"] & \gamma^* S_j \times U_j \times U_i \arrow[u, "id \times h_j \times h_i"']
\end{tikzcd} \quad \square$$

Continuation of the proof. For each action triple $(x,\, y)$ for η, consider a pull-back diagram:

$$\begin{tikzcd}
C \arrow[r, "(u,\,v)"] \arrow[d] & U_i \times U_j \arrow[d, "h_i \times h_j"] \\
B \arrow[r, "(x,\,y)"] & V_{\alpha(i)} \times V_{\alpha(j)}
\end{tikzcd}$$

It is easy to check with the aid of the diagram in Remark 3.2 that the pair $(u\, v)$ is an action triple for σ. Since pulling back an epimorphic family yields an epimorphic family, this finishes the proof. $\square$

We can think of the functor in the previous proposition as a system $(\mathcal{G}_{\mathcal{U}})_{\mathcal{U} \in Cov(\mathcal{E})}$ of topoi and geometric morphisms, or as a system of categories and inverse image functors, indexed by $Cov(\mathcal{E})$. Although $Cov(\mathcal{E})$ is not filtered, we can give a simple description of the colimit of the categories.

4. The category and the topos of covering projections

The objects of this category, denoted $c\mathcal{G}(\mathcal{E})$, are pairs $(X,\, \mathcal{U})$, where $\mathcal{U}$ is a cover, and $X = (X, S \to I, \theta) = (S \to I, \sigma)$ is a covering projection (1.9) trivialized by $\mathcal{U}$.

An arrow $(X,\, \mathcal{U}) \to (Y,\, \mathcal{V})$ is map $X \xrightarrow{f} Y$ in $\mathcal{E}$ such that there exist a common refinement $\mathcal{W} \xrightarrow{(h,\,\alpha)} \mathcal{U}$, $\mathcal{W} \xrightarrow{(l,\,\beta)} \mathcal{V}$, and an arrow $(h,\, \alpha)^*(X) \xrightarrow{(f,\,\vartheta)} (l,\, \beta)^*(Y)$ in $\mathcal{G}_{\mathcal{W}}$, $\upsilon^*(f,\, \vartheta) = f$. From the fact that any two covers have a common refinement it follows that given two maps $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ in $\mathcal{E}$, if f and g are arrows in $c\mathcal{G}(\mathcal{E})$, then so is the composite $X \xrightarrow{gf} Z$.

The hom-sets in the category $c\mathcal{G}(\mathcal{E})$ are the filtered colimit (actually a filtered union) of the hom-sets in the categories $\mathcal{G}_{\mathcal{W}}$, indexed by the $\mathcal{W}$ which follow $\mathcal{U}$ and $\mathcal{V}$ in $cov(\mathcal{E})$, $\mathcal{W} \geq \mathcal{U}$, $\mathcal{W} \geq \mathcal{V}$.

Clearly for each $\mathcal{U} \in Cov(\mathcal{E})$ there is a faithful functor $\mathcal{G}_{\mathcal{U}} \xrightarrow{\lambda^*_{\mathcal{U}}} c\mathcal{G}(\mathcal{E})$, and these functors form a cone for the system $(\mathcal{G}_{\mathcal{U}})_{\mathcal{U} \in Cov(\mathcal{E})}$ (given $X \in \mathcal{G}_{\mathcal{V}}$ and a refinement $\mathcal{U} \xrightarrow{(h,\,\alpha)} \mathcal{V}$, the identity map $X \xrightarrow{id} X$ establish an isomorphism $(X,\, \mathcal{V}) \cong ((h,\, \alpha)^*(X),\, \mathcal{U})$ in $c\mathcal{G}(\mathcal{E})$).

Proposition 4.1.

1. *The category $c\mathcal{G}(\mathcal{E})$ has finite limits and the functors $\lambda^*_{\mathcal{U}}$ preserve them.*
2. *The cone $\mathcal{G}_{\mathcal{U}} \xrightarrow{\lambda^*_{\mathcal{U}}} c\mathcal{G}(\mathcal{E})$ is a colimit of the system of categories and inverse image functors (indexed by the category $Cov(\mathcal{E})$).*

Proof. It needs a straightforward but careful verification which is left to the interested reader. $\square$

Notice that there is a functor $c\mathcal{G}(\mathcal{E}) \xrightarrow{\upsilon^*} \mathcal{E}$ making the following diagram a commutative diagram of faithful functors:

$$\begin{tikzcd} \mathcal{G}_{\mathcal{U}} \arrow[r, "\lambda^*_{\mathcal{U}}"] \arrow[dr, "\upsilon^*_{\mathcal{U}}"'] & c\mathcal{G}(\mathcal{E}) \arrow[d, "\upsilon^*"] \\ & \mathcal{E} \end{tikzcd} \tag{4.2}$$

Consider in the category $c\mathcal{G}(\mathcal{E})$ the Grothendieck topology generated by all the epimorphic families in $\mathcal{G}_{\mathcal{U}}$ for all $\mathcal{U} \in Cov\mathcal{E}$. Notice that this topology is subcanonical. The (small) set of covering projections corresponding to the connected objects of $\mathcal{G}_{\mathcal{U}}$, $\mathcal{U}$ running over any (small) *cofinal* set of coverings (for example, covering sieves, see 4.6 3.) is a "topologically generating" family in the sense of [1, Expose II 3.0.1] (that is, every object in $c\mathcal{G}(\mathcal{E})$ is covered by objects in the family). It follows that the category of sheaves is legitimate and that it is a Grothendieck topos [1, Expose II 4.11]. We shall denote this topos $\mathcal{G}(\mathcal{E})$. There is a full and faithful functor $c\mathcal{G}(\mathcal{E}) \to \mathcal{G}(\mathcal{E})$. Clearly the composite functors $\mathcal{G}_{\mathcal{U}} \longrightarrow c\mathcal{G}(\mathcal{E}) \to \mathcal{G}(\mathcal{E})$ are the inverse image functors of geometric morphisms $\mathcal{G}(\mathcal{E}) \to \mathcal{G}_{\mathcal{U}}$ which determine a cone for the system $(\mathcal{G}_{\mathcal{U}})_{\mathcal{U} \in Cov(\mathcal{E})}$ of topoi and geometric morphisms. There is a commutative diagram of surjective geometric morphisms:

$$\begin{tikzcd} \mathcal{G}_{\mathcal{U}} & \mathcal{G}(\mathcal{E}) \arrow[l, "\lambda_{\mathcal{U}}"'] \\ & \mathcal{E} \arrow[ul, "\upsilon_{\mathcal{U}}"] \arrow[u, "\upsilon"'] \end{tikzcd} \tag{4.3}$$

Theorem 4.4. *The cone $\mathcal{G}(\mathcal{E}) \xrightarrow{\lambda_{\mathcal{U}}} \mathcal{G}_{\mathcal{U}}$ is a limit cone in the 2-category $\mathcal{T}op_{\mathcal{S}}$. That is, $\mathcal{G}(\mathcal{E})$ is the inverse limit of the system of topoi and geometric morphisms $(\mathcal{G}_{\mathcal{U}})_{\mathcal{U} \in Cov(\mathcal{E})}$.*

Proof. It follows immediately from Proposition 4.1 using [1, Expose IV 4.9.4] (referred to as *the basic theorem concerning classifying topos* in [12, Chapter VIII,3]). □

We now consider *covering sieves* as a technical tool in order to exhibit the topos $\mathcal{G}(\mathcal{E})$ as the classifying topos of a progroupoid.

Covering sieves 4.5. Given $C_0 \in \mathcal{S}$, $\emptyset \notin C_0$, a (small) set of generators for the topos $\mathcal{E}$, we say that $I \subset C_0$ is a *sieve* if, given an arrow $C \to D$, with $D \in I$ and $C \in C_0$, then $C \in I$. Further, I is a *covering sieve* if $\Sigma I \to 1$ is epimorphic, with $\Sigma I = \sum_{C \in I} C$. Covering sieves form a (small) poset $sCov(C_0)$ ordered by inclusion. To each covering sieve I, we can associate the cover $\mathcal{R}(I) = (\Sigma I,\ I,\ \pi)$, with $\pi = \sum_{C \in I}(C \to 1)$. In other words, $\mathcal{R}(I) = \{U_C\}_{C \in I}$, with $U_C = C$. Also, to each cover $\mathcal{U} = (U,\ I,\ \eta)$ we can associate the sieve $s(\mathcal{U}) = \{C \in C_0 \mid \exists i \in I \text{ and } C \to U_i\}$.

Proposition 4.6. 1. *The poset $sCov(C_0)$ is filtered.*

2. *There is a functor $sCov(C_0) \xrightarrow{\mathcal{R}} Cov(\mathcal{E})$ defined by $\mathcal{R}(I) = (\Sigma(I), I, \pi)$.*
3. *There is a functor $Cov(\mathcal{E}) \xrightarrow{s} sCov(C_0)$, and the composite $\mathcal{R}(s\mathcal{U})$ refines $\mathcal{U}$. In this sense, the functor $\mathcal{R}$ is cofinal.*
4. *Given any two sets of generators, $C_0,\ D_0$, there are cofinal morphism of posets $sCov(C_0) \longrightarrow sCov(D_0)$, $sCov(D_0) \longrightarrow sCov(C_0)$.*
5. *Given any functor $Cov(\mathcal{E}) \xrightarrow{\mathcal{F}} \mathcal{X}$ into a category $\mathcal{X}$, it determines a pro-object $(\mathcal{F}_I)_{I \in sCov(C_0)}$, $\mathcal{F}_I = \mathcal{F}_{\mathcal{R}(I)}$, for each set of generators C_0, and all these pro-objects are isomorphic as pro-objects.*

Proof. 1. Given two covering sieves $I, J \in s\,Cov(C_0)$, the intersection sieve $I \cap J$ is also covering. This follows from the fact that given $C \in I$, $D \in J$, the product $C \times D$ is covered by objects of C_0.

2. Clearly, given $I,\ J \in sCov(C_0)$, if $I \subset J$, there is a canonical monomorphic refinement $\mathcal{R}(I) \hookrightarrow \mathcal{R}(J)$.
3. That the sieve $s(\mathcal{U})$ is covering follows because every U_i is covered by objects of C_0. The rest of the statement is clear.
4. Given any covering sieve I in $sCov(C_0)$, it generates a sieve in $sCov(D_0)$, $sI = \{D \in D_0 \mid \exists D \to C \text{ with } C \in I\}$, which is covering since the objects in D_0 generate. This defines a morphism of posets. The same holds in the other direction. It is clear that $ss(I) \subset I$. This finishes the proof.
5. Follows from 1 and 4 □

Here it is important to remark that although the refining sieve of a cover is canonical, *the refinement itself is not*. There is no consistent choice of refinements in such a way that the maps $\mathcal{R}(s\,\mathcal{U}) \to \mathcal{U}$ define a transformation natural on $\mathcal{U}$.

The functor $\mathcal{R}$ is not a real cofinal functor (in the sense of [1, Expose I, 8.]) because $Cov(\mathcal{E})$ is not a filtered category. However, for the particular functor $Cov(\mathcal{E})^{op} \xrightarrow{\mathcal{G}} \mathcal{C}at$, it follows from the construction of the colimit in Proposition 4.1:

Proposition 4.7. *Given any set of generators C_0 of $\mathcal{E}$, the canonical functor*

$$Colimit_{I \in sCov(C_0)} \mathcal{G}_{\mathcal{R}(I)} \longrightarrow c\mathcal{G}(\mathcal{E})$$

is an equivalence of categories.

Proof. This colimit can be constructed in the same way as the category $c\mathcal{G}(\mathcal{E})$, and it determines a full subcategory. Then, Proposition 4.6, 3. suffices to show that the inclusion is essentially surjective. □

Given any set of generators of a topos $\mathcal{E}$, consider the system (a pro-object in the 2-category $\mathcal{T}op_{\mathcal{S}}$) $(\mathcal{G}_{\mathcal{R}(I)})_{I \in sCov(C_0)}$, and the inverse limit of topoi and geometric morphisms $Limit_{I \in sCov(C_0)}\, \mathcal{G}_{\mathcal{R}(I)}$ (known to exist, [1, Expose VI, 8.2.11]). This topos does not depend of the chosen set of generators (Proposition 4.6, 4.).

Theorem 4.8.

1. *The restriction of the cone in Theorem 4.4: $\mathcal{G}(\mathcal{E}) \xrightarrow{\lambda_{\mathcal{R}(I)}} \mathcal{G}_{\mathcal{R}(I)}$, is a limit cone in the 2-category $\mathcal{T}op_{\mathcal{S}}$. That is, $\mathcal{G}(\mathcal{E})$ is the inverse limit of the protopos $(\mathcal{G}_{\mathcal{R}(I)})_{I \in sCov(C_0)}$.*

2. *There is a canonical natural transformation (in particular a canonical morphism of pro-objects) $(\mathcal{S}_{/I} \xrightarrow{\varrho_I} \mathcal{G}_{\mathcal{R}(I)})_{I \in sCovS(C_0)}$. The topos $\mathcal{G}(\mathcal{E})$ is equipped with a localic point $Sh(L) \xrightarrow{\varrho} \mathcal{G}(\mathcal{E})$, where L is the inverse limit in the category of localic spaces of the discrete spaces determined by the sets I.*

Proof. 1. Follows from Proposition 4.7.
2. Follows from Proposition 3.1 and the fact that the assignment of the topos of sheaves is a functor that preserves all inverse limits (since it has a left adjoint, the localic reflection [10]). □

5. The fundamental progroupoid of a topos

The statements without proof in this section are justified by the yoga of the theory of classifying topos of localic groupoids established in [14].

A *localic progroupoid* $\mathbf{G}$ is a pro-object $\mathbf{G} = (\mathbf{G}_\alpha)_{\alpha \in \Gamma}$ in the 2-category of localic groupoids. There are natural transformations (in particular, canonical morphisms of pro-objects) $\mathbf{I}_\alpha \to \mathbf{G}_\alpha$, where $\mathbf{I}_\alpha$ are the localic spaces of objects.

Definition 5.1. Given a localic progroupoid $\mathbf{G}$ as above, its classifying topos is the inverse limit topos of the classifying topoi $\beta\mathbf{G}_\alpha$, $\beta\mathbf{G} = Limit_\alpha \beta\mathbf{G}_\alpha$. It is equipped with a localic point $Sh(\mathbf{I}) \to \beta\mathbf{G}$, where $\mathbf{I}$ is the inverse limit of the localic spaces $\mathbf{I}_\alpha$, $\mathbf{I} = Limit_\alpha \mathbf{I}_\alpha$. Notice that $Sh(\mathbf{I})$ is also the inverse limit of the topoi $Sh(\mathbf{I}_\alpha)$, $Sh(\mathbf{I}) = Limit_\alpha Sh(\mathbf{I}_\alpha)$.

When $\mathbf{G}$ is an ordinary (strict) progroup, this is exactly the definition given in [1, Expose IV 2.7.], where the objects of the topos $\beta\mathbf{G}$ are described explicitly.

Let $\mathbf{I} \to \mathbf{gG}$ be the inverse limit of the localic groupoids $\mathbf{G}_\alpha$, $\mathbf{gG} = Limit_\alpha \mathbf{G}_\alpha$. There is a comparison functor $\beta\mathbf{gG} \to \beta\mathbf{G}$ (that is $\beta\, Limit_\alpha \mathbf{G}_\alpha \to Limit_\alpha \beta\mathbf{G}_\alpha$). It is an open problem (plausibly with a negative answer) to know if this is an equivalence. This is related to the failure or not of the point $Sh(\mathbf{I}) \to \beta\mathbf{G}$ to be of effective descent. Since $Sh(\mathbf{I}) \to \beta\mathbf{gG}$ is always of effective descent, we have:

Proposition 5.2. *The comparison morphism $\beta\mathbf{gG} \to \beta\mathbf{G}$ is an equivalence if and only if $Sh(\mathbf{I}) \to \beta\mathbf{G}$ is of effective descent* □

The answer is positive in the classical cases corresponding to the Galois theory of locally connected topoi. Recall that a morphism of groupoids $\mathbf{G} \to \mathbf{H}$ is *composably onto* if it is surjective on commutative triangles (thus, also on arrows and objects) (see [11] 2.7). Extending SGA4 terminology, we say that a progroupoid is *strict* if the transition morphisms are composably onto. In [11] 4.18. it is established that the comparison morphism $\beta\mathbf{gG} \to \beta\mathbf{G}$ is an equivalence for any strict progroupoid $\mathbf{G}$. In the particular case of strict progroups this was first observed in [16], and later stated independently in [15] (where, furthermore, the equivalence is proved for localic progroups whose transition morphisms are open surjections). This equivalence allows us to replace strict progroups by localic prodiscrete groups in the SGA4 Galois theory of locally connected topos.

Given any topos $\mathcal{E}$, consider now the system $(\mathcal{G}_{\mathcal{U}})_{\mathcal{U} \in Cov(\mathcal{E})}$. By the Morita equivalence for etale complete localic groupoids ([3, 2.6], see also [14, 7.7]) it follows:

Proposition 5.3. *The equivalences $\beta\mathbf{G}_{\mathcal{U}} \xrightarrow{\cong} \mathcal{G}_{\mathcal{U}}$ in Theorem 2.20 determine a system of localic groupoids $(\mathbf{G}_{\mathcal{U}})_{\mathcal{U} \in Cov(\mathcal{E})}$ and a natural equivalence $(\beta\mathbf{G}_{\mathcal{U}} \xrightarrow{\cong} \mathcal{G}_{\mathcal{U}})_{\mathcal{U} \in Cov(\mathcal{E})}$ of systems.* □

Given any set of generators C_0, by restriction this determines a localic progroupoid. This progroupoid does not depend on the chosen generators (up to isomorphism of pro-objects, 4.6, 5.), and it is defined to be the fundamental progroupoid of the topos.

Definition 5.4. The fundamental progroupoid $\pi_1(\mathcal{E})$, of the topos $\mathcal{E}$, is defined as $\pi_1(\mathcal{E}) = (\mathbf{G}_{\mathcal{R}(I)})_{I \in sCov(C_0)}$, for any set of generators C_0.

As before, let L be the inverse limit in the category of localic spaces of the discrete spaces determined by the sets I.

Theorem 5.5. *Given any topos $\mathcal{E}$, the topos $\mathcal{G}(\mathcal{E})$ of covering projections is the classifying topos of the fundamental localic progroupoid $\pi_1(\mathcal{E})$, by an equivalence $\beta\pi_1(\mathcal{E}) \xrightarrow{\cong} \mathcal{G}(\mathcal{E})$ which identifies the localic points $Sh(L) \longrightarrow \mathcal{G}(\mathcal{E})$, $Sh(L) \longrightarrow \beta\pi_1(\mathcal{E})$.*

Proof. It only remains to indicate that the theorem follows immediately from the given definitions, Proposition 5.3 and Theorem 4.8. □

6. The representation of torsors

In this last section we prove that the fundamental localic progroupoid $\pi_1(\mathcal{E})$ represents torsors. Given a group $K \in \mathcal{S}$ and a topos $\mathcal{E}$, recall that the category (groupoid) $\mathcal{T}ors^K(\mathcal{E})$ of K-torsors (see below) is equivalent to the category of geometric morphisms from $\mathcal{E}$ to the classifying topos βK, $\mathcal{T}op_{\mathcal{S}}[\mathcal{E}, \beta K] \cong \mathcal{T}ors^K(\mathcal{E})$ [12, Chapter VIII, Theorem 7]. We shall denote by $\mathcal{G}rpd$, $pro\mathcal{G}rpd$, the 2-categories of localic groupoids, localic progroupoids, respectively.

Proposition 6.1. *There is an equivalence of categories* $pro\mathcal{G}rpd[\pi_1(\mathcal{E}),\ K] \cong \mathcal{T}ors^K(c\mathcal{G}(\mathcal{E}))$.

Proof.

$$\begin{array}{cc} pro\mathcal{G}rpd[\pi_1(\mathcal{E}),\ K] & \cong^1 \\ Colimit_{I\in sCov(C_0)}\ \mathcal{G}rpd[\mathbf{G}_{\mathcal{R}(I)},\ K] & \cong^2 \\ Colimit_{I\in sCov(C_0)}\ \mathcal{T}op_{\mathcal{S}}[\beta\mathbf{G}_{\mathcal{R}(I)},\ \beta K] & \cong^3 \\ Colimit_{I\in sCov(C_0)}\ \mathcal{T}op_{\mathcal{S}}[\mathcal{G}_{\mathcal{R}(I)},\ \beta K] & \cong^4 \\ Colimit_{I\in sCov(C_0)}\ \mathcal{T}ors^K(\mathcal{G}_{\mathcal{R}(I)}) & \cong^5 \\ \mathcal{T}ors^K(c\mathcal{G}(\mathcal{E})). & \end{array}$$

(1) holds by definition of morphisms of progroupoids, (2) by the Morita equivalence for etale complete localic groupoids ([3, 2.6], see also [14, 7.7]), (3) by Proposition 5.3, (4) is clear (see [12, Chapter VIII, Theorem 7]), and (5) is an standard property of filtered colimits. $\square$

Our next task will be to show that there is an equivalence of categories $Tors^K(c\mathcal{G}(\mathcal{E})) \cong Tors^K(\mathcal{E})$.

Given a topos $\mathcal{E}$, we shall use letters as variables to describe arrows in $\mathcal{E}$. We shall denote by a central dot "$\cdot$" all group products and group actions. Given a set $S \in \mathcal{S}$, and an element $x \in S$, by the letter x we shall also indicate the corresponding global section $1 \xrightarrow{x} \gamma^* S$ in the topos.

Given a group K in $\mathcal{S}$, and given $x,\ y \in K$, we set $x/y = x \cdot y^{-1}$ and $x \setminus y = x^{-1} \cdot y$. Recall that a *K-torsor* in a topos $\mathcal{E}$ is an object $T \in \mathcal{E}$, $T \to 1$ epi, and an action $\gamma^*K \times T \longrightarrow T$ such that the arrow $\gamma^*K \times T \xrightarrow{\varepsilon} T \times T$ defined by $\varepsilon(x,\ u) = (x \cdot u,\ u)$ is an isomorphism. There is an arrow $T \times T \longrightarrow \gamma^*K$, $(u,\ v) \mapsto v/u$ defined by $\varepsilon^{-1}(u,\ v) = (v/u,\ v)$. Thus, $z \cdot u = v \ \Leftrightarrow\ z = v/u$. It immediately follows the equation $(x \cdot u)/(y \cdot u) \ =\ x/y$.

Clearly any torsor T determines in a canonical way a locally constant object $T = (T,\ K,\ \varepsilon)$ split by the (singleton family) cover $T \to 1$.

Proposition 6.2. 1. *Given any K-torsor in* $\mathcal{E}$, *the locally constant object* $T = (T,\ K,\ \varepsilon)$ *is a covering projection (Definition 2.12), that is, an object in* $\mathcal{G}_T$.

2. *The covering projection* $T = (T,\ K,\ \varepsilon)$ *is a K-torsor of* $\mathcal{G}_T$ *with the same arrow as action. The group product* $K \times K \to K$ *furnishes the function that lifts the action* $\gamma^*K \times T \longrightarrow T$ *into a morphism of covering projections.*

Proof. 1. The corresponding descent data $\gamma^*K \times T \times T \xrightarrow{\sigma} \gamma^*K \times T \times T$ is described by $\sigma(z,\ u,\ v) = (v\,/\,(z \cdot u),\ v,\ u)$ (see 1.8). Any pair of elements $x,\ y \in K$ define an arrow $T \xrightarrow{(x,\,y)} T \times T$ in the topos, $(x,\ y)(u) = (x \cdot u,\ y \cdot u)$. Let $K \xrightarrow{s} K$ be defined by $s(z) = (y/x)/z$ (with an inverse given by $h(z) = z \setminus (y/x)$). It is immediate to check that $(x,\ y,\ s)$ is an action triple (2.1). This proves the statement since the family of arrows $T \xrightarrow{(x,\,y)} T \times T$, all $x,\ y \in K$, is an epimorphic family.

2. γ^*K as an object of $\mathcal{G}_T$ has the constant split structure, and $\gamma^*K \times T$ the (cartesian) product split structure $\gamma^*(K \times K) \times T \xrightarrow{\varepsilon} (\gamma^*K \times T) \times T$, described by $\varepsilon(x,\ y,\ u) = (x,\ y \cdot u,\ u)$. It is immediate to check that the group product $K \times K \to K$ furnishes the function that lifts the action $\gamma^*K \times T \longrightarrow T$ into a morphism of covering projections. $\square$

Given two torsors T, H, an arrow $T \xrightarrow{f} H$ in $\mathcal{E}$ determines a refinement $f = (f,\ id_1)$ of the respective covers, so that H can be viewed as an object $f^*H \in \mathcal{G}_T$.

Proposition 6.3. *An arrow in* $T \xrightarrow{f} H$ *in* $\mathcal{E}$ *between torsors is equivariant (that is, it is a morphism of torsors) if and only if the pair* $(f,\ id_K)$ *is a morphism* $T \longrightarrow f^*H$ *of covering projections (that is, a morphism in* $\mathcal{G}_T$*).*

Proof. It is immediate to check (see Remark 3.2) that the identity function $K \xrightarrow{id} K$ lifts f to a morphism of covering projections if and only if f respects the actions. $\square$

We now consider a cover $\mathcal{U}$ and a torsor in the topos $\mathcal{G}_{\mathcal{U}}$. It consists of a covering projection $T = (T, S \to I, \theta)$, an action $\gamma^*K \times T \longrightarrow T$ which comes together with an action $\{K \times S_i \longrightarrow S_i\}_{i\in I}$ in the topos $\mathcal{S}_{/I}$, such that $\theta_i(x \cdot s,\ u) = x \cdot \theta_i(s,\ u)$. The torsor in $\mathcal{S}_{/I}$ is non-canonically (and it seems choice dependent) isomorphic to the canonical torsor $\{K \times K \longrightarrow K\}_{i\in I}$ ([9, 8.31], [1, Expose IV 7.2.5]). Given a section $I \xrightarrow{s} S$, it determines an isomorphism $\{K \xrightarrow{s_i} S_i,\ \ s_i(x) = x \cdot s_i\}_{i\in I}$. It also determines a refinement (that we denote θ_s) of covers $\theta_s = \{\ U_i \xrightarrow{\theta(s_i,\,-)} T\ \}_{i\in I}$.

Proposition 6.4. *Given any torsor in $\mathcal{G}_{\mathcal{U}}$ as above, the pair $(id_T,\ \{s_i\}_{i\in I})$ establishes an isomorphism $\theta_s^*(T,\ K,\ \varepsilon) \xrightarrow{\cong} (T, S \to I, \theta)$ of torsors in $\mathcal{G}_{\mathcal{U}}$.*

Proof. Notice that the action in both covering projections is the same. It is immediate to check (see Remark 3.2) that $\{s_i\}_{i\in I}$ lifts id_T to a morphism of covering projections. □

Theorem 6.5. *The faithful functor $c\mathcal{G}(\mathcal{E}) \xrightarrow{\upsilon^*} \mathcal{E}$ (4.2) establishes an equivalence of categories $Tors^K(c\mathcal{G}(\mathcal{E})) \xrightarrow{\cong} Tors^K(\mathcal{E})$.*

Proof. It follows from Propositions 6.3 and 6.4 that the torsor defined in Proposition 6.2 determines an inverse $Tors^K(\mathcal{E}) \xrightarrow{\cong} Tors^K(c\mathcal{G}(\mathcal{E}))$. □

It then follows from this theorem and Proposition 6.1.

Theorem 6.6. *Given any topos $\mathcal{E}$ and group $K \in \mathcal{S}$, the fundamental localic progroupoid $\pi_1(\mathcal{E})$ represents K-torsors. That is, there is an equivalence of categories $pro\mathcal{G}rpd[\pi_1(\mathcal{E}),\ K] \cong \mathcal{T}ors^K(\mathcal{E})$.* □

Of course, this equivalence induces a bijection between the sets of equivalence classes of objects.

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Appendix. The particular case of topological spaces

When the topos $\mathcal{E}$ is spatial, the situation is much simpler, since the topos $\mathcal{G}_{\mathcal{U}}$ of covering projections trivialized by a fixed cover $\mathcal{U}$ can be defined simply as the descent topos $\mathcal{D}_{\mathcal{U}}$ in the bottom row of diagram (1.6).

Topologists have dealt successfully with covering projections of non-locally connected topological spaces. In their work, the descent data underneath the notion of covering projection has to be taken into account in one way or another. In the paper of Hernandez Paricio [8] we can see an implicit situation of classical topological descent as described in the introduction to "Categories Fibrees et Descente", [7, Expose VI]. Once the descent datum is made explicit, the category of covering projections of a topological space trivialized by a (fixed) covering is, by its very definition, the classifying topos of a discrete groupoid, and this groupoid can be explicitly constructed as the free category over the Cech nerve of the covering. The assignment of this groupoid is functorial on the filtered poset of covering sieves and determines the fundamental progroupoid of the space.

We now describe briefly all this. Given a topological space B, when B is not locally connected, covering projections cannot be considered as local homeomorphisms of a particular type, but should be considered as local homeomorphisms with an added structure. This is reflected by the fact that not all continuous maps between the underlying sheaves are admitted, but only those that preserve the trivialization structure. This determines a category (topos) $\mathcal{P}_{\mathcal{U}}$, a geometric morphism $Sh(B) \to \mathcal{P}_{\mathcal{U}}$, with faithful but not full inverse image $\mathcal{P}_{\mathcal{U}} \to ShB$) (where $Sh(B)$ is the topos of sheaves over B), and a surjective point $\mathcal{S}_{/I} \to \mathcal{P}_{\mathcal{U}}$, which is not (contrary to the case of a locally connected space) of effective descent.

Not all locally constant sheaves over B should be admitted as covering projections. Consider a sheaf $X \to B$ split by a cover $\mathcal{U} = \{U_i\}_{i\in I}$, $U_i \subset B$, by means of homeomorphisms $S_i \times U_i \xrightarrow{\theta_i} X|_{U_i}$. Given $i, j,\ \ U_i \cap U_j \neq \emptyset$, there is an induced homeomorphism $S_i \times (U_i \cap U_j) \xrightarrow{\theta_i} X|_{U_i\cap U_j} \xrightarrow{\theta_j^{-1}} S_j \times (U_i \cap U_j)$ over $U_i \cap U_j$. The following definition is essentially Definition 2.1 in [8].

Definition A.1. A covering projection split (or trivialized) by $\mathcal{U}$ is a locally constant sheaf X such that the bijections between the fibers $S_i \times \{x\} \to S_j \times \{x\}$ are given by the same function for all the points $x \in U_i \cap U_j$. That is, $\forall x, y \in U_i \cap U_j$, $\theta_j^{-1} \circ \theta_i(-,\ x) = \theta_j^{-1} \circ \theta_i(-,\ y)$.

Covering projections $X \to B$ trivialized by $\mathcal{U}$ define a full subcategory $\mathcal{D}_{\mathcal{U}} \subset \mathcal{P}_{\mathcal{U}}$, $\mathcal{D}_{\mathcal{U}}$ is a topos, and there is a (connected) geometric morphism $\mathcal{P}_{\mathcal{U}} \to \mathcal{D}_{\mathcal{U}}$ with inverse image given by the full inclusion. Thus $\mathcal{D}_{\mathcal{U}}$ is equipped with a surjective point $\mathcal{S}_{/I} \to \mathcal{P}_{\mathcal{U}} \to \mathcal{D}_{\mathcal{U}}$ which is of effective descent. In fact, the collection of homeomorphisms $\theta_j^{-1} \circ \theta_i : S_i \times (U_i \cap U_j) \to S_j \times (U_j \cap U_i)$ defines a situation of classical (topological) descent [7, Expose VI]. The condition in Definition A.1 means that there are bijections $\lambda_{ji} : S_i \to S_j$ which induce the composite homeomorphisms $\theta_j^{-1} \circ \theta_i$. The sheaf X together with the trivialization $\{\theta_i\}_{i\in I}$ can be recovered by descent from the family of topological spaces $X_i = S_i \times U_i$ and the bijections $\lambda_{ji} : S_i \to S_j$. The cover $\mathcal{U}$ determines a simplicial set $\mathcal{U}_\bullet$ (the Cech nerve) whose n-simplexes are given by $\mathcal{U}_n = \{(i_0, i_1, \ldots i_n) \mid U_{i_0} \cap U_{i_1} \cdots \cap U_{i_n} \neq \emptyset\}$ (notice that $\mathcal{U}_0 = I$). In turn, this determines a simplicial topos $\mathcal{S}_{/\mathcal{U}_\bullet}$ by slicing. The family of bijections λ_{ji} is exactly a *descent datum* on the object $S \to I$ in the topos $\mathcal{S}_{/I}$. The topos $\mathcal{D}_{\mathcal{U}}$ is (equivalent to) the descent topos $\mathcal{S}_{\mathcal{U}_\bullet} \longrightarrow \mathcal{D}_{\mathcal{U}}$, and as such, the morphism $\mathcal{S}_{/I} \to \mathcal{D}_{\mathcal{U}}$ is of effective descent. In this case, the topos $\mathcal{D}_{\mathcal{U}}$ is just the descent topos in the bottom row of diagram (1.6). As such, it is the classifying topos of a discrete groupoid $\pi\mathcal{U}$ (whose objects are the index set of the cover). Furthermore, this groupoid can be explicitly constructed as the free category over the nerve of the covering.

In particular, the topos $\mathcal{D}_{\mathcal{U}}$ is locally connected, so *even though covering projections are not locally connected topological spaces, their set of connected components (relative to a trivialization) can be constructed.*

The collection of faithful functors $\mathcal{D}_{\mathcal{U}} \to Sh(B)$ form a cone over the category of refinements, and this allows the construction of the category of all covering projections $Cp(B) \to Sh(B)$ as the colimit of the system of categories $\mathcal{D}_{\mathcal{U}}$ [8] (compare with the construction $c\mathcal{G}(\mathcal{E})$ in Section 4). The system of groupoids $\pi\mathcal{U}$, with $\mathcal{U}$ running over the filtered poset of covering sieves, determines a progroupoid, whose category of actions (as defined in [8]) is equivalent to $Cp(B)$.

When the space B is locally connected, the condition on Definition A.1 is vacuous, $\mathcal{P}_{\mathcal{U}} = \mathcal{D}_{\mathcal{U}}$, and this theory yields the classical Galois theory of locally connected topological spaces.

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