Cyclically $t$-complementary uniform hypergraphs

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A R T I C L E I N F O

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A B S T R A C T

A cyclically $t$-complementary $k$-hypergraph is a $k$-uniform hypergraph with vertex set $V$ and edge set $E$ for which there exists a permutation $\theta \in \text{Sym}(V)$ such that the sets $E, E^{\theta}, E^{\theta^2}, \ldots, E^{\theta^{t-1}}$ partition the set of all $k$-subsets of $V$. Such a permutation $\theta$ is called a $(t, k)$-complementing permutation. The cyclically $t$-complementary $k$-hypergraphs are a natural and useful generalization of the self-complementary graphs, which have been studied extensively in the past due to their important connection to the graph isomorphism problem.

For a prime $p$, we characterize the cycle type of the $(p^r, k)$-complementing permutations $\theta \in \text{Sym}(V)$ which have order a power of $p$. This yields a test for determining whether a permutation in $\text{Sym}(V)$ is a $(p^r, k)$-complementing permutation, and an algorithm for generating all of the cyclically $p^r$-complementing $k$-hypergraphs of order $n$, for feasible $n$, up to isomorphism. We also obtain some necessary and sufficient conditions on the order of these structures. This generalizes previous results due to Ringel, Sachs, Adamus, Orcl, Szymański, Wojda, Zwonek, and Bernaldez.

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1. Introduction

For a finite set $V$ and a positive integer $k$, let $V^{(k)}$ denote the set of all $k$-subsets of $V$. A hypergraph with vertex set $V$ and edge set $E$ is a pair $(V, E)$ in which $V$ is a finite set and $E$ is a collection of subsets of $V$. A hypergraph $(V, E)$ is called $k$-uniform (or a $k$-hypergraph) if $E$ is a subset of $V^{(k)}$. The parameters $k$ and $|V|$ are called the rank and the order of the $k$-hypergraph, respectively. The vertex set and the edge set of a hypergraph $X$ will often be denoted by $V(X)$ and $E(X)$, respectively. Note that a 2-hypergraph is a graph. An isomorphism between $k$-hypergraphs $X$ and $X'$ is a bijection $\phi : V(X) \rightarrow V(X')$ which
induces a bijection from $E(X)$ to $E(X')$. If such an isomorphism exists, the hypergraphs $X$ and $X'$ are said to be isomorphic.

A $k$-hypergraph $X = (V, E)$ is cyclically $t$-complementary if there exists a permutation $\theta$ on $V$ such that the sets $E, E^\theta, E^{\theta^2}, \ldots, E^{\theta^{t-1}}$ partition $V^{(k)}$. We denote the set $E^\theta$ by $E_i$. Note that $E_i^\theta = E_{i+1}$ for $i = 0, 1, \ldots, t-2$ and $E_t^\theta = E_0 = E$. Such a permutation $\theta$ is called a $(t, k)$-complementing permutation, and it gives rise to a family of $t$ isomorphic $k$-hypergraphs $\{X_i = (V, E_i) : i = 0, 1, \ldots, t-1\}$ which partition the complete $k$-hypergraph on $V$, and which are permuted cyclically under the action of $\theta$.

The cyclically $t$-complementary $k$-hypergraphs have been previously defined and studied in the cases where $t = 2$ or $k = 2$, and there is some overlap and some contradiction between the terminology used in these cases. The cyclically 2-complementary 2-hypergraphs are the self-complementary graphs. In 1978, Colbourn and Colbourn [3] showed that one of the most important problems in graph theory, the graph isomorphism problem, is polynomially equivalent to the problem of determining whether two self-complementary graphs are isomorphic.

Since then, there has been a great deal of research into self-complementary graphs. A good reference on self-complementary graphs and their generalizations was written by Farrugia [4]. The cyclically 2-complementary $k$-hypergraphs are the self-complementary $k$-hypergraphs studied in [5,8,10–12], and in the terminology of these papers the $(2, k)$-complementing permutations are their corresponding ‘$k$-complementing permutations’, or ‘anti-isomorphisms’. The cyclically $t$-complementary graphs (2-hypergraphs) are the $t$-complementary graphs, or $t$-c graphs, studied in [1,2] and in the terminology of these papers the $(t, 2)$-complementing permutations are their corresponding ‘$t$-complementing permutations’ or ‘$t$-c permutations’.

Whether or not a permutation $\theta$ is $(t, k)$-complementing depends entirely on the cycle type of $\theta$. The cycle type of the $(2, 2)$-complementing permutations was characterized in [6,7] and the cycle types of the $(2, 3)$- and $(2, 4)$-complementing permutations were characterized in [8] and [9], respectively. Quite recently, these earlier results were generalized to characterize the cycle type of the $(2, k)$-complementing permutations in [5,10,12], and the cycle type of the $(t, 2)$-complementing permutations was determined in [1,2]. In Theorem 3.2, we generalize both of these new results and characterize the cycle type of the $(q, k)$-complementing permutations which have order a power of $p$, where $q = p^r$ is a prime power. We will show that this is sufficient to characterize all of the $(q, k)$-complementing permutations for these $q$, and we obtain necessary and sufficient conditions on the order of a $q$-complementary $k$-hypergraph.

In Section 2, we will prove some useful facts about $(t, k)$-complementing permutations, and then in Section 3, we will use these facts to prove the main result in Theorem 3.2. This yields Corollary 3.3, which gives a method for testing any permutation algorithmically to determine whether it is $(q, k)$-complementing, and a method for generating all of the cyclically $q$-complementary $k$-hypergraphs of order $n$, for feasible $n$. In Section 4, we obtain Corollary 4.1, which gives necessary and sufficient conditions on the order of a $q$-complementary $k$-hypergraph in the case where $q$ is a prime power, and these conditions simplify in the case where $q$ is prime.

2. The $(t, k)$-complementing permutations

We have the following natural characterization of the $(t, k)$-complementing permutations.

Lemma 2.1. Let $V$ be a finite set, let $k$ and $t$ be positive integers, and let $\theta \in \text{Sym}(V)$. Then the following three statements are equivalent:

1. $\theta$ is a $(t, k)$-complementing permutation.
2. $A_j^{\theta^i} \neq A$ for $j \neq 0 \pmod{t}$, for all $A \in V^{(k)}$.
3. The sequence $A, A_j^{\theta}, A_j^{\theta^2}, \ldots, A_j^{(t-1)}$ has length divisible by $t$, for all $A \in V^{(k)}$.

Proof. Suppose $\theta$ is a $(t, k)$-complementing permutation. Then there is a $k$-hypergraph $X = (V, E)$ such that $E_0, E_1, \ldots, E_{t-1}$ partitions $V^{(k)}$, where $E_i = E_i^{\theta^i}$. Let $A \in V^{(k)}$. Then $A \in E_i$ for exactly one $i \in \{0, 1, \ldots, t-1\}$. If $j \neq 0 \pmod{t}$, then $A_j^{\theta^i} \in E_i^{\theta^j} = E_{i+j \pmod{t}} \neq E_i$. Hence $A_j^{\theta^i} \neq A$, and so in particular $A_j^{\theta^i} \neq A$. Hence (1) implies (2).
Suppose (2) holds. Let \( j \) be the length of a sequence in (3). Then \( A^{\theta^j} = A \), and so (2) implies that \( j \equiv 0 \pmod{t} \). Hence (2) implies (3).

Suppose (3) holds. To show that (3) implies (1), we describe a simple algorithm which takes a permutation \( \theta \) satisfying (3) as input, and returns the nonempty set \( \mathcal{H}_\theta \) of all cyclically \( t \)-complementary \( k \)-hypergraphs \( X \) on \( V \) that have \( \theta \) as a \((t, k)\)-complementing permutation. This algorithm was previously described for the case where \( k = 2 \) by Adamus et al. [1].

**Algorithm 2.2.** Let \( \theta \in \text{Sym}(V) \) satisfy (3).

(I) Construct the orbits \( \mathcal{O}_1, \ldots, \mathcal{O}_m \) of \( \theta \) on \( V^{(k)} \). Each orbit \( \mathcal{O}_j \) has the form

\[
A, A^\theta, A^{\theta^2}, A^{\theta^3}, \ldots,
\]

where \( A \in V^{(k)} \), and hence each orbit \( \mathcal{O}_j \) is a sequence in (3).

(II) For each \( j \in \{1, 2, \ldots, m\} \), choose \( i \in \{0, 1, \ldots, t-1\} \) and let \( E^j_i \) denote the set of \( k \)-sets of the form \( A^{\theta^{iz+j}} \) in the orbit \( \mathcal{O}_j \) constructed in (I), where \( z \) is an integer. Since (3) holds, each orbit \( \mathcal{O}_j \) has length divisible by \( t \). Thus, within each orbit \( \mathcal{O}_j, \theta \) maps \( E^j_i \) to \( E^j_{i+1} \) for each \( i = 0, 1, \ldots, t-2 \), and \( \theta \) maps \( E^j_{t-1} \) to \( E^j_0 \).

(III) Let \( E \) be a subset of \( V^{(k)} \) that contains exactly one of the sets \( E^j_0, E^j_1, E^j_2, \ldots, E^j_{t-1} \) constructed in (II) for each \( j \in \{1, 2, \ldots, m\} \). Then \( X = (V, E) \) is a cyclically \( t \)-complementary \( k \)-hypergraph. Moreover, if there are \( m \) orbits of \( \theta \) on \( V^{(k)} \), then there are \( t^m \) different choices for the edge set \( E \), and the \( t^m \) different choices for \( E \) generate the set \( \mathcal{H}_\theta \) of all \( t^m \) cyclically \( t \)-complementary \( k \)-hypergraphs on \( V \) for which \( \theta \) is a \((t, k)\)-complementing permutation. \( \square \)

In the next lemma, we obtain some useful properties of \((t, k)\)-complementing permutations. For a permutation \( \theta \) on a set \( V \), the symbol \(|\theta|\) denotes the order of \( \theta \) in \( \text{Sym}(V) \).

**Lemma 2.3.** Let \( V \) be a finite set, and let \( s, t \) and \( k \) be positive integers such that \( \gcd(t, s) = 1 \).

(1) A permutation \( \theta \in \text{Sym}(V) \) is a \((t, k)\)-complementing permutation if and only if \( \theta^s \) is a \((t, k)\)-complementing permutation.

(2) The order of a \((t, k)\)-complementing permutation is divisible by \( t \).

(3) If \( q = p^i \) is a prime power, every cyclically \( q \)-complementary \( k \)-hypergraph has a \((q, k)\)-complementing permutation with order a power of \( p \).

**Proof.** (1) If \( \theta \in \text{Sym}(V) \) is a \((t, k)\)-complementing permutation, then there is a cyclically \( t \)-complementary \( k \)-hypergraph \( X = (V, E) \) such that the sets \( E_0, E_1, \ldots, E_{t-1} \) partition \( V^{(k)} \), where \( E_i = E^{\theta^j} \). Consider the sequence

\[
E_0, E_s, E_{2s}, E_{3s}, \ldots, E_{(t-1)s},
\]

where each subscript is taken modulo \( t \). If \( i \equiv js \pmod{t} \) for some \( i, j \) where \( 0 \leq i < j \leq t-1 \), then since \( \gcd(s, t) = 1 \) we must have \( i \equiv j \pmod{t} \), a contradiction. Hence the subscripts \( 0, s, 2s, 3s, \ldots, (t-1)s \) are pairwise incongruent modulo \( t \), and hence the sets \( E_0, E_s, E_{2s}, E_{3s}, \ldots, E_{(t-1)s} \) (with subscripts taken modulo \( t \)) also partition \( V^{(k)} \). That is, the sets

\[
E, E^{\theta^s}, E^{(\theta^s)^2}, \ldots, E^{(\theta^s)^{t-1}}
\]

partition \( V^{(k)} \), and so \( \theta^s \) is also a \((t, k)\)-complementing permutation of \( X \).

Conversely, suppose that \( \theta^s \) is a \((t, k)\)-complementing permutation. Then Lemma 2.1 guarantees that each orbit of \( \theta^s \) on \( V^{(k)} \) has cardinality congruent to 0 modulo \( t \). Observe that each orbit of \( \theta^s \) on \( V^{(k)} \) is contained in an orbit of \( \theta \) on \( V^{(k)} \). Also, every \( k \)-subset in an orbit of \( \theta \) on \( V^{(k)} \) must certainly lie in an orbit of \( \theta^s \) on \( V^{(k)} \). Since the orbits of \( \theta^s \) on \( V^{(k)} \) are pairwise disjoint, it follows that every orbit of \( \theta \) on \( V^{(k)} \) is a union of pairwise disjoint orbits of \( \theta^s \) on \( V^{(k)} \), each of which has cardinality divisible by \( t \). Hence every orbit of \( \theta \) on \( V^{(k)} \) has cardinality divisible by \( t \), and so by Lemma 2.1, \( \theta \) is a \((t, k)\)-complementing permutation.

(2) This follows directly from Lemma 2.1(2).
(3) Let \( X = (V, E) \) be a cyclically \( q \)-complementary \( k \)-hypergraph. Then \( X \) has a \((q, k)\)-complementing permutation \( \sigma \in \text{Sym}(V) \), and by part (2), the order of \( \sigma \) is divisible by \( q \), and hence by \( p \). Thus \(|\sigma| = p^a b\) for a positive integer \( a \) and an integer \( b \) such that \( p \) does not divide \( b \). Since \( \gcd(b, q) = 1 \), part (1) implies that \( \theta = \sigma^b \) is also a \((q, k)\)-complementing permutation of \( X \), and its order is \(|\theta| = p^a \). \( \Box \)

3. Cycle types of \((q, k)\)-complementing permutations

For a prime power \( q = p^r \), Theorem 3.2 gives a characterization of the cycle types of the \((q, k)\)-complementing permutations which have order equal to a power of \( p \), in terms of the base-\( p \) representation of \( k \). We will make use of the following technical lemma to prove Theorem 3.2.

**Lemma 3.1** ([5]). Let \( \ell \) and \( p \) be positive integers, where \( p \geq 2 \). Let \( a_0, a_1, \ldots, a_{\ell-1} \) be nonnegative integers such that \( \sum_{i=0}^{\ell-1} a_i p^i \geq p^\ell \). Then there exists a sequence of integers \( c_0, c_1, \ldots, c_{\ell-1} \), where \( 0 \leq c_i \leq a_i \), such that \( \sum_{i=0}^{\ell-1} c_i p^i = p^\ell \). \( \Box \)

To state and prove Theorem 3.2, we require some terminology and notation. We will denote the base-\( p \) representation of an integer \( k \) by \( b(p, k) \), where \( b(p, k) \) is the vector \((b_m, b_{m-1}, \ldots, b_1, b_0) \). That is, \( b(p, k) \) is the vector such that \( k = \sum_{i=0}^{m} b_i p^i \), with \( b_i = 0 \) for \( i > m \). The support of the base-\( p \) representation \( b = b(p, k) \) is the set \( \{i \in \{0, 1, 2, \ldots, m\} : b_i \neq 0\} \), and is denoted by \( \text{supp}(b) \). For positive integers \( m \) and \( n \), let \( n(m) \) denote the unique integer in \( \{0, 1, \ldots, m-1\} \) such that \( n \equiv n(m) \pmod{m} \). For a permutation \( \theta \) on a set \( V \), an invariant set of \( \theta \) is a subset of \( V \) which is fixed setwise by \( \theta \).

**Theorem 3.2.** Let \( V \) be a finite set and let \( k \) be a positive integer such that \( k \leq |V| \). Let \( q = p^r \) be a prime power, and let \( b = b(p, k) = (b_m, b_{m-1}, \ldots, b_2, b_1, b_0) \) be the base-\( p \) representation of \( k \). Let \( \theta \in \text{Sym}(V) \) be a permutation whose order is a power of \( p \). For an integer \( m \geq 0 \), let \( A_m \) denote those points of \( V \) contained in cycles of \( \theta \) of length at most \( p^m \). Then \( \theta \) is a \((q, k)\)-complementing permutation if and only if there is \( \ell \in \text{supp}(b) \) such that

\[ |A_{\ell+r-1}| < k_{[p^{\ell+1}]} \]

**Proof.** (\( \Rightarrow \))

**Claim 1:** If \( \theta \in \text{Sym}(V) \) has order a power of \( p \) and \( |A_{\ell}| \geq k_{[p^{\ell+1}]} \) for all \( \ell \in \text{supp}(b) \), then \( \theta \) has an invariant set of size \( k \).

**Proof of Claim 1:** Suppose that \( \theta \in \text{Sym}(V) \) has order a power of \( p \), and that \( |A_{\ell}| \geq k_{[p^{\ell+1}]} \) for all \( \ell \in \text{supp}(b) \). Every cycle of \( \theta \) has length a power of \( p \). Let \( a_i \) denote the number of cycles of \( \theta \) of length \( p^i \). If \( a_i \geq b_i \) for every \( i \in \text{supp}(b) \), then there would be an invariant set of \( \theta \) of cardinality \( \sum_{i \in \text{supp}(b)} b_ip^i = k \), as claimed. Hence we may assume that, for some \( i \in \text{supp}(b) \), \( a_i < b_i \). Let

\[ L = \{i \in \text{supp}(b) : a_i < b_i\}. \]  

(1)

Then \( L \neq \emptyset \). Since \( L \subseteq \text{supp}(b) \), we have \( |A_{\ell}| \geq k_{[p^{\ell+1}]} \) for all \( \ell \in L \).

Now \( |A_{\ell}| = \sum_{i=\ell}^{\ell} a_i p^i \). Note that \( k_{[p^{\ell+1}]} = \sum_{i=0}^{\ell} b_ip^i \). Thus, by assumption, \( \sum_{i=\ell}^{\ell} a_i p^i \geq \sum_{i=0}^{\ell} b_ip^i \) for all \( \ell \in L \). Let

\[ L = \{\ell_1, \ell_2, \ldots, \ell_z\} \]

where \( \ell_1 < \ell_2 < \cdots < \ell_z \).

- **Claim 1A:** Let \( x \in \{1, 2, \ldots, z\} \). If \( |A_{\ell_x}| \geq \sum_{j=0}^{\ell_x} b_j p^j \) for all \( j \in \{1, 2, \ldots, x\} \), then \( \theta|_{A_{\ell_x}} \) has an invariant set of size \( \sum_{j=0}^{\ell_x} b_j p^j \).

**Proof of Claim 1A:** The proof is by induction on \( x \).
Since \( \ell_1 \) is the smallest element of the set \( L \) defined in (1), it follows that \( a_i \geq b_i \) for \( 0 \leq i \leq \ell_1 - 1 \) and \( a_{\ell_1} < b_{\ell_1} \). Thus (2) implies that

\[
\sum_{i=0}^{\ell_1-1} (a_i - b_i)p^i \geq (b_{\ell_1} - a_{\ell_1})p^{\ell_1}
\]

holds with \( a_i - b_i \geq 0 \) for all \( i = 1, 2, \ldots, \ell_1 - 1 \). Applying Lemma 3.1 \( b_{\ell_1} - a_{\ell_1} \) times, we obtain a sequence \( c_0, c_1, \ldots, c_{\ell_1-1} \) such that \( 0 \leq c_i \leq (a_i - b_i) \) for \( 0 \leq i \leq \ell_1 - 1 \), and

\[
\sum_{i=0}^{\ell_1-1} c_ip^i = (b_{\ell_1} - a_{\ell_1})p^{\ell_1}
\]

Now let \( \hat{a}_i = b_i + c_i \) for \( 1 \leq i \leq \ell_1 - 1 \) and let \( \hat{a}_{\ell_1} = a_{\ell_1} \). Then

\[
0 \leq \hat{a}_i = b_i + c_i \leq b_i + (a_i - b_i) = a_i
\]

for \( 1 \leq i \leq \ell_1 - 1 \) and hence \( 0 \leq \hat{a}_i \leq a_i \) for \( 0 \leq i \leq \ell_1 \). Moreover

\[
\sum_{i=0}^{\ell_1} \hat{a}_ip^i = \sum_{i=0}^{\ell_1-1} b_ip^i + \sum_{i=0}^{\ell_1-1} c_ip^i + a_{\ell_1}p^{\ell_1} = \sum_{i=0}^{\ell_1-1} b_ip^i + (b_{\ell_1} - a_{\ell_1})p^{\ell_1} + a_{\ell_1}p^{\ell_1}
\]

and hence

\[
\sum_{i=0}^{\ell_1} \hat{a}_ip^i = \sum_{i=0}^{\ell_1} b_ip^i.
\]

The sum \( \sum_{i=0}^{\ell_1} \hat{a}_ip^i \) is the sum of the lengths of a collection of cycles of \( \theta|_{A_{\ell_1}} \), and hence it is the size of an invariant set of \( \theta|_{A_{\ell_1}} \), as required.

**Induction Step:** Let \( 2 \leq x \leq z \) and assume that if \( |A_{\ell_j}| \geq \sum_{i=0}^{\ell_j} b_ip^i \) for all \( j \in \{1, 2, \ldots, x - 1\} \), then \( \theta|_{A_{\ell_{x-1}}} \) has an invariant set of size \( \sum_{i=0}^{\ell_{x-1}} b_ip^i \). Now suppose that \( |A_{\ell_j}| \geq \sum_{i=0}^{\ell_j} b_ip^i \) for all \( j \in \{1, 2, \ldots, x\} \). Then certainly \( |A_{\ell_j}| \geq \sum_{i=0}^{\ell_j} b_ip^i \) for all \( j \in \{1, 2, \ldots, x - 1\} \), and so by the induction hypothesis, \( \theta|_{A_{\ell_{x-1}}} \) has an invariant set of size \( \sum_{i=0}^{\ell_{x-1}} b_ip^i \). By the definition of \( L \) in (1), \( a_i \geq b_i \) for \( \ell_{x-1} < i < \ell_x \). Thus \( \theta|_{A_{\ell_{x-1}}} \) has an invariant set of size \( \sum_{i=0}^{\ell_{x-1}} b_ip^i \). This implies that there is a sequence of integers \( c_0, c_1, \ldots, c_{\ell_{x-1}} \) such that \( 0 \leq c_i \leq a_i \) for \( 0 \leq i \leq \ell_{x-1} - 1 \), and

\[
\sum_{i=0}^{\ell_{x-1}} c_ip^i = \sum_{i=0}^{\ell_{x-1}} b_ip^i.
\]

Since \( |A_{\ell_x}| \geq \sum_{i=0}^{\ell_x} b_ip^i \), we have

\[
\sum_{i=0}^{\ell_x} a_ip^i \geq \sum_{i=0}^{\ell_x} b_ip^i.
\]

Since \( \ell_x \in L, a_{\ell_x} < b_{\ell_x} \), so (3) and (4) together imply that

\[
\sum_{i=0}^{\ell_{x-1}} (a_i - c_i)p^i \geq (b_{\ell_x} - a_{\ell_x})p^i.
\]
Since $a_i - c_i \geq 0$ for $0 \leq i \leq \ell_x - 1$, we can apply Lemma 3.1 $b_{\ell_x} - a_{\ell_x}$ times to obtain a sequence of nonnegative integers $d_0, d_1, \ldots, d_{\ell_x - 1}$ such that $0 \leq d_i \leq (a_i - c_i)$ for $0 \leq i \leq \ell_x - 1$, and

$$\sum_{i=0}^{\ell_x-1} d_i p^i = (b_{\ell_x} - a_{\ell_x}) p^{\ell_x}.$$ 

Now let $\hat{a}_i = c_i + d_i$ for $0 \leq i \leq \ell_x - 1$ and let $\hat{a}_{\ell_x} = a_{\ell_x}$. Then one can check that $0 \leq \hat{a}_i \leq a_i$ for $0 \leq i \leq \ell_x - 1$, and

$$\sum_{i=0}^{\ell_x} \hat{a}_i p^i = \sum_{i=0}^{\ell_x} b_i p^i.$$ 

Since $\sum_{i=0}^{\ell_x} \hat{a}_i p^i$ is the sum of the lengths of a collection of cycles of $\theta|_{A_{\ell_x}}$, we conclude that $\theta|_{A_{\ell_x}}$ has an invariant set of size $\sum_{i=0}^{\ell_x} b_i p^i$, as required.

Hence by the principle of mathematical induction, Claim 1A holds for all $x \in \{1, 2, \ldots, z\}$. \(\square\)

Now applying Claim 1A with $x = z$, we observe that since $|A_{j}| \geq \sum_{i=0}^{\ell_x} b_i p^i$ for all $j \in \{1, 2, \ldots, z\}$, $\theta|_{A_{\ell_x}}$ has an invariant set of size $\sum_{i=0}^{\ell_x} b_i p^i$. But since $\ell_x$ is the largest element of $L$, $\theta$ contains $b_i$ cycles of length $p^i$ for all $\ell \in supp(b)$ with $\ell < \ell_x \leq m$, and hence $\theta$ contains an invariant set of size $\sum_{i=0}^{m} b_i p^i = k$. This proves Claim 1.

Now suppose that $\theta \in Sym(V)$ is a $(q, k)$-complementing permutation with order a power of $p$. For an integer $j$, let $A_j^i$ denote the set of elements of $V$ which lie in cycles of $\theta^i$ of length at most $p^i$. Note that $A_j^0 = A_{j,0}$ where $a$ is the largest integer such that $p^a$ divides $j$.

If $|A_{\ell+r-1}| \geq k_{\ell p^{\ell_r+1}}$ for all $\ell \in supp(b)$, then for $j = p^{r-1}$ we have $|A_j^i| = |A_{\ell+r-1}| \geq k_{p^{\ell_r+1}}$ for all $\ell \in supp(b)$. Hence Claim 1 implies that $\theta^i$ has an invariant set of size $k$. But since $q = p^r$, $j = p^{r-1} \neq 0 \pmod{q}$, and so the fact that $\theta^j$ fixes a $k$-subset of $V$ contradicts Lemma 2.1. We conclude that $|A_{\ell+r-1}| < k_{p^{\ell_r+1}}$ for some $\ell \in supp(b)$, as claimed.

$(\Leftarrow)$ Let $\theta \in Sym(V)$ with order a power of $p$ and suppose that there is $\ell \in supp(b)$ such that $|A_{\ell+r-1}| < k_{p^{\ell_r+1}}$. Let $j$ be an integer such that $j \neq 0 \pmod{q}$. Then $j = b^{p^r}b$ for integers $a$ and $b$ where $0 \leq a < b$ and $p$ does not divide $b$. Thus $|A_j^i| = |A_{\ell+r}| \leq |A_{\ell+r-1}| < k_{p^{\ell_r+1}}$. This implies that $\theta^i$ does not have an invariant set of size $k$. Since $j$ was chosen arbitrarily, we conclude that $A_j^i \neq A$ for all $j \neq 0 \pmod{q}$ and all $A \in V(k)$, and so Lemma 2.1 implies that $\theta$ is a $(q, k)$-complementing permutation. \(\square\)

Lemma 2.3 and Theorem 3.2 together yield the following characterization of $(q, k)$-complementing permutations.

**Corollary 3.3.** Let $k$ be a positive integer, let $q = p^r$ be a prime power, let $b = b(p, k)$ be the base-$p$ representation of $k$, and let $V$ be a finite set. A permutation $\sigma \in Sym(V)$ is a $(q, k)$-complementing permutation if and only if $|\sigma| = j p^r$ for some integers $i$ and $j$ such that $i \geq 1$ and $\gcd(p, j) = 1$, and $\theta = \sigma^j$ satisfies the condition of Theorem 3.2 for some $\ell \in supp(b)$. \(\square\)

Corollary 3.3 and the conditions of Theorem 3.2 can be used to test a permutation algorithmically to determine whether it is a $(q, k)$-complementing permutation.

If $q = p^r$ is a prime power, Lemma 2.3(3) guarantees that every cyclically $q$-complementing $k$-hypergraph has a $(q, k)$-complementing permutation which has order a power of $p$. Hence we can generate all of the cyclically $q$-complementing $k$-hypergraphs of order $n$, up to isomorphism, by applying Algorithm 2.2 to find $H_{\theta}$ for every permutation $\theta$ in $\text{Sym}(n)$ satisfying the conditions of Theorem 3.2. Moreover, if we just wish to generate at least one representative of each isomorphism class of cyclically $q$-complementing $k$-hypergraphs of order $n$, it suffices to apply Algorithm 2.2 to one permutation $\theta$ from each conjugacy class of permutations in $\text{Sym}(n)$ satisfying the conditions of Theorem 3.2.
4. Necessary and sufficient conditions on order

In this section, we present necessary and sufficient conditions on the order \( n \) of a cyclically \( q \)-complementary \( k \)-hypergraph when \( q = p^r \) is a prime power. Since Lemma 2.3(3) guarantees that every cyclically \( q \)-complementary \( k \)-hypergraph has a \((q, k)\)-complementing permutation with order equal to a power of \( p \), Theorem 3.2 immediately implies the following necessary and sufficient conditions on the order of these structures.

**Corollary 4.1.** Let \( k \) and \( n \) be positive integers, \( k \leq n \), let \( q = p^r \) be a prime power, and let \( b \) be the base-\( p \) representation of \( k \). There exists a cyclically \( q \)-complementary \( k \)-hypergraph of order \( n \) if and only if there is \( \ell \in \text{supp}(b) \) such that

\[
n_{[p^{\ell+1}]} < k_{[p^{\ell+1}]}. \tag{5}
\]

**Corollary 4.2.** Let \( k \) and \( n \) be positive integers, \( k \leq n \), let \( p \) be a prime, and let \( b \) be the base-p representation of \( k \). There exists a cyclically \( p \)-complementary \( k \)-hypergraph of order \( n \) if and only if

\[
n_{[p^{\ell+1}]} < k_{[p^{\ell+1}]} \quad \text{for some } \ell \in \text{supp}(b).
\]

**Proof.** Set \( q = p^1 \) in Corollary 4.1. Then \( r = 1 \) and so the only choice for \( a \in \{0, 1, \ldots, r - 1\} \) is \( a = 0 \). Thus there exists a cyclically \( p \)-complementary \( k \)-hypergraph if and only if condition (5) holds with \( a = 0 \) for some \( \ell \in \text{supp}(b) \).

When the rank \( k \) is within \( p - 1 \) of a multiple of a power of a prime \( p \), then Corollary 4.2 yields the following more transparent necessary and sufficient conditions on the order of a cyclically \( p \)-complementary \( k \)-hypergraph.

**Corollary 4.3.** Let \( \ell \) be a positive integer and let \( p \) be prime.

1. If \( k = b_\ell p^\ell \) for \( 0 < b_\ell < p \), then there exists a cyclically \( p \)-complementary \( k \)-hypergraph of order \( n \) if and only if \( n_{[p^{\ell+1}]} < k \).
2. If \( k = b_\ell p^\ell + b_0 \) where \( 0 < b_0, b_\ell < p \), then there exists a cyclically \( p \)-complementary \( k \)-hypergraph of order \( n \) if and only if \( n_{[p]} < b_0 \) or \( n_{[p^{\ell+1}]} < k \).

**Proof.** 1. In this case \( \text{supp}(b) = \{\ell\} \), and so Corollary 4.2 implies that there exists a cyclically \( p \)-complementary \( k \)-hypergraph of order \( n \) if and only if

\[
n_{[p^{\ell+1}]} < k_{[p^{\ell+1}]} \tag{6}
\]

Since \( k = b_\ell p^\ell < p^{\ell+1} \), \( k_{[p^{\ell+1}]} = k \), and so (6) is equivalent to \( n_{[p^{\ell+1}]} < k \).

2. In this case \( \text{supp}(b) = \{0, \ell\} \) and so Corollary 4.2 implies that there exists a cyclically \( p \)-complementary \( k \)-hypergraph of order \( n \) if and only if \( n_{[p]} < k_{[p]} \) or \( n_{[p^{\ell+1}]} < k_{[p^{\ell+1}]} \). Since \( k = b_\ell p^\ell + b_0 \), \( k_{[p]} = b_0 \) and \( k_{[p^{\ell+1}]} = k \), the result follows.

In the case where \( k = \sum_{i=0}^{s} (p - 1)p^{\ell+i} \) for a nonnegative integer \( s \), the condition of Corollary 4.2 holds for the largest integer in the support of the base-p representation of \( k \), as the next result shows.

**Corollary 4.4.** Let \( r, s \) and \( \ell \) be nonnegative integers, let \( p \) be prime, and suppose that \( k = \sum_{i=0}^{s} (p - 1)p^{\ell+i} \). Then there exists a cyclically \( p \)-complementary \( k \)-hypergraph of order \( n \) if and only if \( n_{[p^{\ell+j+1}]} < k \).

**Proof.** Suppose that there exists a cyclically \( p \)-complementary \( k \)-hypergraph of order \( n \), and let \( b \) be the base-\( p \) representation of \( k \). Then

\[
\text{supp}(b) = \{\ell, \ell + 1, \ldots, \ell + s\},
\]

and so Corollary 4.2 guarantees that

\[
n_{[p^{\ell+j+1}]} < k_{[p^{\ell+j+1}]} \tag{7}
\]
for some \( j \in \{0, 1, 2, \ldots, s\} \). If (7) holds for some \( j < s \), then the fact that
\[
n_{[p^j + (j+1)+1]} \leq (p-1)p^{j+1} + n_{[p^{j+1}]}
\]
implies that
\[
n_{[p^j + (j+1)+1]} < (p-1)p^{j+1} + k_{[p^{j+1}]}. \tag{8}
\]
Now since \((p-1)p^{j+1} + k_{[p^{j+1}]} = (p-1)p^{j+1} + \sum_{i=0}^{j} (p-1)p^{j+i} = k_{[p^{j+1}+1]}\), (8) implies that
\[
n_{[p^j + (j+1)+1]} < k_{[p^{j+1}+1]},
\]
and hence (7) also holds for \( j + 1 \). Thus, by induction on \( j \), the fact that (7) holds for some \( j \in \{0, 1, \ldots, s\} \) implies that (7) holds for \( j = s \). Hence \( n_{[p^s + s+1]} < k_{[p^{s+1}+1]} = k \).

Conversely, Corollary 4.2 guarantees that there exists a cyclically \( p \)-complementary \( k \)-hypergraph of order \( n \) for every integer \( n \) such that \( n_{[p^s]} < k \).

\section*{Corollary 4.5.} If \( k = p^\ell - 1 \), then there exists a cyclically \( p \)-complementary \( k \)-hypergraph if and only if \( n_{[p^\ell]} < k \).

\subsection*{Proof.} Since \( k = \sum_{i=0}^{\ell-1} (p-1)p^i \), the result follows directly from Corollary 4.4.

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\section*{References}


